MAT 157Y – Term exam #3: Solutions

No warranty: There may be lots of typos!

(1)

a) $\int_0^\infty x^3 e^{-x^2}\mathrm{d}x$. We first consider the indefinite integral $\int x^3 e^{-x^2}\mathrm{d}x$. Substitute $x^2 = u$, and use integration by parts to get

$$\int x^3 e^{-x^2}\mathrm{d}x = \frac{1}{2}\int ue^{-u}\mathrm{d}u = -\frac{1}{2}ue^{-u} + \frac{1}{2}\int e^{-u}\mathrm{d}u = -\frac{u+1}{2}\,e^{-u} = -\frac{x^2+1}{2}\,e^{-x^2}.$$

(Another approach: We *guess* a primitive of the form $p(x)e^{-x^2}$, and get an equation for p by taking the derivative.) From the primitive, we obtain the definitive integral as

$$\int_0^\infty x^3 e^{-x^2}\mathrm{d}x = \big(-\frac{x^2+1}{2}\,e^{-x^2}\big)\Big|_0^\infty = \frac{1}{2}.$$

b) $\int \frac{1}{\sin x}\mathrm{d}x$. This was discussed in class: We substitute $\tan(x/2) = u$. Then $\mathrm{d}x = \frac{2}{1+u^2}\mathrm{d}u$ and $\sin(x) = \frac{2u}{1+u^2}$, so

$$\int \frac{1}{\sin x}\mathrm{d}x = \frac{1}{u}\mathrm{d}u = \log|u| = \log|\tan(x/2)|.$$

c) $\int \frac{9x+2}{x^2-5x-6}\mathrm{d}x$. The denominator can be written as $(x+1)(x-6)$. Using a partial fractions decomposition, we obtain

$$f(x) = \frac{9x+2}{x^2-5x-6} = \frac{9x+2}{(x+1)(x-6)} = \frac{1}{x+1} + \frac{8}{x-6}.$$

(One reads off the coefficents of the partial fractions decomposition as discussed in class: for instance, the coefficient of $(x+1)^{-1}$ is obtained by putting $x=-1$ in $(x+1)f(x) = \frac{9x+2}{x-6}$. I don't insist that this step is discussed in detail.) Hence the integral is

$$\int \frac{9x+2}{x^2-5x-6}\mathrm{d}x = \log|x+1| + 8\log|x-6|.$$

d)

$$\begin{aligned}\int \sin^2(x)\,e^x\mathrm{d}x &= \int \frac{1-\cos(2x)}{2}e^x\mathrm{d}x \\ &= \frac{1}{2}e^x - \frac{1}{2}\int \cos(2x)e^x\mathrm{d}x.\end{aligned}$$

For the remaining integral, we use a double integration by parts:

$$\int \cos(2x)e^x\mathrm{d}x = \frac{1}{2}\sin(2x)e^x - \frac{1}{2}\int \sin(2x)e^x = \frac{1}{2}\sin(2x)e^x + \frac{1}{4}\cos(2x)e^x - \frac{1}{4}\int \cos(2x)e^x$$

Hence

$$\frac{5}{4}\int \cos(2x)e^x\mathrm{d}x = \frac{1}{2}\sin(2x)e^x + \frac{1}{4}\cos(2x)$$

and finally

$$\int \cos(2x)e^x\mathrm{d}x = (\frac{2}{5}\sin(2x) + \frac{1}{5}\cos(2x))e^x.$$

Again, an alternative method is to guess a primitive of the form $(A\sin(2x)+B\sin(2x))e^x$, and find A, B by differentiating.

(2) We use the following simple observation discussed in class: Suppose f, g are integrable over each $[a,b]$, and $g \geq f \geq 0$. Then $\int_a^b g \geq \int_a^b f \geq 0$ for all b. Hence if $\int_a^\infty g$ exists, then so does $\int_a^\infty f$.
a) The integral $\int_0^\infty \exp(-\frac{1}{x^2})\,\mathrm{d}x$ does *not* exist: Since $\lim_{x\to\infty}\exp(-\frac{1}{x^2}) = \exp(0) = 1$, there exists x_0 with $\exp(-\frac{1}{x^2}) \geq \frac{1}{2}$ for $x \geq x_0$. Hence $\int_{x_0}^\infty \exp(-\frac{1}{x^2})\,\mathrm{d}x$ does not exist.

b) The integral $\int_1^\infty \frac{1}{x^2}\log(\log(x))\mathrm{d}x$ exists:
It is convenient to first substitute $u = \log(x)$, i.e. $x = e^u$, $\mathrm{d}x = e^u\mathrm{d}u$. The integral becomes

$$\int_0^\infty e^{-2u}\log(u)e^u\mathrm{d}u = \int_0^\infty \log(u)e^{-u}\mathrm{d}u.$$

Now there are no convergence problems at the lower boundary, since for $0 < u \leq 1$, $-\log(u)e^{-u} \leq -\log(u)$ and $\int_0^1 \log(u)(u\log(u)-u)\big|_0^1 = -1$ exists. For the upper boundary we can e.g. use the (very rough) estimate

$$\log(u) \leq u \leq e^{u/2}$$

for $u \geq u_0$ to see that the integral exists. (It's also enough to use $\log(u) \leq u$, since we know that $\int_0^\infty ue^{-u}\mathrm{d}u$ exists.)

(3)
a) $f(x) = (x^x)^x$. We have $(x^x)^x = x^{x^2} = \exp(x^2\log(x))$. Hence, by chain rule

$$f'(x) = \exp(x^2\log(x))(2x\log(x)+x) = x^{x^2+1}(2\log(x)+1).$$

b) Using the fundamental theorem of calculus and the chain rule, $f'(x) = \sin(\sqrt{x})\exp(\sin(x)) - 5x^4\sin(\sqrt{x^5})\exp(\sin(x^5))$.
c) $f(x) = \exp(e^x\log\log(x))$ hence

$$f'(x) = \exp(e^x\log\log(x))(e^x\log\log(x) + \frac{e^x}{x\log x}) = \log(x)^{(e^x)}(e^x\log\log(x) + \frac{e^x}{x\log x}).$$

(4) a) Using the fundamental theorem of calculus, and the assumption $f(x) \leq h(x)$,

$$(\log h)'(x) = \frac{h'(x)}{h(x)} = \frac{f(x)g(x)}{h(x)} \leq g(x).$$

b) Integrating this result from a to x, we obtain

$$\log h(x) - \log A = \log h(x) - \log h(a) \leq \int_a^x g(t)\mathrm{d}t.$$

Hence

$$f(x) \leq h(x) \leq A\exp(\int_a^x g(t)\mathrm{d}t)$$

c) Integrating $f' = fg$ with $f(a) = 0$, we obtain $f(x) = \int_a^x f(t)g(t)\mathrm{d}t$ for all t. Hence Gronwall's inequality holds for *all* $A > 0$:

$$0 \leq f(x) \leq A\exp(\int_a^x g(t)\mathrm{d}t)$$

for *all* A. Thus, for any given x, $f(x)$ is less than any positive constant. This just means $f(x) = 0$.