MAT 157Y – Term exam #2

Solve each of the following 4 problems. Each problem is worth 15 points.

Write your answers on the lined sides of the exam booklets, the blank sides won't be graded. Make sure to write your name and student number on each booklet. If you run out of space and need a second booklet, please ask the tutors. You have 1 hr 50 min.

Tools allowed: None.

Good luck!

(1) a) Show that $\cos\colon (0,\pi) \to \mathbb{R}$ is one-to-one, and compute the derivative of the inverse function $\arccos\colon (-1,1) \to \mathbb{R}$.

b) Compute $f'(\pi)$ where

$$f(x) = \left(\sin\left(\frac{x}{4}\right)\right)^{12}.$$

(2) Graph the function $f(x) = \frac{x^2}{x-1}$, making sure to indicate: the zeroes of f, the critical points, local maxima and minima, regions where f is increasing/decreasing, behavior at infinity and at points where f is undefined, convexity/concavity.

(3) Axiom (P13) states that if a non-empty subset $A \subseteq \mathbb{R}$ has an upper bound, then A also has a least upper bound. We proved that in this case, this least upper bound $\sup(A)$ is unique.

a) State the definition of an upper bound and a least upper bound of a non-empty set A.

b) Let $A = \mathbb{N}$ be the natural numbers. Prove that $\mathbb{N}$ does not have an upper bound. (Your proof shouldn't use any results whose proof relies on this fact.) Hint: Is a is a least upper bound then $a - 1$ is not an upper bound.

(4) There exists a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ with the properties $f' = f$ and $f(0) = 1$. (This so-called *exponential function* will be constructed later in this course.)

a) Prove that $f(x)f(-x) = 1$ for all $x \in \mathbb{R}$.

b) Prove that $f(x) > 0$ for all $x \in \mathbb{R}$, and that f is everywhere increasing and convex.

c) Prove that the function f satisfies

$$f(ax) = f(x)^a \text{ for all } x \in \mathbb{R}, a \in \mathbb{Q}.$$

(This also holds for $a \in \mathbb{R}$, but sofar we have only defined fractional powers.)

d) Find the derivative $(f^{-1})'$ of the inverse function.

Hint: For some parts of this problem, use that a function $g\colon \mathbb{R} \to \mathbb{R}$ is constant if and only if $g'(x) = 0$ everywhere.