MAT 363S – Assignment #5

Due in class on March 22, 2004

Problem #1: [6 points] Let $S \subset \mathbb{R}^3$ be a regular surface, and $\sigma : U \to S$ a surface patch, defining local coordinates u^1, u^2. For any vector field Y on S, defined on the domain of σ, let $\frac{DY}{\mathrm{d}u^1}$ be the covariant derivative along the coordinate lines $u^2 = \text{const}$, and similarly $\frac{DY}{\mathrm{d}u^2}$ be the covariant derivative along the coordinate lines $u^1 = \text{const}$. In general, the covariant partial derivatives need not commute. Let

$$S_{ij}(Y) := \frac{D}{\mathrm{d}u^i}\frac{DY}{\mathrm{d}u^j} - \frac{D}{\mathrm{d}u^j}\frac{DY}{\mathrm{d}u^i}.$$

For $r = 1, 2$ Let $\sigma_r = \frac{\partial \sigma}{\partial u^r}$. Find a formula for $S_{ij}(\sigma_k) \cdot \sigma_l$ in terms of Christoffel symbols and their derivatives. Do you recognize the answer?

Problem #2: [6 points] Consider the surface of revolution,

$$\sigma(u, v) = (\rho(u) \cos v,\ \rho(u) \sin v,\ z(u))$$

where $u \mapsto (\rho(u), 0, z(u))$ is a unit speed curve with $\rho > 0$ everywhere.

a) Give a formula for the geodesic curvature of a parallel

$$\gamma(t) = (\rho(u) \cos t,\ \rho(u) \sin t,\ z(u)),$$

for any given u.

b) Let $X(t) \in R^3$ be a parallel vector field along $\gamma(t)$, tangent to the given surface. Let the $\psi(t)$ denote the angle between $X(t)$ and $\dot{\gamma}(t)$. Find

$$\Delta\psi(t) = \psi(t + 2\pi) - \psi(t).$$

(Note that ψ, as a continuous function, is only determined up to multiples of 2π, but $\psi(t + 2\pi) - \psi(t)$ is independent of that choice.)

Hint: In class we solved this problem for $S = S^2$. Try to generalize the argument given there.

Problem #3: [8 points] Let $S \subset \mathbb{R}^3$ be an oriented surface. A regular curve $C \subset S$ is called a *line of curvature* if at each point of C, its tangent line is a principal direction. It is called an *asymptotic curve* if for any vector $X \in T_pS$ tangent to C, $\mathcal{F}_{II}(X, X) = 0$ where $\mathcal{F}_{II}$ is the second fundamental form. Disprove (by giving a counterexample) or prove the following statements:

a) If C is both a line of curvature and a geodesic, then it is a plane curve.

b) If C is both a geodesic and a plane curve, then it is a line of curvature.

c) If C is both a line of curvature and a plane curve, then it is a geodesic.

d) If C is both an asymptote and a geodesic, then it is a straight line.