MAT 157Y – Term exam #2: Solutions

(1) a) Since $\cos'(x) = -\sin(x) < 0$ for all $x \in (0, \pi)$, the function cos is decreasing, and in particular one-to-one. Write $y = \sin(x)$, so that $\arccos(y) = x$. We have, by the formula of the derivative of an inverse function,

$$\arccos'(y) = \frac{1}{-\sin(x)} = \frac{1}{-\sqrt{1-\cos^2(x)}} = -\frac{1}{\sqrt{1-y^2}}.$$

Here we used that $\sin(x) = \sqrt{1-\cos^2(x)}$, with the positive square root for $x \in (0, \pi)$.

b) Using the Chain rule,

$$f'(x) = 12\Big(\sin\big(\frac{x}{4}\big)\Big)^{11}\frac{1}{4}\cos(\frac{x}{4}) = 3\Big(\sin\big(\frac{x}{4}\big)\Big)^{11}\cos(\frac{x}{4}).$$

But $\sin(\pi/4) = \cos(\pi/4) = 2^{-1/2}$. Hence the result is

$$f'(\pi) = 3\ 2^{-12/2} = 3\ 2^{-6} = 3\ 4^{-3} = \frac{3}{64}.$$

(2) Solution: We compute, using quotient rule

$$\begin{aligned} f'(x) &= \frac{2x(x-1)-x^2}{(x-1)^2} \\ &= \frac{x^2-2x}{(x-1)^2} \\ &= 1 - \frac{1}{(x-1)^2} \\ f''(x) &= \frac{2}{(x-1)^3}. \end{aligned}$$

Hence, the zeroes of f are $x = 0$, and the critical points are $x = 0, 2$. The values of f at the critical points are $f(0) = 0$, $f(2) = 4$. The values of the second derivative are, $f''(0) = -2$, $f''(2) = 2$. Hence, $x = 2$ is a local minimum, while $x = 0$ is a local maximum. We have $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ for $0 < x < 1$ and for $1 < x < 2$, and $f'(x) > 0$ for $x > 2$. Also $f''(x) > 0$ for $x > 1$ (f is convex in this region) and $f''(x) < 0$ for $x < 1$ (f is concave in this region). Next,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x-1} = +\infty$$

while

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

For the behavior at infinity, we observe that

$$f(x) - x = \frac{x^2}{x-1} - x = \frac{x^2 - x(x-1)}{x-1} = \frac{x}{x-1} = \frac{1}{1-\frac{1}{x}}$$

Hence

$$\lim_{x\to\pm\infty} (f(x) - x) = 1$$

thus f approaches the line $y = x + 1$ for both $x \to \pm\infty$. Now draw the graph..

(3) a) See textbook or lecture notes. b) See textbook or lecture notes: Suppose $\mathbb{N}$ is bounded above. Then there exists a least upper bound $a \in \mathbb{R}$. Since $a - 1$ is not an upper bound, there exists $n \in \mathbb{N}$ with $n > a - 1$. But then $n + 1 \in \mathbb{N}$ with $n + 1 > a$, so a is not an upper bound. Contradiction.

(4) (a) Let $g(x) = f(-x)f(x)$. We have to show $g(x) = 1$. But

$$g'(x) = -f'(-x)f(x) + f(-x)f'(x) = -f(-x)f(x) + f(-x)f(x) = 0,$$

so g is constant: $g(x) = g(0) = 1$ for all x.

(b) The identity $f(x)f(-x) = 1$ proved in (a) shows in particular that $f(x) \neq 0$. Since $f(0) = 1 > 0$, and f is continuous, it follows that $f(x) > 0$ for all x. This also implies $f'(x) = f(x) > 0$, so f is increasing, and $f''(x) = f(x) > 0$, so f is convex.

(c) Let $h(x) = f(ax)f(x)^{-a}$. We have to show that $h(x) = 1$ for all x. Taking the derivative,

$$h'(x) = af'(ax)f(x)^{-a} + f(ax)\big(-a\,f(x)^{-a-1}f'(x)\big) = af(ax)f(x)^{-a} - af(ax)f(x)^{-a} = 0.$$

Hence h is constant: $h(x) = h(0) = 1$.

(d) Using the formula for the derivative of an inverse function we have, writing $y = f(x)$ i.e. $x = f^{-1}(y)$,

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f(x)} = \frac{1}{y}.$$