MAT 157Y Assignment 1 Solutions

October 11, 2005

Problem 1 *Determine a, b, c such that*

$$\sin^5(\alpha) = a\sin(\alpha) + b\sin(3\alpha) + c\sin(5\alpha) \tag{1}$$

for all α.

Solution: Using the identities

$$\begin{aligned} \sin(a+b) &= \sin(a)\cos(b) + \sin(b)\cos(a) \\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \end{aligned}$$

repeatedly, we have,

$$\begin{aligned} \sin(3\alpha) &= \sin(\alpha + 2\alpha) \\ &= \sin(\alpha)\cos(2\alpha) + \sin(2\alpha)\cos(\alpha) \\ &= \sin(\alpha)\left(\cos^2(\alpha) - \sin^2(\alpha)\right) + 2\sin(\alpha)\cos^2(\alpha) \\ &= 3\sin(\alpha) - 4\sin^3(\alpha). \end{aligned}$$

Similarly, (but with a bit more work) one finds

$$\sin(5\alpha) = 5\sin(\alpha) - 20\sin^3(\alpha) + 16\sin^5(\alpha).$$

Substituting this into (1) we have

$$\sin^5(\alpha) = (a + 3b + 5c)\sin(\alpha) + (-4b - 20c)\sin^3(\alpha) + 16c\sin^5(\alpha). \tag{2}$$

Thus, by comparing both sides of (2), we must have

$$\begin{aligned} a + 3b + 5c &= 0 \\ -4b - 20c &= 0 \\ 16c &= 1, \end{aligned}$$

from which we deduce that

$$\begin{aligned} a &= 5/8 \\ b &= -5/16 \\ c &= 1/16 \end{aligned}$$

Problem 2 *Compute* $\tan(\pi/12)$.

Solution: We again make use of the angle addition identities used in problem one above, and the fact that $\pi/12 = \pi/3 - \pi/4$:

$$\begin{aligned}
\tan(\pi/12) &= \tan(\pi/3 - \pi/4) \\
&= \frac{\sin(\pi/3 - \pi/4)}{\cos(\pi/3 - \pi/4)} \\
&= \frac{\sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4)}{\cos(\pi/3)\cos(\pi/4) + \sin(\pi/3)\sin(\pi/4)} \\
&= \frac{(\sqrt{3}/2)(1/\sqrt{2}) - (1/2)(1/\sqrt{2})}{(1/2)(1/\sqrt{2}) + (\sqrt{3}/2)(1/\sqrt{2})} \\
&= 2 - \sqrt{3}
\end{aligned}$$

Problem 3 *For* $a, b \in \mathbb{R}$ *and* $\epsilon > 0$ *prove the inequality,*

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2.$$

When does equality hold?

Solution: Since $x^2 \geq 0$ for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
0 &\leq (a - \frac{b}{2\epsilon})^2 \qquad (3) \\
&= a^2 - \frac{ab}{\epsilon} + \frac{b^2}{4\epsilon^2},
\end{aligned}$$

whence, by multiplying by $\epsilon > 0$ and adding ab to both sides, we obtain

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2,$$

as required. From (3), it is clear that equality holds iff $a = \frac{b}{2\epsilon}$.

Problem 4 *For any natural number* $k \in \mathbb{N} = \{1, 2, 3, \cdots\}$, *with* $k > 1$, *let*

$$\mathbb{Z}_k := \{0, 1, \ldots, k-1\}.$$

For any integer $x \in \mathbb{Z}$, *there is a unique integer* $y \in \mathbb{Z}_k$ *such that the difference* $x - y$ *is an integer multiple of* k. *One writes "*$y = x \mod k$*". Let* $\mathbb{Z}_k$ *be equipped with the following addition and multiplication operations,*

$$a \oplus b := (a + b) \mod k, \quad a \odot b := (a \cdot b) \mod k.$$

Show that all of the axioms (P1)-(P9) are satisfied for $\mathbb{Z}_k$ *(with these operations* $\oplus, \odot$ *of addition and multiplication), with the possible exception of (P7). Show that (P7) holds for* $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$ *but does* not *hold for* $\mathbb{Z}_4$.

Solution: We will use the following notation: given any $x \in \mathbb{Z}$, let $[x]$ denote the unique element of $\mathbb{Z}_k$ such that $[x] = x \mod k$. It will be understood throughout that the operations $\oplus, \odot$ are defined only for elements of $\mathbb{Z}_k$, while $+, \cdot$ are used for elements of $\mathbb{Z}$.

In this notation, for any $x, y \in \mathbb{Z}$, the definitions of the operations $\oplus, \odot$ can be written

$$[x] \oplus [y] = [x + y], \quad \text{and} \quad [x] \odot [y] = [x \cdot y].$$

With this in mind, we check the axioms (P1)-(P9) for $\mathbb{Z}_k$ as follows:

(P1) Let $x, y, z \in \mathbb{Z}$. Then we have

$$\begin{aligned} [x] \oplus ([y] \oplus [z]) &= [x] \oplus [y + z] \\ &= [x + (y + z)] \\ &= [(x + y) + z] \\ &= [x + y] \oplus [z] \\ &= ([x] \oplus [y]) \oplus [z]. \end{aligned}$$

(P2) $0 \in \mathbb{Z}$ corresponds to $[0] \in \mathbb{Z}_k$. For any $[a] \in \mathbb{Z}_k$, we have

$$[a] \oplus [0] = [a + 0] = [a] = [0 + a] = [0] \oplus [a].$$

(P3) Let $a \in \mathbb{Z}$. We need to find $b \in \mathbb{Z}$ such that $[a] \oplus [b] = [0] = [b] \oplus [a]$. Clearly if $a = 0$ we may take $b = 0$ as well, and (P3) holds trivially. If $a \neq 0$, then $k - a$ is in the set $\mathbb{Z}_k$, and we have

$$[a] \oplus [k - a] = [a + (k - a)] = [k] = [(k - a) + a] = [k - a] \oplus [a],$$

and since $k - 0$ is of course divisible by k, we have $[k] = [0]$, as required.

(P4) For any $a, b \in \mathbb{Z}$ we have

$$[a] \oplus [b] = [a + b] = [b + a] = [b] \oplus [a].$$

(P5) For any $a, b, c \in \mathbb{Z}$ we have

$$[a] \odot ([b] \odot [c]) = [a] \odot [b \cdot c] = [a \cdot (b \cdot c)] = [(a \cdot b) \cdot c] = [a \cdot b] \odot c = ([a] \odot [b]) \odot [c].$$

(P6) The number $1 \in \mathbb{Z}$ corresponds to an element $[1] \in \mathbb{Z}_k$, with $[1] \neq [0]$, since $k > 1 \Rightarrow 0 \neq 1 \mod k$, and we have, for any $a \in \mathbb{Z}$,

$$[a] \odot [1] = [a \cdot 1] = [a] = [1 \cdot a] = [1] \odot [a].$$

(P8) For any $a, b \in \mathbb{Z}$ we have

$$[a] \odot [b] = [a \cdot b] = [b \cdot a] = [b] \odot [a].$$

(P9) For any $x, y, z \in \mathbb{Z}$, we have

$$\begin{aligned} [x] \odot ([y] \oplus [z]) = [x] \odot [y + z] = [x \cdot (y + z)] &= [(x \cdot y) + (x \cdot z)] \\ = [x \cdot y] \oplus [x \cdot z] &= ([x] \odot [y]) \oplus ([x] \odot [z]). \end{aligned}$$

Finally, we have still to check the validity of (P7) for the four special cases given.
$\mathbb{Z}_2$: Here, the only non-zero element is $1 = [1]$, and we have $[1] \odot [1] = [1 \cdot 1] = [1]$.
$\mathbb{Z}_3$: One checks in this case that both non-zero elements 1 and 2 are their own inverses:
$[1] \odot [1] = [1]$ as before, and
$[2] \odot [2] = [2 \cdot 2] = [4] = [1]$.
$\mathbb{Z}_5$: The non-zero elements here are 1,2,3 and 4, and we exhibit inverses for each as follows:

$$[1] \odot [1] = [1], [2] \odot [3] = [6] = [1] = [3] \odot [2], \text{ and } [4] \odot [4] = [16] = [1].$$

For the case of $k = 4$, we check that for $2 \in \mathbb{Z}_4$ we have

$$[2] \odot [0] = [0], [2] \odot [1] = [2], [2] \odot [2] = [4] = [0], [2] \odot [3] = [6] = [2].$$

Since there is no product with $[2]$ giving $[1]$, and $[2] \neq [0]$, we conclude that (P7) is not valid in the case of $k = 4$.

Remark 1 The notation used in the solution to problem 4 above is certainly not the only one that can be used, although it is possibly the most convenient. However, in whatever way one chooses to write up the solution to this problem, a number of key points must be observed:

- It must be stated clearly which symbols represent elements of $\mathbb{Z}$, and which ones represent elements of $\mathbb{Z}_k$.
- Since the operations $\oplus, \odot$ are defined on a different set than $+, \cdot$, one must be careful not to mix them, or to apply them to elements of the wrong set; if both do appear in the same expression, one should take care to point out why this is valid.
- It should be clear from the solution how the axioms for $\mathbb{Z}_k$ are inherited from those of $\mathbb{Z}$.

Remark 2 A common mistake in the solution to problem 3 is to start with the inequality one is asked to verify, and conclude that it holds because from it, one can derive another statement that is clearly true.

While technically valid in the case of problem 3, since all the steps used are reversible (i.e. "if and only ifs"), this approach is considered bad mathematical form, and in more intricate situations can lead to invalid proofs and logically incorrect statements.

This technique of "working backwards" should not necessarily be avoided altogether, since it can often provide insight while trying to solve a problem; however, when writing up solutions for submission, care should be taken to write all steps in the correct order.