# MAT 157Y Assignment 2 Solutions

October 7, 2005

**Problem 1** *In the lecture we proved* $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. *Give a similar formula for*

$$\sum_{k=1}^{n}(-1)^k k^2.$$

*(You are not asked to explain how you discovered your formula. But you should prove your answer.)*

**Solution:**
**Claim:**

$$\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}. \tag{1}$$

**Proof:** (By induction on $n$)
When $n = 1$, we have

$$\sum_{k=1}^{1}(-1)^k k^2 = (-1)^1 \cdot 1^2 = -1 = (-1)^1 \frac{1(1+1)}{2}.$$

Assume, for some $m \geq 1$ that (1) holds; that is,

$$\sum_{k=1}^{m}(-1)^k k^2 = (-1)^m \frac{m(m+1)}{2}, \tag{2}$$

and consider the case when $n = m+1$. We have:

$$\begin{aligned}
\sum_{k=1}^{m+1}(-1)^k k^2 &= \sum_{k=1}^{m}(-1)^k k^2 + (-1)^{m+1}(m+1)^2 \\
&= (-1)^m \frac{m(m+1)}{2} + (-1)^{m+1}(m+1)^2, \quad \text{by (2)} \\
&= (-1)^m (m+1)\left(m/2 - (m+1)\right) \\
&= (-1)^m (m+1)\left(\frac{-m-2}{2}\right) \\
&= (-1)^{m+1} \frac{(m+1)((m+1)+1)}{2}.
\end{aligned}$$

Hence, (1) holds when $n = m+1$, and thus it holds for all $n \in \mathbb{N}$ by the principle of mathematical induction.

**Remark 1** One can derive the formula (1) algebraically using the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{3}$$

by writing $\sum(-1)^k k^2 = -\sum(2l+1)^2 + \sum(2l)^2$ and considering cases where $n$ is odd or even.

**Problem 2** *Let $k \in \mathbb{N}$ be a given natural number. Prove that every integer $n \in \mathbb{Z}$ can be* uniquely *written in the form*

$$n = qk + r$$

*where $q \in \mathbb{Z}$ and $0 \leq r \leq k-1$.*

**Solution:** We first prove existence, starting with $n \in \mathbb{N}$, by induction.

We may assume $k > 1$, since for $k = 1$ we have $n = n(1) + 0$ for any $n$.

When $n = 1$ we have

$$1 = 0k + 1,$$

so $q = 0$ and $r = 1 \leq k-1$.

Now, suppose for some $n = m$, $m \geq 1$, that there exist $\overline{q}$ and $\overline{r}$, with $0 \leq \overline{r} \leq k-1$, such that

$$m = \overline{q}k + \overline{r}. \tag{4}$$

If $\overline{r} < k-1$, then $0 \leq \overline{r}+1 \leq k-1$, and we're done, by adding 1 to both sides of (4) (with $r = \overline{r}+1$, and $q = \overline{q}$). If $\overline{r} = k-1$, then we have

$$m + 1 = \overline{q}k + \overline{q} + 1 = \overline{q}k + k = (\overline{q}+1)k + 0,$$

and again we're done, with $r = 0$ and $q = \overline{q} + 1$.

Thus, the result holds for $n \in \mathbb{N}$, by induction.

Now, clearly the result holds for $n = 0$ (with q=r=0), so it remains to prove existence for $n < 0$.

If $n < 0$, let $m = -n$. If $k$ divides $n$, we're done, with $q = n/k$ and $r = 0$.

If not, then since $m > 0$, there exist $q, r \in \mathbb{Z}$, with $0 < r < k$, such that

$$-n = m = qk + r \Rightarrow n = -qk - r = (-q-1)k + (k-r),$$

where $0 < k - r < k$, and thus we have obtained existence for all $n \in \mathbb{Z}$.

We now prove uniqueness. Suppose there are integers $q_1, q_2, r_1, r_2$, with $0 \leq r_1, r_2 < k$, such that

$$n = q_1 k + r_1 = q_2 k = r_2, \tag{5}$$

and, without loss of generality, $r_1 > r_2$. Then (5) gives

$$r_1 - r_2 = k(q_2 - q_1).$$

Thus, $k$ divides $r_1 - r_2$, but $0 \leq r_1 - r_2 \leq r_1 < k$, so this is possible if and only if $r_1 - r_2 = 0$; that is, $r_1 = r_2$, whence $q_1 = q_2$ as well, thus completing our proof.

**Problem 3** *In this problem, we use the fact (proved later in this course) that for any $a \in \mathbb{R}$ with $a \geq 0$, there exists a unique square root $\sqrt{a} \geq 0$.*
*a) Prove that $\sqrt{3}$ is irrational. (You may use the statement of problem(2), for $k = 3$.)*
*b) Prove that $\sqrt{2} + \sqrt{3}$ and $\sqrt{2}\sqrt{3}$ are irrational.*
*c) Are there two irrational numbers $a, b$ such that both $a+b$ and $ab$ are rational? (Justify your answer!)*

**Solution:**
a) We proceed as in the proof of the irrationality of $\sqrt{2}$. Suppose that $\sqrt{3}$ can be written as

$$\sqrt{3} = \frac{p}{q}, \tag{6}$$

with $p/q$ in lowest terms (i.e. $p$ and $q$ have no common factor). Then from (6) we have

$$p^2 = 3q^2. \tag{7}$$

Thus, $p^2$ is divisible by 3. Now, by Problem 2, the integer $p$ can be written in the form $p = 3s + r$, where $r = 0, 1, \text{or } 2$. Considering $p^2 = 9s^2 + 6sr + r^2$, we see that $p^2$ is divisible by 3 if and only if $r = 0$, since $r^2 = p^2 - 9s^2 - 6sr$ must be divisible by 3 if $p^2$ is, and neither of $1^2 = 1$ or $2^2 = 4$ are divisible by 3.

Therefore, $p^2 = 9s^2$, and (7) becomes

$$9s^2 = 3q^2 \Rightarrow q^2 = 3s^2,$$

whence by the same reasoning $q$ is divisible by 3, since $q^2$ is. Thus $p$ and $q$ have the common factor 3, a contradiction.

Since we arrived at our contradiction by assuming that $\sqrt{3}$ is rational, it must in fact be irrational.

b)For part (b), we first prove two easy lemmas:

**Lemma 1** *If $x$ is irrational, and $y$ is rational, then $x + y$ is irrational.*

**Proof:** Suppose not. Then, since $x + y$ is rational, we have $x = (x + y) - y$, the difference of two rational numbers. But the difference of any two rational numbers is rational, contradicting our assumption that $x$ is irrational.

**Lemma 2** *If $x$ is irrational, and $y$ is rational, with $y \neq 0$, then $xy$ is irrational.*

The proof is similar to that of the previous lemma, and left as an easy exercise.

We now prove that $\sqrt{2} + \sqrt{3}$ is irrational:

We know that $\sqrt{2}$ and $\sqrt{3}$ irrational. Suppose that their sum is rational. Then by Lemma 2 we know that $-2\sqrt{2}$ is irrational, and hence, by Lemma 1, and our assumption that $\sqrt{2}+\sqrt{3}$ is rational, we have that

$$-2\sqrt{2}+(\sqrt{2}+\sqrt{3})=-\sqrt{2}+\sqrt{3}$$

is irrational. Thus, by another application of Lemma 2 and the use of our assumption,

$$(-\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})=-2+3=1$$

is irrational, which is a contradiction, since 1 is clearly rational.

To prove that $\sqrt{2}\sqrt{3}$ is irrational, one shows first (use uniqness of the square root) that $\sqrt{2}\sqrt{3}=\sqrt{6}$. From this observation, one may proceed in a fashion similar to that of part (a), making use once again of the statement of problem 2, but having to consider the possible remainders 0,1,2,3,4 or 5.

(c) As we know, $\sqrt{2}$ is irrational, and making use of Lemma 2, we have that $-\sqrt{2}=-1\cdot\sqrt{2}$ is irrational as well. Thus, with $a=\sqrt{2}$ and $b=-\sqrt{2}$, both $a$ and $b$ are irrational, and yet $a+b=0$ and $ab=2$ are both rational.