# MAT 157Y – Term exam #1 Solutions

**(1)** We first observe that

$$\begin{aligned}\binom{\alpha}{k}+\binom{\alpha}{k-1} &= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}+\frac{\alpha(\alpha-1)\cdots(\alpha-k+2)}{(k-1)!}\\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)+\alpha(\alpha-1)\cdots(\alpha-k+2)k}{k!}\\ &= \frac{(\alpha+1)\alpha(\alpha-1)\cdots(\alpha-k+2)}{k!}\\ &= \binom{\alpha+1}{k}.\end{aligned}$$

We now proceed by induction. Let $\mathcal{P}(n)$ denote the statement,

$$\sum_{k=0}^{n-1}\binom{\alpha+k}{k}=\binom{\alpha+n}{n-1}.$$

Then $\mathcal{P}(1)$ is true since

$$\sum_{k=0}^{0}\binom{\alpha+k}{k}=\binom{\alpha}{0}=1=\binom{\alpha+1}{0}.$$

Suppose $\mathcal{P}(n)$ is true. Then

$$\begin{aligned}\sum_{k=0}^{n}\binom{\alpha+k}{k} &= \sum_{k=0}^{n-1}\binom{\alpha+k}{k}+\binom{\alpha+n}{n}\\ &= \binom{\alpha+n}{n-1}+\binom{\alpha+n}{n}\\ &= \binom{\alpha+n+1}{n}.\end{aligned}$$

which shows that $\mathcal{P}(n+1)$ is true. Hence by induction, $\mathcal{P}(n)$ is true for all $n$.

**(2)** a) We proceed by induction. Let $\mathcal{P}(n)$ be the statement that there are polynomials $f_n, g_n$ such that

$$\cos(nx) = f_n(\cos(x)), \quad \sin(nx) = \sin(x)\, g_n(\cos(x))$$

for all $x$. Then $\mathcal{P}(1)$ is trivially true, with $f_1(x) = x$ and $g_1(x) = 1$. Suppose $\mathcal{P}(n)$ is true. Then

$$\begin{aligned}
\cos((n+1)x) &= \cos(nx)\cos(x) - \sin(nx)\sin(x) \\
&= f_n(\cos(x))\cos(x) - g_n(\cos(x))\sin^2(x) \\
&= f_n(\cos(x))\cos(x) - g_n(\cos(x)) + g_n(\cos(x))\cos^2(x) \\
&= f_{n+1}(\cos(x))
\end{aligned}$$

with

$$f_{n+1}(x) = x f_n(x) + (x^2 - 1) g_n(x)$$

a polynomial of degree $n+1$, and similarly

$$\begin{aligned}
\sin((n+1)x) &= \sin(nx)\cos(x) + \cos(nx)\sin(x) \\
&= \sin(x) g_n(\cos(x))\cos(x) + f_n(\cos(x))\sin(x) \\
&= \sin(x) g_{n+1}(\cos(x))
\end{aligned}$$

with

$$g_{n+1}(x) = x g_n(x) + f_n(x),$$

a polynomial of degree $n$.

b) Using the recursion formulas from part a, we obtain

$$\begin{aligned}
f_1(x) &= x, & g_1(x) &= 1, \\
f_2(x) &= 2x^2 - 1, & g_2(x) &= 2x, \\
f_3(x) &= 4x^3 - 3x, & g_3(x) &= 4x^2 - 1, \\
f_4(x) &= 8x^4 - 8x^2 + 1, & g_4(x) &= 8x^3 - 4x.
\end{aligned}$$

**(3)** We begin by determining the zeroes of this function: $\sin(\frac{\pi}{\sqrt{|x|}})$ vanishes if and only if $\frac{1}{\sqrt{|x|}} = k \in \mathbb{N}$, i.e. for $|x| = \frac{1}{k^2}$. Thus, the zeroes are at $x$ with

$$|x| = 1, \frac{1}{4}, \frac{1}{9}, \ldots.$$

Next, $\sin(\frac{\pi}{\sqrt{|x|}})$ equals $+1$ if $\frac{1}{\sqrt{|x|}} = \frac{1}{2}, \frac{5}{2}, \cdots$, i.e.

$$|x| = 4, \frac{4}{25}, \frac{4}{81}, \cdots$$

while it is equal to $-1$ at $\frac{1}{\sqrt{|x|}} = \frac{3}{2}, \frac{7}{2}, \cdots$, i.e.

$$|x| = \frac{4}{9}, \frac{4}{49}, \ldots.$$

Now one can draw the picture, keeping in mind also that the function $f(x)$ is odd. Your picture should indicate the following features:

- the graph is trapped between the lines $y = x$ and $y = -x$,
- $f$ is an odd function,
- $f(x) > 0$ for $x > 1$,
- $f$ has zeroes at $x = \pm 1, \pm\frac{1}{4}, \pm\frac{1}{9}$,
- $f(x) = -x$ for $x = \pm\frac{4}{9}$, and $f(x) = x$ for $x = \pm\frac{4}{25}$,
- the function oscillates infinitely often as $x \to 0$.

**(4)** a) Let $A = \{n \in \mathbb{N} | \ n \geq 1\}$. This is an inductive subset, because $1 \in A$, and since $n \geq 1$ implies $n + 1 \geq 2 \geq 1$. Thus $A = \mathbb{N}$.

b) Let $A$ be as suggested. Note $A \subset \mathbb{N}$, since $1 \in \mathbb{N}$, and since $x - 1 \in \mathbb{N}$ implies $x = (x-1)+1 \in \mathbb{N}$ (since $\mathbb{N}$ is inductive).

We want to show $A$ is inductive. Since $1 \in A$, it suffices to show that $n \in A$ implies $n + 1 \in A$. By definition of $A$, the condition $n \in A$ means that $n = 1$ or $n - 1 \in \mathbb{N}$. Case 1: $n = 1$. Then $(n + 1) - 1 = 1 \in \mathbb{N}$, thus $n + 1 \in A$. Case 2: $n - 1 \in \mathbb{N}$. Then also $(n + 1) - 1 = (n - 1) + 1 \in \mathbb{N}$, and it follows that $n + 1 \in A$.

This proves $A = \mathbb{N}$. But if $x \in \mathbb{N} - \{1\} = A - \{1\}$, then $x - 1 \in \mathbb{N}$, thus $x - 1 \geq 1$ (by part (a)) or equivalently $x \geq 2$. Q.E.D.

**(5)** a) We say that the limit $\lim_{x \to a} f(x)$ exists and write $\lim_{x \to a} f(x) = l$ if

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \forall x : \ \ |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon.$$

It is equally fine if you work with the condition

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \forall x : \ \ 0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon.$$

b) The expected limit is $f(0) = -\frac{1}{4}$. We verify:

$$\begin{aligned} \Big|\frac{x^2 - 1}{x^2 + 4} - \frac{1}{4}\Big| &= \Big|\frac{4(x^2 - 1) - (x^2 + 4}{4(x^2 + 4)}\Big| \\ &= \Big|\frac{3x^2}{4(x^2 + 4)}\Big| \\ &\leq 3x^2. \end{aligned}$$

Given $\epsilon > 0$ take $\delta = \min(\frac{\epsilon}{3}, 1)$. Then for $|x| < \delta$, we have

$$3x^2 = 3|x||x| < 3\frac{\epsilon}{3} \cdot 1 = \epsilon,$$

and hence

$$\Big|\frac{x^2 - 1}{x^2 + 4} - \frac{1}{4}\Big| < \epsilon.$$