MAT 157Y, 2005/06 – Term exam #4

Solve each of the following 4 problems. Each problem is worth 15 points.

Write your answers on the lined sides of the exam booklets, the blank sides won't be graded. Make sure to write your name and student number on each booklet. If you run out of space and need a second booklet, please ask the tutors. **TOOLS ALLOWED: NONE.**

For each problem, **JUSTIFY YOUR ANSWER** – just giving the final result is not enough!

You have 1 hr 50 min. **GOOD LUCK!**

In case you need any of these:

$$4! = 24,\ 5! = 120,\ 6! = 720,\ 7! = 5040,\ 8! = 40430,\ 9! = 362880.$$

(1) Find a rational number a (expressed in the form $a = \frac{p}{q}$) such that

$$|\sin(1) - a| < \frac{1}{3791}.$$

Indicate whether your a is larger or smaller than $\sin(1)$.

(2) a) Compute the 6th order Taylor polynomial at 0 of

$$f(x) = \exp\left(\sqrt[3]{1+x^3} - 1\right) - \frac{x^3}{3}.$$

(You may want to write $f(x) = g(x^3)$, and consider a Taylor polynomial (up to what order ?) of $g(u)$.)

b) Decide whether f has a local minimum, local maximum, or neither at 0.

(Even without part a), you may earn partial credit for b) by explaining the "general" approach.)

(3) a) Find the following limits:

$$\lim_{n\to\infty} \sqrt[n]{n}, \qquad \lim_{n\to\infty} (1 + \frac{a}{n})^n \quad (a \in \mathbb{R})$$

b) Decide convergence or divergence for each of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{\log(n)}{n^2}, \quad \sum_{n=2}^{\infty} \frac{1}{\log n}$$

(4) Let $f_n \colon A \to \mathbb{R}$ be a sequence of functions defined on $A \subset \mathbb{R}$.

a) State the definition of *uniform convergence* of the sequence f_n against a function $f \colon A \to \mathbb{R}$.

b) Prove that if a sequence of integrable functions $f_n \colon [a,b] \to \mathbb{R}$ converges uniformly to an integrable function $f \colon [a,b] \to \mathbb{R}$, then

$$\lim_{n\to\infty} \Big(\int_a^b f_n(x) \mathrm{d}x \Big) = \int_a^b \big(\lim_{n\to\infty} f_n(x) \big) \mathrm{d}x$$

c) Give an example of a sequence of continuous functions $f_n \colon [a,b] \to \mathbb{R}$, converging pointwise to a continuous function $f \colon [a,b] \to \mathbb{R}$, but with

$$\lim_{n\to\infty} \Big(\int_a^b f_n(x) \mathrm{d}x \Big) \neq \int_a^b \big(\lim_{n\to\infty} f_n(x) \big) \mathrm{d}x$$