MAT 157Y – Term exam #3

Solve each of the following 4 problems. Each problem is worth 15 points.

Write your answers on the lined sides of the exam booklets, the blank sides won't be graded. Make sure to write your name and student number on each booklet. If you run out of space and need a second booklet, please ask the tutors. You have 1 hr 50 min.

Tools allowed: None.

Good luck!

In each following problems, justify your answer – just giving the final result is not enough! Also, the use of complex numbers is *not* permitted since we did not yet discuss this in class!

(1) Compute the following integrals.

a)

$$\int_0^\infty x^3 e^{-x^2} \mathrm{d}x$$

b)

$$\int \frac{1}{\sin x} \mathrm{d}x$$

c)

$$\int \frac{9x+2}{x^2-5x-6} \mathrm{d}x$$

d)

$$\int \sin^2(x)\, e^x \mathrm{d}x$$

(2) Decide whether or not the following improper integrals exist. Make sure to consider convergence at both ends of the integral.

a)

$$\int_0^\infty \exp(-\frac{1}{x^2})\, \mathrm{d}x$$

b)

$$\int_1^\infty \frac{1}{x^2} \log(\log(x)) \mathrm{d}x$$

(Hint: You may wish to make a convenient substitution.)

(3) Find the derivatives of the following functions:

a)

$$f(x) = (x^x)^x, \quad x > 0$$

b)

$$f(x) = \int_{x^5}^{x} \sin(\sqrt{t}) \exp(\sin(t)) \mathrm{d}t, \quad x > 0$$

c)

$$f(x) = (\log(x))^{(e^x)}, \quad x > 0$$

(4) Let $f, g \colon [a, \infty) \to \mathbb{R}$ be continuous functions, both of which are ≥ 0 everywhere.

i) Suppose that there exists a constant $A > 0$ such that

$$(*) \qquad f(x) \leq A + \int_a^x f(t)g(t)\mathrm{d}t, \qquad \text{for all } x \geq a.$$

Letting $h(x) = A + \int_a^x f(t)g(t)\mathrm{d}t$, show that

$$(\log h)'(x) \leq g(x), \qquad \text{for all } x \geq a.$$

ii) Show that $(*)$ implies *Gronwall's inequality*:

$$f(x) \leq A \exp\left(\int_a^x g(t)\mathrm{d}t\right) \quad \text{for all } x \geq a$$

(with the same constant A).

iii) Show that if f, g are continuous functions on $[a, \infty)$, both of which are ≥ 0 everywhere, and if f satisfies a differential equation

$$f' = fg \quad \text{with} \quad f(a) = 0,$$

then $f = 0$.

Note: Each part depends on the preceding part. If you can't figure out (i) or (ii), you might still do the rest of the problem assuming *the result from (i) respectively (ii).*