MAT 157Y, 2005/06 – Term exam #4 – Solutions

(1) Find a rational number a (expressed in the form $a = \frac{p}{q}$) such that

$$|\sin(1) - a| < \frac{1}{3791}.$$

Indicate whether your a is larger or smaller than $\sin(1)$.

We take the sixth order Taylor polynomial of $\sin(x)$, with the Lagrange form of the remainder term. Since $\sin^{(7)} = -\cos$ this reads,

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cos(t)\frac{x^7}{5040},$$

where t is between 0 and x. Putting $x = 1$,

$$\sin(1) = 1 - \frac{1}{6} + \frac{1}{120} - \cos(t)\frac{1}{5040} = \frac{101}{120} - \cos(t)\frac{1}{5040}.$$

Put $a = \frac{101}{120}$. Since $|\cos(t)\frac{1}{5040}| \leq \frac{1}{5040} < \frac{1}{3791}$, this gives the desired approximation of $\sin(1)$. Moreover, since $0 \leq t \leq 1 < \frac{\pi}{2}$, we have $\cos(t) > 0$, hence

$$a - \sin(1) = \cos(t)\frac{1}{5040} > 0.$$

(2) a) Compute the 6th order Taylor polynomial at 0 of

$$f(x) = \exp\left(\sqrt[3]{1 + x^3} - 1\right) - \frac{x^3}{3}.$$

Let $g(u) = \exp\left(\sqrt[3]{1 + u} - 1\right) - \frac{u}{3}$. Consider the Taylor expansions, up to second order in u,

$$\begin{aligned}
\sqrt[3]{1+u} &= 1 + \frac{1}{3}u - \frac{1}{9}u^2 + \cdots \\
\exp(\sqrt[3]{1+u} - 1) &= \exp(\frac{1}{3}u - \frac{1}{9}u^2 + \cdots) \\
&= 1 + (\frac{1}{3}u - \frac{1}{9}u^2 + \cdots) + \frac{1}{2}(\frac{1}{3}u - \frac{1}{9}u^2 + \cdots)^2 + \cdots \\
&= 1 + \frac{1}{3}u - \frac{1}{18}u^2 + \cdots \\
\exp(\sqrt[3]{1+u} - 1) - \frac{u}{3} &= 1 - \frac{1}{18}u^2 + \cdots
\end{aligned}$$

Hence

$$f(x) = 1 - \frac{1}{18}x^6 + \cdots.$$

b) Decide whether f has a local minimum, local maximum, or neither at 0.

General principle: The question whether f has a local minimum, local maximum, or neither at some given point x_0 is decided by the first non-trivial term in the Taylor expansion at x_0. That is, let $n > 0$ be the smallest number such that $f^{(n)}(x_0) \neq 0$. Then, if n is even and $f^{(n)}(x_0) > 0$, there is a local minimum, if n is even and $f^{(n)}(x_0) < 0$ there is a local maximum, if n is odd there is no local extremum. In our case, we read off $f^{(6)}(0) < 0$, hence there is a local maximum.

(3) a)

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} \exp(\log(n)\frac{1}{n}) = \exp(\lim_{n\to\infty} \log(n)\frac{1}{n}) = \exp(0) = 1.$$

$$\lim_{n\to\infty} (1+\frac{a}{n})^n = \lim_{n\to\infty} \exp(n\log(1+\frac{a}{n})) = \exp(\lim_{n\to\infty} n\log(1+\frac{a}{n})) = \exp(\lim_{h\to 0} \frac{\log(1+ah)}{h}) = \exp(a).$$

In both cases we used the continuity of the exponential function.

b) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ is converges by the Leibnitz criterion, since

$$\frac{1}{1+\sqrt{n}} \geq \frac{1}{1+\sqrt{n+1}}.$$

The series $\sum_{n=1}^{\infty} \frac{\log(n)}{n^2}$ converges by the comparison test, since

$$\frac{\log(n)}{n^2} < \frac{1}{n^{3/2}}.$$

The series $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges by the comparison test, since

$$\frac{1}{\log n} > \frac{1}{n}.$$

(4) This was all done in class. For part c), take any continuous function $f(x)$ with the properties that $f \geq 0$, and $f(x) = 0$ for $x \leq 0$ or $x \geq 1$. Let $f_n(x) = nf(nx)$. Then $f_n(x) \to 0$ for all x, but $\int_0^1 f_n = \int_0^1 f = C > 0$.