SIGN CONVENTIONS

1. Lie groups and Lie algebras

The prototype of a Lie algebra is the Lie algebra associated to any associative algebra, with bracket the commutator. Vector fields on manifolds can be viewed as opearting on functions, and their Lie bracket in that sense is just the commutator, $[X, Y] = X \circ Y - Y \circ X$. For any Lie group G, the associated Lie algebra $\mathfrak{g}$ is the Lie algebra of left-invariant vector fields, where vector fields are viewed as operators on functions. A Lie group action on a manifold M is a group homomorphism $G \to \mathrm{Diff}(M)$. For any such action, one obtains an action on the algebra of functions, i.e. a group homomorphism $G \to \mathrm{Aut}(C^\infty(M))$ by $(g.f)(x) = f(g^{-1}.x)$.

1.1. **Flows of vector fields.** A *flow* is the same thing as an $\mathbb{R}$-action, $\mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto F_t$. The corresponding action on functions is thus, $(t.f)(x) = f(F_t^{-1}(x))$. We define the vector field for the given flow by

$$\boxed{(Xf)(x) = \frac{\partial}{\partial t}\Big|_{t=0} f(F_t^{-1}x).}$$

The definition of the flow of a vector field results in the following convention for the Lie derivative,

$$\boxed{L_X = \frac{\partial}{\partial t}\Big|_{t=0} (F_t^{-1})^*.}$$

Alternatively, $(F_t^{-1})^* \circ L_X = \frac{\partial}{\partial t}(F_t^{-1})^*$.

The flow for the vector field $X = \frac{\partial}{\partial x}$ on the real line is

$$F_t(x) = x - t$$

(with a minus sign!). More generally, the flow of a vector field $X = \sum_i a^i(x) \frac{\partial}{\partial x^i}$ is given by the differential equation

$$\dot{x}^i = -a^i(x(t)).$$

We can also consider *time dependent* vector fields, where X depends on t as well. If $X = \sum_i a^i(x,t) \frac{\partial}{\partial x^i}$ then the corresponding ODE should be $\dot{x}_i = -a^i(x(t), t)$. In coordinate-free fashion, we define the flow F_t of X in terms of the flow $\widetilde{F}_t$ of the time-independent vector field $\widetilde{X} = X - \frac{\partial}{\partial s}$ on $M \times \mathbb{R}$, i.e. $\widetilde{F}_t(x, s) = (F_t(x), s + t)$. Indeed, in local coordinates this gives

$$\dot{x}_i = -a_i(x(t), s(t)), \quad \dot{s} = 1.$$

Date: October 10, 2015.

This is consistent with

$$\boxed{(F_t^{-1})^* \circ L_{X_t} = \frac{\partial}{\partial t}(F_t^{-1})^*.}$$

(Note that, unlike the case of constant vector fields, the operators on the left don't commute.) For example, the vector field $X_t = t\frac{\partial}{\partial x}$ has $F_t(x) = x - \frac{t^2}{2}$, hence $F_t^{-1}(x) = x + \frac{t^2}{2}$. It's easy to verify the identities above for this case.

1.2. **Lie algebra of G as left-invariant vector fields.** Consider $G = \mathrm{GL}(n,\mathbb{R})$. If $\xi \in \mathfrak{gl}(n,\mathbb{R})$, consider $\exp(t\xi) \in \mathrm{GL}(n,\mathbb{R})$ (exponential of matrices). Then ξ defines a left-invariant vector field ξ^L with flow $g \mapsto g\exp(-t\xi)$, and a right-invariant vector field ξ^R with flow $g \mapsto \exp(-t\xi)g$.

$$(\xi^L f)(g) = \frac{\partial}{\partial t}\Big|_{t=0} f\big(g\exp(t\xi)\big), \quad (\xi^R f)(g) = \frac{\partial}{\partial t}\Big|_{t=0} f\big(\exp(t\xi)g\big).$$

We should verify that for $G = \mathrm{GL}(n,\mathbb{R})$, the Lie bracket on $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$ corresponds to the commutator of matrices. Thus, let $\xi, \eta \in \mathfrak{gl}(n,\mathbb{R})$. Clearly, $[\xi^L, \zeta^L]$ is again a left-invariant vector field. To determine what it is, we may use a *linear* function f (e.g., matrix elements $f(A) = A_{ij}$), and evaluate at the group unit. We calculate,

$$\begin{aligned}
\xi^L(\eta^L(f))(e) &= \frac{\partial}{\partial t}|_{t=0}\eta^L(f)(\exp t\xi) \\
&= \frac{\partial}{\partial t}|_{t=0}\frac{\partial}{\partial s}|_{s=0} f(\exp(t\xi)\exp(s\eta)) \\
&= \frac{\partial}{\partial t}|_{t=0}\frac{\partial}{\partial s}|_{s=0} f\Big(I + t\xi + s\eta + \frac{t^2}{2}\xi^2 + ts\xi\eta + \frac{s^2}{2}\eta^2 + \dots\Big) \\
&= f(\xi\eta).
\end{aligned}$$

Hence $\xi^L(\eta^L(f))(e) - \eta^L(\xi^L(f))(e) = f([\xi,\eta]) = [\xi,\eta]^L(f)(e)$. This verifies

$$[\xi^L, \eta^L] = [\xi,\eta]^L.$$

By a similar calculation, $[\xi^R, \eta^R] = -[\xi,\eta]^R$. This motivates defining the Lie bracket for general Lie groups in terms of left-invariant vector fields, in such a way that $\xi \mapsto \xi^L$ is a Lie algebra homomorphism. One defines the exponential map $\exp\colon \mathfrak{g} \to G$ by

$$F_t(g) = g\exp(-t\xi),$$

where F_t is the flow of ξ^L. (One may also use ξ^R to define exp; the two definitions coincide.)

Remark 1.1. Some authors prefer that a left G-action on a manifold defines a right-action on functions, by $(\phi g)(x) = \phi(gx)$. Likewise a right-action on M defines a left-action on $C^\infty(M)$. (We generally prefer working with left actions throughout, but there can be situations where right actions are 'natural': For example, if one considers actions of semi-groups.) Note that with this convention, a group homomorphism (resp.

anti-homomorphism) $G \to \mathrm{Diff}(M)$ induces a group anti-homomorphism (resp. homomorphism) $G \to \mathrm{Aut}(C^\infty(M))$.

For an abelian group, left and right actions coincide. But in any case one obtains a sign change in the definition of flow of a vector field: $(Xf)(x) = \frac{\partial}{\partial t}|_{t=0} f(F_t(x))$. In particular, the flow of $\frac{\partial}{\partial x}$ on $\mathbb{R}$ is now $x \mapsto x + t$. For $\mathrm{GL}(n, \mathbb{R})$, the flow $g \mapsto \exp(t\xi)g$ corresponds to a right invariant vector field as $(\xi^R f)(g) = \frac{\partial}{\partial t}|_0 f(\exp(t\xi)g)$, while similarly $g \mapsto g\exp(t\xi)$ corresponds to a left invariant vector field ξ^L. Thus ξ^L defines a left-action on $C^\infty(G)$ corresponding to the right action on $M = G$, $a \mapsto ag$. Likewise ξ^R defines a right-action on $C^\infty(G)$ corresponding to the left-action on G, $a \mapsto ga$. Note that the definition of left-invariant vector field is just as before; the same calculation as above show $[\xi^L, \eta]^L = [\xi, \eta]^L$, as one would expect from a left action.

2. Symplectic forms

Consider he phase space $\mathbb{R}^2$, with coordinates q, p. The Hamiltonian for a particle in a potential V is $H = \frac{1}{2}p^2 + V(q)$. Hamilton's equation decree that the flow defined by H is the solution of the differential equation

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

By our conventions, this is the flow of the vector field

$$X_H = \frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q}.$$

The standard symplectic form on $\mathbb{R}^{2n}$ is

$$\omega = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

Thus, our sign convention for a Hamiltonian vector field is

$$\boxed{\iota(X_H)\omega = -\mathrm{d}H.}$$

If we put $z = q + \mathrm{i}p$ then $\omega = \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}$. In polar coordinates $q = r\cos\phi$, $p = r\sin\phi$, $\omega = r\mathrm{d}r\mathrm{d}\phi$. We define Poisson brackets by

$$\boxed{\{F, H\} = \omega(X_F, X_H).}$$

With this sign convention

$$\boxed{X_{\{F,H\}} = [X_F, X_H], \quad \{H, \cdot\} = L_{X_H}}$$

We check:

$$\mathrm{d}\{F, H\} = \mathrm{d}(\omega(X_F, X_H)) = \mathrm{d}\iota(X_H)\iota(X_F)\omega = L_{X_H}\iota(X_F)\omega = \iota([X_H, X_F])\omega.$$

so $X_{\{F,H\}} = [X_F, X_H]$, and

$$L_{X_H}F = \iota_{X_H}\mathrm{d}F = -\omega(X_F, X_H) = -\{F, H\} = \{H, F\}.$$

Explicitly, on $\mathbb{R}^{2n}$,

$$\boxed{\{F,H\} = \sum_{i=1}^{n}\Big(\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i}\frac{\partial F}{\partial p_i}\Big).}$$

In particular, $X_{q^i} = \frac{\partial}{\partial p_i}$, $\quad X_{p_i} = -\frac{\partial}{\partial q^i}$, and hence $\{q^i, p_j\} = \delta^i_j$.

The Poisson bracket of functions F, H depends only on theit differentials. This defines a bi-vector field

$$\boxed{P(\mathrm{d}F, \mathrm{d}H) = \{F, H\}.}$$

For the standard symplectic form, one gets

$$\boxed{P = \sum_{i=1}^{n}\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i}.}$$

Note that the maps $P^\sharp$ and $\omega^\flat$ are related by

$$P^\sharp = -(\omega^\flat)^{-1}.$$

Remark 2.1. There doesn't seem to be a deep reason for this sign convention for P; one could also agree that $P^\sharp$ is the inverse to $\omega^\flat$, which then results in $P(\mathrm{d}F, \mathrm{d}H) = -\{F, H\}$.

3. Group actions

For any G-action on a manifold M, we obtain a G-action on $C^\infty(M)$ by $g.f = (g^{-1})^* f$. Thus if $g = \exp(t\xi)$ is a 1-parameter group, the infinitesimal action is given by $\xi.f = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\exp(-t\xi)^* f$. Thus if we define

$$\boxed{\xi_M(f) = \frac{d}{dt}|_{t=0}\exp(-t\xi)^* f,}$$

we obtain a Lie homomorphism $\mathfrak{g} \mapsto \mathfrak{X}(M)$. For the standard (= counterclockwise) $S^1 = \mathbb{R}/\mathbb{Z}$ action on $\mathbb{C}$ given by $t.z = e^{2\pi i t}z$, the generating vector field corresponding to $1 \in \mathbb{R}$ is obtained from the calculation

$$(1_{\mathbb{C}} f)(z, \overline{z}) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(e^{-2\pi i t}z, e^{2\pi i t}\overline{z}) = -2\pi i z\frac{\partial f}{\partial z} + 2\pi i\overline{z}\frac{\partial f}{\partial \overline{z}},$$

thus

$$1_{\mathbb{C}} = -2\pi i z\frac{\partial}{\partial z} + 2\pi i\overline{z}\frac{\partial}{\partial \overline{z}}.$$

We have

$$\iota(1_{\mathbb{C}})\omega = \pi(z\mathrm{d}\overline{z} + \overline{z}\mathrm{d}z) = \pi\mathrm{d}|z|^2.$$

We define the moment map $\Phi: M \to \mathfrak{g}^*$ for a Hamiltonian G-action to be an equivariant map with $X_{\langle\Phi,\xi\rangle} = -\xi_M$, that is,

$$\boxed{\iota(\xi_M)\omega = -\mathrm{d}\langle\Phi, \xi\rangle.}$$

For example, the moment map for the S^1 action on $\mathbb{C}$ is $-\pi(q^2+p^2)$. The extra sign is natural, because the flow of the harmonic oscillator gives a rotation *opposite* to the standard circle action (again, since $\dot{q}>0$ if $p>0$).

Remark 3.1. This is also natural from the point of view of "quantization": The induced action on the space of holomorphic functions $(e^{i\phi}.f)(z) = f(e^{-i\phi}z)$ has spectrum the non-positive integers.

Let $\mathbb{C}^n$ have the symplectic form $\frac{\mathrm{i}}{2}\sum_i \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i$. The moment map for the T-action on $\mathbb{C}^n$, with weights $\alpha_1,\ldots,\alpha_n \in \mathfrak{t}^*$, is

$$\boxed{\langle \Phi(z), \xi\rangle = -\pi \sum_{i=1}^n \langle \alpha_i, \xi\rangle |z_i|^2.}$$

4. Coadjoint orbits, Poisson bracket on $\mathfrak{g}^*$

Consider the coadjoint action,

$$g.\mu = (\mathrm{Ad}_{g^{-1}})^*\mu.$$

Any $\xi\in\mathfrak{g}$ defines a function on $\mathfrak{g}^*$, $f_\xi\colon \mu\mapsto\langle\mu,\xi\rangle$. The action on such functions is then

$$g.f_\xi = f_{\mathrm{Ad}_g \xi}.$$

(Check: $(g.f_\xi)(\mu) = f_\xi(g^{-1}.\mu) = \langle \mathrm{Ad}_g^*\mu, \xi\rangle = \langle \mu, \mathrm{Ad}_g \xi\rangle$.) Infinitesimally, $L_{\xi_{\mathfrak{g}^*}} f_\eta = f_{[\xi,\eta]}$.

The symplectic form on a coadjoint orbit $\mathcal{O}\subset\mathfrak{g}^*$ should be such that the action is Hamiltonian, with moment map the inclusion. We claim that this gives

$$\boxed{\omega(\xi_{\mathcal{O}}(\mu), \eta_{\mathcal{O}}(\mu)) = \langle \mu, [\xi,\eta]\rangle.}$$

We check:

$$\begin{aligned}
\omega(\xi_{\mathcal{O}}, \eta_{\mathcal{O}}) &= \iota(\eta_{\mathcal{O}})\iota(\xi_{\mathcal{O}})\omega \\
&= -\iota(\eta_{\mathcal{O}})\mathrm{d}f_\xi|_{\mathcal{O}} \\
&= -(\iota(\eta_{\mathfrak{g}^*})\mathrm{d}f_\xi)\Big|_{\mathcal{O}} \\
&= -(L(\eta_{\mathfrak{g}^*})f_\xi)\Big|_{\mathcal{O}} \\
&= f_{[\xi,\eta]}|_{\mathcal{O}}.
\end{aligned}$$

The Poisson structure on $\mathfrak{g}^*$ is defined in such a way that the inclusion of an orbit is a Poisson map. This gives

$$\boxed{\{f,g\}(\mu) = \langle \mu, [\mathrm{d}f(\mu), \mathrm{d}g(\mu)]\rangle.}$$

It is enough to check for $f(\mu) = \langle\mu,\xi\rangle$ and $g(\mu)=\langle\mu,\eta\rangle$. We have $X_f(\mu) = -\xi_{\mathcal{O}}(\mu)$, and similarly for X_g. Thus

$$\{f,g\}(\mu) = \omega(X_f, X_g)(\mu) = \omega(\xi_{\mathcal{O}}, \eta_{\mathcal{O}})(\mu) = \langle\mu,[\xi,\eta]\rangle,$$

as claimed. Let e_i be a basis of $\mathfrak{g}$, with dual basis $e^i \in \mathfrak{g}^*$, and let μ_i be the corresponding coordinates on $\mathfrak{g}^*$. Let c_{ij}^k be the structure constants defined as $[e_i, e_j] = \sum_k c_{ij}^k e_k$. Then the Poisson bivector is

$$\boxed{P_{\mathfrak{g}^*} = \tfrac{1}{2} \sum_{ijk} c_{ij}^k \ \mu_k \frac{\partial}{\partial \mu_i} \wedge \frac{\partial}{\partial \mu_j}.}$$

Indeed, for $f = \mu_i$ and $g = \mu_j$ we have $\mathrm{d}f = e_i, \ \mathrm{d}g = e_k$, hence the above formula gives $P(\mathrm{d}\mu_i, \mathrm{d}\mu_j) = \langle \mu, [e_i, e_j] \rangle = \sum_k c_{ij}^k \mu_k$.

4.1. **Symplectic volume of coadjoint orbits.** The symplectic volume of a coadjoint orbit, for G compact, may be computed as follows. Vhoose a positive Weyl chamber $\mathfrak{t}_+^* \subset \mathfrak{t}^* \subset \mathfrak{g}^*$. Choose an invariant inner product on $\mathfrak{g}$, thus identifying $\mathfrak{g} \cong \mathfrak{g}^*$.

Proposition 4.1. *For $\mu \in \mathfrak{t}_+^*$,*

$$\boxed{\mathrm{Vol}(G.\mu) = \prod\nolimits'_{\alpha > 0} \frac{\langle \alpha, \mu \rangle}{\langle \alpha, \rho \rangle}.}$$

Here the product is over all those roots such that $\alpha \cdot \mu \neq 0$.

Proof. We'll explain a proof for the case that μ is regular. Let $\{e_\alpha, f_\alpha, \ \alpha \in \mathfrak{R}_+\}$ be an orthonormal basis of $\mathfrak{t}^\perp$, such that $\mathrm{ad}_\xi(e_\alpha) = 2\pi \langle \alpha, \xi \rangle f_\alpha$. Let ω_0 be the symplectic form on $\mathfrak{t}^\perp$ given by $\omega_0(e_\alpha, f_\alpha) = 1$. The symplectic form on $G.\mu$ at μ is given by

$$\omega(e_\alpha, f_\alpha) = \langle \mu, [e_\alpha, f_\alpha] \rangle = 2\pi \langle \mu, \alpha \rangle = 2\pi \langle \mu, \alpha \rangle \omega_0(e_\alpha, f_\alpha)..$$

Hence, the two Liouville forms differ by a factor $\prod_{\alpha > 0} 2\pi \langle \alpha, \mu \rangle$. The volume with respect to ω_0 is just the Riemannian volume for the inner product on $\mathfrak{t}^\perp$. It follows that

$$\mathrm{Vol}(G.\mu) = \prod_{\alpha > 0} 2\pi \langle \alpha, \mu \rangle \ \ \mathrm{vol}(G/T).$$

One may compute $\mathrm{vol}(G/T)$ explicitly to

$$\mathrm{vol}(G/T) = \frac{1}{\prod_{\alpha > 0} 2\pi \langle \alpha, \rho \rangle}.$$

so that

$$\mathrm{Vol}(G.\mu) = \prod_{\alpha > 0} \frac{\langle \alpha, \mu \rangle}{\langle \alpha, \rho \rangle}.$$

□

In particular, the symplectic volume of the coadjoint orbit through ρ is equal to 1. This last fact may be seen, for example, by computing the Fourier transform of the DH-measure of the coadjoint orbit, and taking the limit $\xi \to 0$.

5. Equivariant de Rham theory

Let M be a G-manifold. Recall that the Weil algebra $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge \mathfrak{g}^*$ has a differential,

$$\mathrm{d}^W = y^a(L_a \otimes 1) + (v^a - \tfrac{1}{2} f^a_{bc} y^b y^c)\iota_a.$$

Let M be a G-manifold. Then projecting out the $\wedge\mathfrak{g}^*$ part should give a homomorphism of differential algebras,

$$(W^{\mathfrak{g}} \otimes \Omega(M))_{\mathrm{basic}} \cong \Omega_G(M).$$

The term $v^a \iota_a$ in the Weil differential determines the Cartan differential to be

$$\mathrm{d}_G = \mathrm{d} - v^a \iota^M_a = \mathrm{d} - \iota(\xi_M).$$

The corresponding equivariant symplectic form is

$$\omega_G(\xi) = \omega - \langle \Phi, \xi \rangle.$$

(Sometimes one does not insist on this correspondence with the Weil algebra, and uses other choice $\mathrm{d}_G = \mathrm{d} + \lambda \iota(\xi_M)$. Then $\omega_G(\xi) = \omega + \lambda\langle \Phi, \xi\rangle$.

Let M be a compact oriented T-manifold with isolated fixed points. The correct statement of the localization formula reads,

$$\int_M \alpha(\xi) = (-1)^{\dim M/2} \sum_p \frac{\alpha(\xi)_{[p]}}{\prod \langle a, \xi\rangle}$$

where the product is over the (real) weights of the T-action. To see this, consider $M = \mathbb{R}^2 \cong \mathbb{C}$ with standard orientation given by $\mathrm{d}x\mathrm{d}y$. Let S^1 act on $\mathbb{C}$ with weight $a \in \mathbb{Z} \neq 0$. The generating vector field corresponding to $\xi \in \mathbb{R}$ is $\xi_{\mathbb{R}^2} = -2\pi a \xi \frac{\partial}{\partial \phi}$. Let $\chi \in C^\infty(\mathbb{R})$ be equal to 1 for $t < 1$ and equal to 0 for $t > 2$, and define an equivariantly closed form by

$$\alpha(\xi) = \mathrm{d}_G(\chi(r)\mathrm{d}\phi) = \chi'(r)\mathrm{d}r\mathrm{d}\phi + 2\pi a \xi \chi(r).$$

The integral of $\chi'(r)\mathrm{d}r\mathrm{d}\phi = \frac{\chi'(r)}{r}\mathrm{d}x\mathrm{d}y$ is equal to -2π (note that it should come out negative, since we may arrange $\chi'(r) \leq 0$). On the other hand, $\alpha(\xi)_{[0]} = 2\pi\langle a, \xi\rangle$. Comparing, we find the above formula.

The formula for the Fourier transform of coadjoint orbits reads,

$$\int_{\mathcal{O}} e^{\omega + 2\pi i \langle \Phi, \xi\rangle} = \frac{\sum_{w\in W} (-1)^{l(w)} e^{2\pi i \langle w\mu, \xi\rangle}}{\prod 2\pi i \langle \alpha, \xi\rangle}$$

(Indeed, this is a special case of the localization formula applied to $e^{\omega_G(\zeta)}$ with $\zeta = -2\pi \mathrm{i}\xi$.) The factors may be verified by observing that the limit $\xi \to 0$ should recover the symplectic volume. Set $\xi = s\rho$ and let $s \to 0$. Write $t = \exp(s\mu)$. Then

$$\begin{aligned}\prod_{\alpha\in\mathfrak{R}_+} 2\pi\langle\alpha,\rho\rangle\ \mathrm{vol}(\mathcal{O}) &= \lim_{s\to 0^+}(is)^{-\#\mathfrak{R}_+}\sum_{w\in W}(-1)^{l(w)}e^{2\pi is\langle w\mu,\rho\rangle}\\ &= \lim_{s\to 0^+}(is)^{-\#\mathfrak{R}_+}\prod_{\alpha\in\mathfrak{R}_+}(t^{\alpha/2}-t^{-\alpha/2})\\ &= \lim_{s\to 0^+} s^{-\#\mathfrak{R}_+}\prod_{\alpha\in\mathfrak{R}_+}2\sin(\pi s\langle\alpha,\mu\rangle)\\ &= \prod_{\alpha\in\mathfrak{R}_+}2\pi\langle\alpha,\mu\rangle.\end{aligned}$$

5.1. **Euler class.** The Euler class (form) is defined in such a way that it is the pullback of the Thom class. For $\mathbb{R}^2 = \mathbb{C}$, with the circle action of weight a, the equivariant Thom form is

$$\mathrm{Th}(\xi) = \frac{-1}{2\pi}\chi'(r)\mathrm{d}r\mathrm{d}\phi - a\xi\chi(r),$$

where $\chi \in C^\infty(\mathbb{R})$ is a bump function, equal to 1 for t close to 0 and equal to 0 for large t. The pull-back to 0 is $-a\xi$. Thus, we see:

Lemma 5.1. *Let T act on $\mathbb{C}^n$ with weights $a_1,\ldots,a_n$. The equivariant Euler form for this action is*

$$\mathrm{Eul}(\mathbb{C}^n,\xi) = (-1)^n\prod\langle a_i,\xi\rangle.$$

For non-isolated fixed points, the localization formula becomes

$$\boxed{\int_M \alpha(\xi) = \sum_F\int_F\frac{\iota_F^*\alpha(\xi)}{\mathrm{Eul}(\nu_F,\xi)}.}$$

(Actually, the formula holds for any F containing the fixed point set of ξ.) Note that the signs have been absorbed into the Euler form. The formula for the Fourier transform of coadjoint orbits generalizes to the following version of Duistermaat-Heckman:

$$\boxed{\langle\mathfrak{m}, e^{-2\pi\mathsf{i}\langle\mu,\xi\rangle}\rangle = \int_M e^{\omega-2\pi i\langle\Phi,\xi\rangle} = \sum_F\int_F\frac{e^{\omega_F-2\pi i\langle\Phi_F,\xi\rangle}}{\mathrm{Eul}(\nu_F,2\pi\mathsf{i}\xi)}.}$$

(Note that we now have the minus signs in the enumerator.)

6. Group-valued moment maps

Let $\theta^L = g^{-1}\mathrm{d}g,\ \ \theta^R = \mathrm{d}gg^{-1}$ be the left-/right invariant Maurer Cartan forms. They satisfy

$$\mathrm{d}\theta^L = -\tfrac{1}{2}[\theta^L,\theta^L],\ \ \mathrm{d}\theta^R = \tfrac{1}{2}[\theta^R,\theta^R].$$

Let $\eta = \frac{1}{12}B(\theta^L,[\theta^L,\theta^L])$. The equivariant extension of η reads

$$\eta_G(\xi) = \eta - \tfrac{1}{2}B(\theta^L+\theta^R,\xi).$$

We check: Since $\xi_G = \xi^L - \xi^R$,

$$\iota(\xi_G)\eta = \frac{1}{4}B(\xi,[\theta^L,\theta^L]) - \frac{1}{4}B(\xi,[\theta^R,\theta^R]) = -\tfrac{1}{2}\mathrm{d}B(\xi,\theta^L+\theta^R)$$

so

$$(\mathrm{d}-\iota(\xi_G))\eta = \tfrac{1}{2}\mathrm{d}B(\xi,\theta^L+\theta^R) = (\mathrm{d}-\iota(\xi_G))B(\xi,\theta^L+\theta^R).$$

The moment map condition for a space with G-valued moment map should be $\iota(\xi_M)\omega = -\frac{1}{2}B(\theta^L+\theta^R,\xi)$, so that its linearization reproduces the standard theory. We thus require

$$\mathrm{d}_G\omega = \Phi^*\eta_G.$$

The localization formula for DH-measures of group-valued moment maps states that [1]

$$\langle \mathfrak{m},\chi_\lambda\rangle = \dim V_\lambda \sum_{F\subset\mathcal{F}(\lambda+\rho)} \int_F \frac{e^{\omega_F}\Phi_F^{\lambda+\rho}}{\mathrm{Eul}(\nu_F,-2\pi\mathsf{i}(\lambda+\rho))}$$

We'd prefer a formula inolving $\chi_{*\lambda} = \overline{\chi}_\lambda$. Recall that $*(\lambda+\rho) = *\lambda+\rho = -w_0(\lambda+\rho)$. The fixed point set of $-(\lambda=\rho)$ is, of course, the same as the fixed point set of $(\lambda+\rho)$, and w_0 moves the fixed point set by the corresponding element of $N_G(T)$. Because of all this, I think we also have

$$\boxed{\langle \mathfrak{m},\overline{\chi}_\lambda\rangle = \dim V_\lambda \sum_{F\subset\mathcal{F}(\lambda+\rho)} \int_F \frac{e^{\omega_F}\Phi_F^{-(\lambda+\rho)}}{\mathrm{Eul}(\nu_F,2\pi\mathsf{i}(\lambda+\rho))}}$$

Example 6.1. Let's verify for a conjugacy class $\mathcal{C} = G.\exp\xi$. For simplicity, assume that $\exp\xi$ is regula so that the volume is

$$\mathrm{Vol}\,\mathcal{C} = \frac{\sum_w (-1)^{l(w)}\ t^{w\rho}}{\prod_{\alpha>0} 2\pi\mathsf{i}\langle\rho,\alpha\rangle}$$

Clearly, we must have $\langle\mathfrak{m},\overline{\chi}_\lambda\rangle = \overline{\chi}_\lambda(\exp\xi)$. And this is exactly what we get from the localization formula:

$$\langle \mathfrak{m},\overline{\chi}_\lambda\rangle = \dim V_\lambda \sum_{w\in W} \frac{w(t)^{-(\lambda+\rho)}}{\prod 2\pi\mathsf{i}\langle -w(\alpha),\lambda+\rho\rangle} = \frac{\dim V_\lambda}{\prod 2\pi\mathsf{i}\langle\alpha,\lambda+\rho\rangle}\sum_w (-1)^{l(w)} t^{w(*\lambda+\rho)}.$$

here replaced w^{-1} with ww_0^{-1} and used $-w_0(\lambda+\rho) = *\lambda+\rho$.

`One should verify this for conjugacy classes.`

7. Pre-quantization

In this section, we take the equivariant differential $\mathrm{d}+\lambda\iota(\xi_M)$. Later we specialize to $\lambda=-1$.

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[1] The sign in the Euler class wasn't there in our papers; perhaps a matter of definition of Euler class??

7.1. **Circle bundles.** Let $\pi : P \to M$ be a $S^1 = \mathbb{R}/\mathbb{Z}$-bundle over a symplectic manifold M. An invariant connection 1-form $\theta \in \Omega^1(P)$ has curvature $\mathrm{d}\theta$. We call P a pre-quantum circle bundle if $\mathrm{d}\theta = \pi^*\omega$.

Suppose M is a Hamiltonian G-manifold, and that G acts on P, preserving θ. Then

$$0 = L(\xi_P)\theta = \iota(\xi_P)\pi^*\omega + \mathrm{d}\iota(\xi_P)\theta = -\pi^*\mathrm{d}\langle\Phi, \xi\rangle + \mathrm{d}\iota(\xi_P)\theta.$$

One calls P a G-equivariant pre-quantum circle bundle if

$$\iota(\xi_P)\theta = \pi^*\langle\Phi, \xi\rangle.$$

Equivalently,

$$\xi_P = \mathrm{Lift}(\xi_M) + \pi^*\langle\Phi, \xi\rangle\frac{\partial}{\partial t},$$

where $\frac{\partial}{\partial t}$ is the generator of the S^1-action with $\iota(\frac{\partial}{\partial t})\theta = 1$. In terms of the equivariant curvature

$$F_G^\theta(\xi) = \mathrm{d}_G\theta = \mathrm{d}\theta + \lambda\iota(\xi_P)\theta,$$

the pre-quantum condition reads $F_G^\theta = \pi^*\omega_G$.

To fix signs, consider the example of a pre-quantized coadjoint orbit $\mathcal{O} = G.\mu$ for μ a weight. The pre-quantum line bundle should be $L_\mu = G \times_{G_\mu} \mathbb{C}_\mu$, with the unique left-invariant connection, and the pre-quantum circle bundle is the unit circle bundle in L_μ. It's easy to see that the left-invariant connection on $P = G \times_{G_\mu} \mathrm{U}(1)$ is represented by the 1-form $\psi = \langle\theta^L, \mu\rangle$ on G. The equivariant curvature of ψ is

$$\mathrm{d}_G\psi = \mathrm{d}\psi + \lambda\iota(-\xi^R)\psi = \mathrm{d}\psi - \lambda\langle\xi, \mathrm{Ad}_g^*(\mu)\rangle.$$

Since the moment map for a coadjoint orbit is the inclusion, this indicates that the moment map condition should read

$$\mathrm{d}_G\theta = -\omega_G.$$

7.2. **Hermitian line bundles.** Next, let $L \to M$ be a complex Hermitian line bundle. We want to arrange the pre-quantization condition in such a way that for $M = \mathbb{C}$, the space of square integrable holomorphic sections for the canonical Hermitian connection is non-empty.

A connection ∇ on L is a map on $\Gamma^\infty(M, L)$ with the usual derivation property. The curvature of ∇ is the 2-form valued endomorphism defined as $F_\nabla = \nabla^2 \in \Omega^2(M, \mathrm{End}(L))$. For a complex line bundle, one usually identifies $\mathrm{Aut}(L)$ with $C^\infty(M, \mathbb{C}^\star)$ (acting by multiplication) and $\mathrm{End}(L)$ with $C^\infty(M, \mathbb{C})$, so F_∇ becomes a complex-valued 2-form. In local coordinates, if

$$\nabla\sigma = \mathrm{d}\sigma + A\sigma,$$

for a connection 1-form $A \in \Omega^1(M, \mathrm{End}(L)) = \Omega^1(M, \mathbb{C})$, $\nabla^2\sigma = A\mathrm{d}\sigma + \mathrm{d}(A\sigma) = (\mathrm{d}A)\sigma$, so

$$\nabla^2 = \mathrm{d}A.$$

Consider $M = \mathbb{C}$ with trivial line bundle, with its standard holomorphic structure and with a fiber metric $\langle \lambda_z, \lambda_z \rangle = h(z, \overline{z})|\lambda_z|^2$. It is well-known that the corresponding Hermitian connection 1-form is given by

$$A = h^{-1}\partial h$$

(see e.g. [BGV], p.137 or [Wells], p. 79). Hence the curvature is

$$\mathrm{d}A = \overline{\partial}\partial \log(h).$$

Suppose $h = \exp(-\pi|z|^2)$, so that the space of square integrable sections of L is non-empty. Then

$$\mathrm{d}A = -\pi\overline{\partial}\partial|z|^2 = \pi \mathrm{d}z \wedge \mathrm{d}\overline{z} = \frac{2\pi}{i}\omega$$

This motivates the pre-quantization condition,

$$\omega = \frac{i}{2\pi}F_\nabla = c_1(\nabla).$$

(The factor 2π is not so important, but the sign is.) The sign convention for the first Chern form seems standard. (See e.g. [Wells] p.93).

Given a G-action on L, leaving ∇ invariant, one defines the equivariant covariant derivative

$$\nabla_G = \nabla + \lambda\iota(\xi_M)$$

(recall that $\lambda = -1$ is appearantly the most natural choice) and the equivariant curvature

$$F_{\nabla,G} = \nabla_G^2 - \lambda\, L_{\xi_L}$$

where L_{ξ_L} is the infinitesimal action on sections. (Note that without this term the expression on the right hand side would not be C^∞-linear). The equivariant curvature is equivariantly closed: $\nabla_G F_{\nabla,G} = 0$. The equivariant moment map condition reads,

$$\lambda(\nabla_{\xi_M} - L_{\xi_L})\sigma = \frac{2\pi}{i}\langle \Phi, \xi\rangle\sigma,$$

so if we make the natural choice $\lambda = -1$,

$$\nabla_{\xi_M} - L_{\xi_L} = 2\pi i\langle \Phi, \xi\rangle\sigma.$$

8. Components of a linear map

Let V be a vector space and $A : V \to V$ a linear map. In a given basis e_a of V we define the components of A as

$$Ae_a = A^b{}_a e_b.$$

With this convention,

$$(AB)e_a = (AB)^b{}_a a e_b = A(B(e_a)) = B^c{}_a A(e_c) = B^c{}_a A^b{}_c e_b,$$

that is,

$$A^b{}_c B^c{}_a = (AB)^b{}_a.$$

Also, the dual map $A^*: V^* \to V^*$ has components in the corresponding dual basis e^a,

$$(A^*)_a{}^b = A^b{}_a.$$

Check: $(A^*)_a{}^b e^a = A^*(e^b)$, so

$$(A^*)_c{}^b = \langle (A^*)_a{}^b e^a, e_c\rangle = \langle A^*(e^b), e_c\rangle = \langle e^b, A(e_c)\rangle = A_c{}^r\langle e^b, e_r\rangle = A_c{}^b.$$

Let $\mathfrak{g}$ be a Lie algebra, with structure constants f^c_{ab} in a basis e_a defined as

$$[e_a, e_b] = f^c{}_{ab} e_c.$$

For $\xi = \xi^r e_r \in \mathfrak{g}$ we have $[\xi, e_a] = \xi^r f^s{}_{ra} e_s$, so

$$(\mathrm{ad}_\xi)^b{}_a = \xi^c f^b{}_{ra}.$$

The corresponding formula for the co-adjoint action $\mathrm{ad}^*_\xi = -(\mathrm{ad}_\xi)^*$ involves a minus sign:

$$(\mathrm{ad}^*_\xi)_a{}^b = -\xi^c f^b{}_{ra}.$$

9. Orientation on the boundary

The usual definition of the integral of a differential form $\alpha = f(x_1, \ldots, x_n)\mathrm{d}x_1 \cdot \mathrm{d}x_n$ is as an iterated integral:

$$\int \alpha = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\mathrm{d}x_1 \Big) \mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

For an oriented manifold with boundary, one wants to define the orientation on the boundary in such a way that Stokes' theorem holds: $\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha$. Suppose the boundary is defined by $x_1 = 0$, and the manifold is on the side $x_1 \geq 0$. Then if $\mathrm{d}x_1 \cdots \mathrm{d}x_n$ defines the orientation of M, the orientation of ∂M is given by $-\mathrm{d}x_2 \cdots \mathrm{d}x_n$. Put differently, if Λ is the volume form on M, and X an outward-pointing vector field along ∂M, the orinetation on the boundary is defined by $\iota_X \Lambda$.

10. Integration over fibers

If $\pi: E \to M$ is a fiber bundle, one defines integration over the fibers $\pi_*: \Omega^*_{cp}(E) \to \Omega^*(M)$ in such a way that

$$\int_E \pi^*\alpha \wedge \beta = \int_M \alpha \wedge \pi_*\beta.$$

Suppose $E = M \times F$ where M, F are oriented. Let E be equipped with the product orientation. If α is a form on M and β a form on F (viewed as a form on E), we have

$$\int_E \pi^*\alpha \wedge \beta = \int_M \alpha \int_F \beta.$$

Thus integration over the fiber of a form on the fiber just integrates out the fiber variables. Thus

$$\pi_*(\pi^*\alpha \wedge \beta) = \alpha \wedge \pi_*\beta.$$

(Note that this depended on the choice of orientation on E as the product orienation on $M \times F$ rather than $F \times M$.) Taking $\alpha = 1$ we have the composition rule,

$$f_*^E = f_*^M \circ \pi_*$$

where $f^E : E \to \mathrm{pt}$ is the map to a point. Push-forward is a chain map: $\pi_* \mathrm{d}\beta = \mathrm{d}\pi_*\beta$, as can be seen from the definition: For α a k-form,

$$\begin{aligned} f_*^M(\alpha \wedge \pi_*\mathrm{d}\beta) &= f_*^E(\pi^*\alpha \wedge \mathrm{d}\beta) \\ &= (-1)^k f_*^E(\mathrm{d}\pi^*\alpha \wedge \beta) \\ &= (-1)^k f_*^M(\mathrm{d}\alpha \wedge \pi_*\beta) \\ &= f_*^M \alpha \wedge \mathrm{d}\pi_*\beta. \end{aligned}$$

Now suppose F has a boundary, so E is a fiber bundle with boundary ∂E. Let $\pi^{\partial E}$ be the boundary projection. Then we have the following generalization of Stokes' theorem:

$$\pi_*\mathrm{d}\beta = \mathrm{d}\pi_*\beta + (-1)^{\dim M}\pi_*^{\partial E}\beta.$$

We check, using the usual Stokes' theorem:

$$\begin{aligned} f_*^M(\alpha \wedge \pi_*\mathrm{d}\beta) &= f_*^E(\pi^*\alpha \wedge \mathrm{d}\beta) \\ &= (-1)^k f_*^E(\mathrm{d}(\pi^*\alpha \wedge \beta) - \pi^*\mathrm{d}\alpha \wedge \beta) \\ &= (-1)^k f^{\partial E}(\pi^*\alpha \wedge \beta) - (-1)^k f_*^E(\pi^*\mathrm{d}\alpha \wedge \beta) \\ &= (-1)^k f_*^M \alpha \wedge \pi_*^{\partial E}\beta + f_*^M \alpha \wedge \mathrm{d}\pi_*\beta. \end{aligned}$$

We could have avoided the extra sign, if we orient the fiber bundle as $F \times M$ rather than $M \times F$.