MAT 157Y – Term exam #1

Solve 4 of the following 5 problems. Each problem is worth 15 points. If you solve more than 4 problems indicate very clearly which ones you want graded; otherwise a random one will left out at grading and it may be your best one!

Write your answers on the lined sides of the exam booklets, the blank sides won't be graded. Make sure to write your name and student number on each booklet. If you run out of space and need a second booklet, please ask the tutors. You have 1 hr 50 min.

Tools allowed: None.

Good luck!

(1) For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ let $\binom{\alpha}{k}$ denote the generalized binomial coefficient, defined by

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

Put $\binom{\alpha}{0} = 1$. Prove that for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} \binom{\alpha+k}{k} = \binom{\alpha+n}{n-1}$$

Hint: Verify first that $\binom{\alpha+1}{k} = \binom{\alpha}{k} + \binom{\alpha}{k-1}$ for all $k \in \mathbb{N}$.

(2) a) Prove that for all $n \in \mathbb{N}$, there are polynomials f_n, g_n such that

$$\cos(nx) = f_n(\cos(x)), \quad \sin(nx) = \sin(x)\, g_n(\cos(x))$$

for all x. What are the degrees of f_n, g_n?

b) Write down the polynomials f_n, g_n for $n \leq 4$.

(3) To the best of your understanding, draw the graph of the function (with domain the set of $x \in \mathbb{R}$ with $x \neq 0$)

$$f(x) = x \sin\left(\frac{\pi}{\sqrt{|x|}}\right)$$

in the region $-2 \leq x \leq 2$. You should devote **plenty of space** to your drawing.

(4) A subset $A \subseteq \mathbb{R}$ is called *inductive* if it has the following two properties:

(1) $1 \in A$,
(2) If $x \in A$ then $x + 1 \in A$.

The natural numbers $\mathbb{N}$ can be defined as the unique inductive subset of $\mathbb{R}$ that is contained in every inductive subset of $\mathbb{R}$.

a) Show directly from this definition of $\mathbb{N}$ (i.e. by considering a suitable subset A) that $n \geq 1$ for all $n \in \mathbb{N}$.

b) Show directly from this definition of $\mathbb{N}$ that there does not exist any element $x \in \mathbb{N}$ with $1 < x < 2$. Hint: Consider $A = \{1\} \cup \{x | \ x - 1 \in \mathbb{N}\}$.

(5) a) State carefully the $\epsilon - \delta$ definition of a limit $\lim_{x \to a} f(x)$ of a function f.

b) Show directly from the $\epsilon - \delta$ definition that the limit $\lim_{x \to 0} f(x)$ of the function

$$f(x) = \frac{x^2 - 1}{x^2 + 4}$$

exists.