MAT 157Y – Assignment #3

(Sketch of) Solutions.

(1)

(a) For $i = 1, \ldots, n$ define a polynomial

$$g_i(x) = \prod_{j \neq i} (x - x_j),$$

where the notation means the product over all $(x - x_j)$ such that $j \neq i$. (E.g., if $n = 4$, then $g_2(x) = (x - x_1)(x - x_3)(x - x_4)$.) Since there are $n - 1$ factors, each g_i is a polynomial of degree $n - 1$. Since a product is zero if and only if one of the factors is zero, the set of zeroes g_i are exactly the points x_j with $j \neq i$. In particular, $g_i(x_i) \neq 0$. Let

$$f_i(x) := \frac{g_i(x)}{g_i(x_i)}.$$

Then $f_i(x_i) = 1$ while $f_i(x_j) = 0$ for all $j \neq i$.

(b) With f_i as in part (a), define

$$f(x) = \sum_{i=1}^{n} a_i f_i(x).$$

Then $f(x_j) = \sum_{i=1}^{n} a_i f_i(x_j) = a_j f_j(x_j) = a_j$, as desired.

(2) We have

$$f(f(x)) = \frac{a\frac{ax+b}{cx+1} + b}{c\frac{ax+b}{cx+1} + 1} = \frac{a(ax+b) + b(cx+1)}{c(ax+b) + (cx+1)}.$$

This equals the identity function $I(x) = x$ if and only if

$$a(ax+b) + b(cx+1) = x(c(ax+b) + (cx+1))$$

Since two polynomials are equal if and only if the coefficients are equal, this yields, by comparing coefficiants of $x^2,\ x,\ x^0$,

$$0 = ac + c, \quad a^2 + bc = bc + 1, \quad ab + b = 0,$$

that is

$$(a+1)c = 0, \ a^2 = 1, \ (a+1)b = 0.$$

Case 1: $a \neq -1$. Then the first and third equation give $b = c = 0$, and the second equation gives $a = 1$. In this case, f is simply the identity function.

Case 2: $a = -1$. Then all three quations hold, for arbitrary values of b, c. That is, the function

$$f(x) = \frac{-x + b}{cx + 1}$$

has the property $f \circ f = I$ for arbitrary b, c.

(3)

(a) Given f, define functions E, O by

$$E(x) = \frac{f(x) + f(-x)}{2}, \quad O(x) = \frac{f(x) - f(-x)}{2}.$$

Then $E(x) = E(-x)$ (so E is even) and and $O(x) = -O(-x)$ (so O is odd), and $E(x) + O(x) = f(x)$ as desired.

(b) Let $f = \tilde{E} + \tilde{O}$ where $\tilde{E}$ is even and $\tilde{O}$ is odd. Let E, O be the functions defined in (a). Then

$$\begin{aligned} E(x) &= \frac{f(x) + f(-x)}{2} \\ &= \frac{\tilde{E}(x) + \tilde{O}(x) + \tilde{E}(-x) + \tilde{O}(-x)}{2} \\ &= \frac{\tilde{E}(x) + \tilde{O}(x) + \tilde{E}(x) - \tilde{O}(x)}{2} \\ &= \tilde{E}(x) \end{aligned}$$

showing that $\tilde{E} = E$, and hence also $\tilde{O} = f - \tilde{E} = f - E = O$.

(4)

$$h = I_X \circ h = (g \circ f) \circ h = g \circ (f \circ h) = g \circ I_Y = g.$$