MAT 363S – Assignment #2

Due in class on February 4, 2004

As explained in class, the definition of a regular surface is a special case of the definition of an m-dimensional submanifold of $\mathbb{R}^n$. In particular, we have the notion of a regular curve as a subset $C \subset \mathbb{R}^2$ (independent of the choice of parametrization).

Problem #1:[7 points] Let $C \subset \mathbb{R}^2$ be a regular curve in the $x-y$ plane. Suppose C is invariant under reflection at the y-axis, i.e. $(x,y) \in C \Leftrightarrow (-x,y) \in C$. Let $S \subset \mathbb{R}^3$ be the subset obtained by rotating C about the y-axis. Show that S is a regular surface.

Hint: If $\phi(t)$ is a smooth function with $\phi(t) = \phi(-t)$, then the function $\psi(x,y) = \phi(\sqrt{x^2+y^2})$ is again smooth. (If you want to use this fact, give a short explanation, using e.g. the Taylor expansion of ϕ.)

Problem #2: [5 points] Determine the tangent planes of the surface

$$S = \{(x,y,z) |\, x^2+y^2-z^2 = 1\}$$

at points of the form $(x,y,0) \in S$. Show that they all contain the z-axis.

Problem #3: [8 points] One way to define a system of coordinates for the sphere S^2, given by $x^2+y^2+z^2=1$, is to use *stereographic projection* from the point $(0,0,1)$,

$$g : \mathbb{R}^3 \backslash \{(0,0,1)\} \to \mathbb{R}^2,$$

where g carries $p \in \mathbb{R}^3 \backslash \{(0,0,1)\}$ to the point of intersection of the xy-plane with the straight line through p and the point $(0,0,1)$. Let π be the restriction of g to $S^2 \backslash \{(0,0,1)\}$.

(a) Show that $g(x,y,z) = \frac{(x,y)}{1-z}$.

(b) Show that π is invertible, and let $\sigma = \pi^{-1} : \mathbb{R}^2 \to S^2 \backslash \{(0,0,1)\}$. Show that $\sigma(u,v) = (x,y,z)$ where

$$x = \frac{2u}{1+u^2+v^2}, \quad y = \frac{2v}{1+u^2+v^2}, \quad z = \frac{u^2+v^2-1}{1+u^2+v^2}.$$