MAT 363S – Assignment #3

Due in class on February 25, 2004

Problem #1:[7 points] a) Show that

$$\sigma(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u)$$

is a regular surface patch for the 2-sphere.

b) Show that the first fundamental form of S^2 in this coordinates is given by $\operatorname{sech}^2 u(\mathrm{d}u^2 + \mathrm{d}v^2)$.

c) A *loxodrome* is a curve on S^2 intersecting the meridians at a fixed angle, α. Show that loxodromes correspond under σ to straight lines in the $u-v$-plane.

d) A *loxodromic triangle* in S^2 is a triangle cut out by three loxodromes. Assuming that the triangle does not contain the poles $\pm(0, 0, 1) \in S^2$, show that the sum of interior angles in a loxodromic triangle is π.

Problem #2: [5 points] A diffeomorphism $\Phi : S_1 \to S_2$ is called *area-preserving* if for each region $R \subset S_1$, the areas of R and $\Phi(R)$ are equal. Show that a conformal map is area preserving if and only if it is an isometry.

A *vector field on a regular surface* $S \subset \mathbb{R}^3$ is a smooth map $X : S \to \mathbb{R}^3$ such that $X(p) \in T_pS$ for all $p \in S$. (For instance, for any surface patch σ the derivatives σ_u, σ_v are vector fields on the range of σ.)

Problem #3: [8 points] Let $S \subset \mathbb{R}^3$ be a regular surface, and $f : S \to \mathbb{R}$ a smooth map.

a) Show that there exists a unique vector field $\operatorname{grad}(f)$ on S, called the *gradient vector field*, with the property

$$\operatorname{grad}(f) \cdot \dot{\gamma}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)),$$

for all curves $\gamma(t) \in S$.

b) Let $\sigma : U \to S$ be a regular surface patch, defining local coordinates u, v on S, and let E, F, G be the corresponding coefficients of the first fundamental form. Show that in these coordinates u, v,

$$\operatorname{grad}(f) = \frac{(f_u G - f_v F)\sigma_u + (f_v E - f_u F)\sigma_v}{EG - F^2}$$

c) Suppose $\operatorname{grad}(f) \neq 0$ at all points of a level set $C = f^{-1}(a)$. Show that C is a regular curve in S.