CHAPTER 3

Group-valued moment maps

1. Finite-dimensional description of Hamiltonian loop group spaces

Let $(\mathcal{M}, \omega, \Psi)$ be a Hamiltonian LG-space with proper moment map $\Psi : \mathcal{M} \to L\mathfrak{g}^*$. The based loop group $\Omega G \subset LG$ consisting of loops that are trivial at the origin of S^1, acts *freely* on $L\mathfrak{g}^*$ and the quotient map is just the holonomy map,

$$\mathrm{Hol} : L\mathfrak{g}^* \to G, \quad \mu \mapsto \mathrm{Hol}_{S^1}(\mu).$$

By equivariance, ΩG also acts freely on $\mathcal{M}$ and we can form the quotient $M = \mathcal{M}/\Omega G$. The original action of LG descends to an action of $G = LG/\Omega G$, and the map Ψ descends to an equivariant map $\Phi : M \to G$. We thus have a diagram,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Psi} & L\mathfrak{g}^* \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Phi} & G \end{array}$$

Consider $\mathcal{O} = LG.\mu \subset L\mathfrak{g}^*$ and the corresponding conjugacy class $\mathcal{C} = G.g \subset G$ where $g = \mathrm{Hol}(\mu)$. Then

$$\mathcal{M}_\mu = \Psi^{-1}(\mathcal{O})/LG = \Phi^{-1}(\mathcal{C})/G =: M_g.$$

We see that M behaves much like a space with a group valued moment map $\Phi : \mathcal{M} \to G$ – except it is not clear what happens to the symplectic form on $\mathcal{M}$??

More concretely, we would like to construct the symplectic structure of the moduli space of flat G-bundles on a closed oriented surface of genus h as a symplectic quotient,

$$\mathcal{M}(\Sigma) = G^{2h} /\!\!/ G = \Phi^{-1}(e)/G,$$

where G^{2h} is a "Hamiltonian G-space with group-valued moment map"

$$\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{j=1}^{h} [a_j, b_j].$$

More generally, given a surface of genus h with r boundary components, and conjugacy classes $\mathcal{C}_1 = G.g_1, \ldots, \mathcal{C}_r = G.g_r$, the moduli space $\mathcal{M}_G(\Sigma; \mathcal{C}_1, \ldots, \mathcal{C}_r)$ of flat G-bundles for which the restriction to the jth boundary component is the bundle corresponding to the conjugacy class $\mathcal{C}_j$, can be written

$$\mathcal{M}_G(\Sigma; \mathcal{C}_1, \ldots, \mathcal{C}_r) = G^{2h} \times \mathcal{C}_1^- \times \cdots \times \mathcal{C}_r^- /\!\!/ G = \Phi^{-1}(e)/G,$$

where $\mathcal{C}_j^- = G.g_j^{-1}$ and

$$\Phi(a_1, b_1, \ldots, a_h, b_h, d_1, \ldots, d_r) = \prod_{j=1}^{h} [a_j, b_j] \prod_{k=1}^{r} d_k.$$

Our task will be to construct a 2-form on the space $G^{2h} \times \mathcal{C}_1^- \times \cdots \times \mathcal{C}_r^-$ for which Φ is some kind of moment map, and the quotient $\Phi^{-1}(e)/G$ is "symplectic reduction".

2. Definition of a group-valued moment map

Let us try to invent the concept of a Hamiltonian G-manifold (M, ω, Φ) with an equivariant moment map taking values in G. We assume that G comes equipped with an invariant inner product, which we use to identify $\mathfrak{g} \cong \mathfrak{g}^*$.

For a $\mathfrak{g}^* \cong \mathfrak{g}$-valued moment map, the moment map condition was $\iota(\xi_M)\omega + \mathrm{d}\Phi \cdot \xi = 0$. If G is abelian – say, an N-torus $G = (\mathbb{R}/\mathbb{Z})^N$, the moment map condition also makes sense for G-valued maps $\Phi : M \to (\mathbb{R}/\mathbb{Z})^N$. (One sometimes thinks of such a map as a "multi-valued" map into $\mathfrak{g} = \mathbb{R}^N$.) More accurately, we should write the moment map condition for a torus valued moment map in the form

$$\iota(\xi_M)\omega + \Phi^*\gamma \cdot \xi = 0 \tag{1}$$

where $\gamma \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form of the torus. (On an abelian group, left- and right-multiplication coincide, so θ^L and θ^R are the same.

Does this condition also make sense for non-abelian groups? We have to be careful: Since $\omega(\xi_M, \xi_M) = 0$, the form γ should have the property,

$$\iota(\xi_G)\gamma \cdot \xi = 0 \tag{2}$$

for all $\xi \in \mathfrak{g}$. (In the abelian case, this is trivial since $\xi_G = 0$.) Unfortunately, for non-abelian groups neither θ^L nor θ^R have this property. Instead we have (using $\xi_G = \xi^L - \xi^R$ and $\theta^R = \mathrm{Ad}_g\, \theta^L$),

$$\begin{aligned} \iota(\xi_G)\theta^L &= (1 - \mathrm{Ad}_{g^{-1}})\xi, \\ \iota(\xi_G)\theta^R &= (\mathrm{Ad}_g - 1)\xi. \end{aligned}$$

However, multiples of their sum would do: The inner product of

$$\iota(\xi_G)(\theta^L + \theta^R) = (\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi$$

with ξ is zero since $\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}$ is an anti-symmetric operator! We take this multiple to be $1/2$, to recover the standard definition for the torus case. That is, our moment map condition for G-valued moment maps is

$$\iota(\xi_M)\omega + \tfrac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi = 0.$$

Unfortunately, this condition is incompatible with ω being closed and G-invariant. Indeed, the infinitesimal invariance condition $0 = L(\xi_M)\omega = \mathrm{d}\iota(\xi_M)\omega + \iota(\xi_M)\mathrm{d}\omega$ tells us that

$$\begin{aligned}
\iota(\xi_M)(\mathrm{d}\omega) &= \tfrac{1}{2}\Phi^*\mathrm{d}(\theta^L+\theta^R)\cdot\xi \\
&= -\frac{1}{4}\Phi^*\xi\cdot([\theta^L,\theta^L]-[\theta^R,\theta^R]) \\
&= -\frac{1}{4}\Phi^*\xi\cdot(1-\mathrm{Ad}_g)[\theta^L,\theta^L] \\
&= -\frac{1}{4}\Phi^*(1-\mathrm{Ad}_{g^{-1}})\xi\cdot[\theta^L,\theta^L] \\
&= -\frac{1}{4}\Phi^*(\iota(\xi_G)\theta^L)\cdot[\theta^L,\theta^L] \\
&= -\frac{1}{12}\Phi^*\iota(\xi_G)(\theta^L\cdot[\theta^L,\theta^L]) \\
&= -\iota(\xi_M)\Phi^*\eta,
\end{aligned}$$

where

$$\eta=\frac{1}{12}\theta^L\cdot[\theta^L,\theta^L]\in\Omega^3(G).$$

This leads us to the following condition for the differential of ω:

$$\mathrm{d}\omega+\Phi^*\eta=0.$$

Finally, our moment map condition is also incompatible with ω being non-degenerate. Indeed, the moment map condition can be written

$$\iota(\xi_M(m))\omega_m+\tfrac{1}{2}\Phi^*\theta^R\cdot(\mathrm{Ad}_{\Phi(m)}+1)\xi=0,$$

which clearly shows that any $\xi_M(m)$ with $(\mathrm{Ad}_{\Phi(m)}+1)\xi$ is in the kernel. We require that the kernel is not bigger than that and arrive at the following

Definition 2.1. A Hamiltonian G-manifold with group-valued moment map (in short, G-valued Hamiltonian G-space) is a triple (M,ω,Φ) consisting of a G-manifold M, and invariant 2-form ω, and an equivariant smooth map $\Phi: M\to G$ such that

(a) *Relative cocycle condition:* $\mathrm{d}\omega+\Phi^*\eta=0$,
(b) *Moment map condition:* $\iota(\xi_M)\omega+\frac{1}{2}\Phi^*(\theta^L+\theta^R)\cdot\xi=0=0$,
(c) *Minimal degeneracy condition:* The kernel of ω_m consists of all $\xi_M(m)$ such that $(\mathrm{Ad}_{\Phi(m)}+1)\xi=0$.

Before giving a first example, let us comment on the first two axioms in some more detail. In deriving the above axioms we have proved most of the following

Lemma 2.2. *The 3-form η is closed and bi-invariant. We have the identities,*

$$\begin{aligned}
d\eta &= 0 \\
\iota(\xi_G)\eta+\tfrac{1}{2}d(\theta^L+\theta^R)\cdot\xi &= 0 \\
\iota(\xi_G)(\theta^L+\theta^R)\cdot\xi &= 0.
\end{aligned}$$

PROOF. It is clear that η is left-invariant. Under right-translation, θ^L transforms by the adjoint action, so η is invariant by invariance of the inner product on $\mathfrak{g}$. The condition $\mathrm{d}\eta = 0$ can be checked directly, and also follows because any bi-invariant form on a Lie group is closed. $\square$

The first two conditions can be combined into one, using the notion of equivariant de Rham theory: Let

$$\Omega^k_G(M) = \bigoplus_{2i+j=k} (S^i\mathfrak{g}^* \otimes \Omega^j(M))^G$$

be the complex of equivariant differential forms. Elements in $\Omega_G(M)$ can be viewed as equivariant polynomial maps $\xi \mapsto \alpha(\xi)$ from $\mathfrak{g}$ into the space of differential forms. The complex $\Omega^\star(M)$ carries an equivariant differential of degree 1,

$$(\mathrm{d}_G\alpha)(\xi) := (\mathrm{d} + \iota(\xi_M))\alpha(\xi).$$

d_G square to zero exactly because we are restricting to *equivariant* maps, and because $(\mathrm{d} + \iota(\xi_M))^2 = L(\xi_M)$ is the Lie derivative. The group G carries an equivariant 3-form,

$$\eta_G(\xi) = \eta + \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi \in \Omega^3_G(G),$$

where G acts on itself by conjugation. The above Lemma just says that $\mathrm{d}_G\omega_G = 0$, and the first two conditions for a group-valued moment map just mean that

$$\mathrm{d}_G\omega + \Phi^*\eta_G = 0.$$

This is analogous to the moment map condition for $\mathfrak{g}^*$-valued moment maps, which may be written $\mathrm{d}_G\omega + \Phi^*\eta_G^{\mathfrak{g}^*} = 0$ where $\eta^{\mathfrak{g}^*} = \mathrm{d}_G \operatorname{Id}_{\mathfrak{g}^*}$, where we view the identity map of $\mathfrak{g}^*$ as an element $\operatorname{Id}_{\mathfrak{g}^*} \in (S^1\mathfrak{g}^*)^G \subset \Omega^2_G(\mathfrak{g}^*)$.

Now for the first and most important example:

THEOREM 2.3. *Any conjugacy class $\mathcal{C} \subset G$ is uniquely a G-valued Hamiltonian G-manifold $(\mathcal{C}, \omega, \Phi)$ in such a way that the moment map is the inclusion $\Phi : \mathcal{C} \hookrightarrow G$.*

PROOF. The 2-form is completely determined by the moment map condition:

$$\omega_g(\xi_{\mathcal{C}}(g), \zeta_{\mathcal{C}}(g)) = \tfrac{1}{2}(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi \cdot \zeta.$$

To check $\mathrm{d}\omega + \Phi^*\eta = 0$, it suffices to verify that its contractions with $\xi_{\mathcal{C}}$ are zero. This follows from the G-invariance:

$$\begin{aligned} \iota(\xi_M)(\mathrm{d}\omega + \Phi^*\eta) &= -\mathrm{d}\iota(\xi_M)\omega + \Phi^*(\iota(\xi_G)\eta) \\ &= \Phi^*(\tfrac{1}{2}\mathrm{d}(\theta^L + \theta^R) \cdot \xi + \iota(\xi_G)\eta) = 0. \end{aligned}$$

It remains to check the minimal degeneracy condition. Given $g \in \mathcal{C}$ let $\xi_{\mathcal{C}}(g) \in \ker \omega_g$. Since $(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}) = (\mathrm{Ad}_g + 1)(1 - \mathrm{Ad}_{g^{-1}})$, the equation $\omega(\xi_{\mathcal{C}}, \zeta_{\mathcal{C}})|_g = 0$ for all $\zeta \in \mathfrak{g}$ gives

$$(\mathrm{Ad}_g + 1)\xi \cdot (1 - \mathrm{Ad}_g)\zeta = 0,$$

that is,

$$(\mathrm{Ad}_g + 1)\xi \in \mathrm{im}(\mathrm{Ad}_g - 1)^{\perp} = \ker(\mathrm{Ad}_g - 1) = \mathfrak{g}_g.$$

Since $(\mathrm{Ad}_g + 1)$ is invertible (in fact, equal to 2) on $\mathfrak{g}_g = \ker(\mathrm{Ad}_g - 1)$ this shows $\xi \in \ker(\mathrm{Ad}_g + 1) \oplus \mathfrak{g}_g$. □

This example already shows that spaces with group-valued moment map cannot always come from ordinary symplectic manifolds, since there are many examples of conjugacy classes that do not admit symplectic structures. In fact, the exercise below shows that conjugacy classes can be odd-dimensional or non-orientable. We will see later that this does not occur if G is 1-connected, but even then it may happen that $H^2(\mathcal{C}) = 0$.

EXERCISE 2.4 (Conjugacy classes).

(a) Show that
$$\{A \in \mathrm{O}(2) |\ \det(A) = -1\}$$
is a 1-dimensional conjugacy class of $\mathrm{O}(2)$.

(b) Show that
$$\{A \in \mathrm{SO}(3) |\ \mathrm{tr}(A) = -1\}$$
is a conjugacy class of $\mathrm{SO}(3)$, diffeomorphic to the real projective plane $\mathbb{R}P(2)$.

(c) Show that the $\mathrm{SO}(5)$-conjugacy class of the diagonal matrix $A = \mathrm{diag}(1, -1, -1, -1, -1) \in \mathrm{SO}(5)$ is isomorphic to $\mathrm{SO}(5)/\mathrm{O}(4) \cong S^4/\mathbb{Z}_2 = \mathbb{R}P(4)$. Conclude that the conjugacy class $\mathcal{C}$ of any pre-image of A in the simply connected double cover $\mathrm{Spin}(5)$ of $\mathrm{SO}(5)$ is isomorphic to $\mathcal{C} = \mathrm{Spin}(5)/\mathrm{Spin}(4) = \mathrm{SO}(5)/\mathrm{SO}(4) = S^4$. Since $H^2(S^4) = 0$, this conjugacy class does not admit a symplectic structure.

3. Fusion

Now that we have seen that conjugacy classes are spaces with group-valued moment map, we would like to be able to form products. That is, given two Hamiltonian G-spaces with group-valued moment maps (M_j, ω_j, Φ_j) we would like to define a product $\tilde{M} = M_1 \circledast M_2$ which is the direct product of the manifolds, with diagonal G-action, and moment map the pointwise product $\tilde{\Phi} = \Phi_1 \Phi_2$ of the two moment maps. However, we cannot just take the sum of the two 2-forms since $(\Phi_1\Phi_2)^* \eta_G$ is not simply the sum of the two $\Phi_j^* \eta_G$. Instead we have:

LEMMA 3.1. *Let* $\mathrm{Mult}_G : G \times G \to G$ *be group multiplication, and* $\mathrm{pr}_j : G \times G \to G$, $j = 1, 2$ *projections to the first and second factor. Then*

$$\mathrm{Mult}_G^* \eta_G = \mathrm{pr}_1^* \eta_G + \mathrm{pr}_2^* \eta_G - \tfrac{1}{2} d_G(\mathrm{pr}_1^* \theta^L \cdot \mathrm{pr}_2^* \theta^R)$$

(where d_G *is the equivariant differential for the action on* $G \times G$ *given by conjugation of each factor). In detail,*

$$\mathrm{Mult}_G^* \eta = \mathrm{pr}_1^* \eta + \mathrm{pr}_2^* \eta - \tfrac{1}{2} d(\mathrm{pr}_1^* \theta^L \cdot \mathrm{pr}_2^* \theta^R)$$

and

$$\mathrm{Mult}_G^*(\theta^L+\theta^R)\cdot\xi=(\mathrm{pr}_1^*(\theta^L+\theta^R)+\mathrm{pr}_2^*(\theta^L+\theta^R))\cdot\xi-\iota(\xi_G)(\mathrm{pr}_1^*\theta^L\cdot\mathrm{pr}_2^*\theta^R).$$

PROOF. A straightforward calculation which is left as an exercise. □

THEOREM 3.2 (Fusion). *Let $(M,\omega,(\Phi_1,\Phi_2))$ be a group valued Hamiltonian $G\times G$-space. Let $\tilde{M}$ be M with diagonal action, $\tilde{\Phi}=\Phi_1\Phi_2$, and*

$$\tilde{\omega}=\omega+\tfrac{1}{2}(\Phi_1^*\theta^L\cdot\Phi_2^*\theta^R).$$

Then $(\tilde{M},\tilde{\omega},\tilde{\Phi})$ is a group-valued Hamiltonian G-manifold.

PROOF. By the moment map condition for ω, $\mathrm{d}_{G\times G}\omega+(\Phi_1,\Phi_2)^*\eta_{G\times G}=0$. Passing to the diagonal action this becomes $\mathrm{d}_G\omega+\Phi_1^*\eta_G+\Phi_2^*\eta_G=0$. Therefore,

$$\mathrm{d}_G\tilde{\omega}+(\Phi_1\Phi_2)^*\eta_G=\mathrm{d}_G\omega+\Phi_1^*\eta_G+\Phi_2^*\eta_G=0$$

To check the minimal degeneracy condition, let $v\in T_mM$ be in the kernel of $\tilde{\omega}_m$. We have to show that $v=\xi_{\tilde{M}}(m)=(\xi,\xi)_M(m)$ where $(1+\mathrm{Ad}_{\Phi_1\Phi_2(m)})\xi=0$.

Our first goal is to show that v is contained in $G\times G$-orbit directions. Since the kernel of ω is contained in $G\times G$-orbit directions, we need to show that for suitable $\xi_1,\xi_2\in\mathfrak{g}$,

$$\begin{aligned}\iota(v)\omega\overset{!}{=}\iota((\xi_1,\xi_2)_M)\omega &= -\tfrac{1}{2}\Phi_1^*(\theta^L+\theta^R)\cdot\xi_1-\tfrac{1}{2}\Phi_2^*(\theta^L+\theta^R)\cdot\xi_2\\ &= -\tfrac{1}{2}\Phi_1^*\theta^L\cdot(\mathrm{Ad}_{\Phi_1^{-1}}+1)\xi_1-\tfrac{1}{2}\Phi_2^*\theta^R\cdot(\mathrm{Ad}_{\Phi_2}+1)\xi_2.\end{aligned}$$

On the other hand, $\iota(v)\tilde{\omega}=0$ gives an equation,

$$\iota(v)\omega=-\tfrac{1}{2}\Phi_2^*\theta^R\cdot\iota(v)\Phi_1^*\theta^L+\tfrac{1}{2}\Phi_1^*\theta^L\cdot\iota(v)\Phi_2^*\theta^R. \tag{3}$$

So, what we need to show is that for suitable $\xi_1,\xi_2\in\mathfrak{g}$,

$$\iota(v)\Phi_2^*\theta^R=-(\mathrm{Ad}_{\Phi_1^{-1}}+1)\xi_1,\quad \iota(v)\Phi_1^*\theta^L=(\mathrm{Ad}_{\Phi_2}+1)\xi_2. \tag{4}$$

Let $\zeta\in\ker(\mathrm{Ad}_{\Phi_1}+1)$ so that $(\zeta,0)_M\in\ker\omega$. Taking the contraction of (3) with $(\zeta,0)_M$ we find

$$0=-\tfrac{1}{2}(\iota(v)\Phi_2^*\theta^R)\cdot\Phi_1^*\iota(\zeta_G)\theta^L=-\tfrac{1}{2}(\iota(v)\Phi_2^*\theta^R)\cdot(1-\mathrm{Ad}_{\Phi_1^{-1}})\zeta.$$

Since $(1-\mathrm{Ad}_{\Phi_1^{-1}})$ is invertible on $\ker(\mathrm{Ad}_{\Phi_1}+1)$ this shows

$$\iota(v)\Phi_2^*\theta^R\in\ker(\mathrm{Ad}_{\Phi_1}+1)^\perp=\mathrm{im}(\mathrm{Ad}_{\Phi_1}+1),$$

so that we solve the first equation in (4). A similar argument applies to the second equation. Using $v=(\xi_1,\xi_2)_M$ in Equation (4) we find two equations,

$$\begin{aligned}(\mathrm{Ad}_{\Phi_2}-1)\xi_2 &= -(\mathrm{Ad}_{\Phi_1^{-1}}+1)\xi_1,\\ (1-\mathrm{Ad}_{\Phi_1^{-1}})\xi_1 &= (\mathrm{Ad}_{\Phi_2}+1)\xi_2.\end{aligned}$$

Adding the two equations gives $\xi_1=\xi_2=:\xi$, and then either equation shows $(1+\mathrm{Ad}_{\Phi_1\Phi_2})\xi=0$. □

If $M = M_1 \times M_2$ is a direct product, we denote its fusion by $\tilde{M} =: M_1 \circledast M_2$.

EXERCISE 3.3. Show that the fusion operation is associative: $(M_1 \circledast M_2) \circledast M_3 = M_1 \circledast (M_2 \circledast M_3)$. We can therefore omit parantheses for multiple fusion products.

As an application, we see now that

$$M = \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_r$$

has the structure of a Hamiltonian G-space with group-valued moment map Φ, given by the product of the inclusion maps. We next want to show that symplectic reduction works as usual, so that $M/\!\!/G = \Phi^{-1}(e)/G$ is symplectic.

REMARK 3.4. The fusion product of group-valued Hamiltonian G-spaces is a classical analog of the fusion of positive energy loop group representations. (Hence the choice of terminology.)

4. Reduction

THEOREM 4.1 (Reduction). *Let (M, ω, Φ) be a Hamiltonian G-space with group-valued moment map. Then the identity element $e \in G$ is a regular value of Φ if and only if the G-action on $\Phi^{-1}(e)$ is locally free. The pull-back of ω to the identity level set $\Phi^{-1}(e)$ is basic, and descends to a symplectic form ω_{red} on $M/\!\!/G := \Phi^{-1}(e)/G$.*

PROOF. Let $Z = \Phi^{-1}(e)$ and $\iota_Z : Z \hookrightarrow M$ the inclusion. Using the identification $T_eG = \mathfrak{g}$, the moment map condition at $m \in \Phi^{-1}(e)$ reads

$$\omega(\xi_M(m), v) + \xi \cdot \mathrm{d}_m\Phi(v) = 0, \quad v \in T_mM.$$

This shows

$$\ker(\mathrm{d}_m\Phi) = T_m(G.m)^\omega \tag{5}$$

and $\mathrm{im}(\mathrm{d}_m\Phi) \subseteq \mathfrak{g}_m^\perp$. But ω_m is non-degenerate because the equation $(\mathrm{Ad}_g + 1)\xi = 0$ has no solution if g is close to the identity element. Hence, a standard dimension count shows

$$\mathrm{im}(\mathrm{d}_m\Phi) = \mathfrak{g}_m^\perp. \tag{6}$$

From (6), we see that $\mathrm{d}_m\Phi$ is surjective if and only if $\mathfrak{g}_m = 0$, that is, G_m is discrete. Suppose this is the case. Since the pull-back of η_G to $\{e\}$ vanishes,

$$\mathrm{d}_G\iota_Z^*\omega = \iota_Z^*\mathrm{d}_G\omega = -\iota_Z^*\Phi^*\eta_G = -(\Phi|_Z)^*\iota_e^*\eta_G = 0.$$

That is, $\iota_Z^*\omega$ is closed and horizontal. Since ω is invariant, it follows that $\iota_Z^*\omega$ is the pull-back of a closed form ω_{red} on $M/\!\!/G$. Since

$$T_mZ = \ker(\mathrm{d}_m\Phi) = T_m(G.m)^\omega$$

the kernel of $\iota_Z^*\omega$ consists of orbit directions, so ω_{red} is non-degenerate. □

REMARK 4.2. More generally, given a conjugacy class $\mathcal{C} \subset G$, we find that the G-action on $\Phi^{-1}(\mathcal{C})$ is locally free if and only if the action on the identity level set of $M \circledast \mathcal{C}^-$ is locally free. We can define a symplectic quotient $M_\mathcal{C}$ by

$$M_\mathcal{C} = (M \circledast \mathcal{C}^-)/\!\!/G.$$

5. More examples

Our goal is to give $G^{2h} \times \mathcal{C}_1^- \times \cdots \times \mathcal{C}_r^-$ the structure of a Hamiltonian G-space with group-valued moment map, where the action is the conjugation action on each factor, and the moment map is a product of Lie group commutators times the product of inclusion maps $\mathcal{C}_j^- \hookrightarrow G$. We have defined the 2-forms on the conjugacy classes, and also learned how to form products. To get the 2-form on G^{2h}, we will observe in this section that any group G *is* a conjugacy class for a disconnected group.

5.1. Twisted conjugacy classes. To put everything into proper context, let us first talk about the notion of *twisted conjugacy classes.* Suppose G is a compact Lie group and ψ a finite order automorphism of G. Thus ψ generates a subgroup $\mathbb{Z}_k \subset \mathrm{Aut}(G)$ where k is the order of ψ. One defines the ψ-twisted adjoint action by

$$\mathrm{Ad}_h^\psi(g) = (\psi.h)gh^{-1},$$

and class its orbits *twisted conjugacy classes.* The ψ-twisted conjugacy class of g is a homogeneous space

$$\mathcal{C}^\psi = \mathrm{Ad}_G^\psi .g = G/\{h \in G | \psi.h = \mathrm{Ad}_g(h)\}.$$

We note that the twisted conjugacy class is invariant under ψ, since

$$\psi.g = (\psi.g)gg^{-1} = \mathrm{Ad}_g^\psi(g).$$

If ψ is an involution (that is, $\psi = \psi^{-1}$), the twisted conjugacy class of the identity element e is the compact symmetric space $\mathrm{Ad}_G^\psi .e = G/G^\psi$. (The general symmetric space corresponding to ψ is of the form G/H where H is any subgroup between G^ψ and the identity component of G^ψ.) For example, the Lagrangian Grassmannian $\mathrm{U}(n)/\mathrm{O}(n)$ is such a symmetric space. Furthermore G is a symmetric space

$$G = (G \times G)/(G \times G)^\psi$$

where $\psi.(g_1, g_2) = (g_2, g_1)$. Now let $\hat{G} = \mathbb{Z}_k \ltimes G$ denote the semi-direct product of the group generated by ψ with G. Thus $\hat{G}$ is generated by ψ and by elements of G, subject to the relation $\psi g \psi^{-1} = \psi.g$.

LEMMA 5.1. *The twisted conjugacy class $\mathcal{C}^\psi$ of $g \in G$ is naturally isomorphic to the conjugacy class of $\psi^{-1}g \in \hat{G}$.*

PROOF. We have

$$\mathrm{Ad}_h(\psi^{-1}g) = h\psi^{-1}gh^{-1} = \psi^{-1}(\psi.h)gh^{-1} = \psi^{-1}\,\mathrm{Ad}_h^{\psi}(g)$$

showing that $G.(\psi^{-1}g) \cong \mathrm{Ad}_G^{\psi}(g)$. Since

$$\mathrm{Ad}_\psi(\psi^{-1}g) = \psi\psi^{-1}g\psi^{-1} = g\psi^{-1} = \psi^{-1}(\psi.g) = \psi^{-1}\,\mathrm{Ad}_g^{\psi}(g),$$

we see that in fact $\hat{G}.(\psi^{-1}g) \cong \mathrm{Ad}_G^{\psi}(g)$. □

Thus any ψ-twisted conjugacy class of G is a group-valued Hamiltonian $\hat{G}$-space.

Consider in particular the case of symmetric spaces $M = G/G^\psi$. The natural embedding

$$\Phi:\, M \hookrightarrow \psi G, \quad gG^\psi \mapsto \psi(\psi.g)g^{-1}$$

is the moment map for the action of $\mathbb{Z}_2 \ltimes G$. The 2-form is just zero: Indeed it vanishes at $\psi \in \hat{G}$ since $\psi^{-1} = \psi$, and hence by invariance vanishes everywhere.

Now take the fusion product of M with itself. The moment map now becomes

$$(g_1G^\psi, g_2G^\psi) \mapsto g_1\psi g_1^{-1}g_2\psi g_2^{-1} = g_1\psi(g_1)^{-1}\psi(g_2)g_2^{-1} = g_1\psi(g_1^{-1}g_2)g_2^{-1}.$$

taking values in G. Restricting the action, we can view $M \circledast M$ with diagonal G-action as a group-valued Hamiltonian G-space. Note that the 2-form is now non-zero since it acquires the fusion terms.

5.2. Double. This construction is particularly interesting for $M = G$ viewed as a symmetric space for $G \times G$. The moment map embeds M as the anti-diagonal in $\psi(G \times G)$. Thus

$$\Phi(a) = \psi(a^{-1}, a) = (a, a^{-1})\psi.$$

The $G \times G$-action reads, in terms of this embedding,

$$(g_1, g_2).a = g_1ag_2^{-1},$$

while ψ acts on G as inversion, $\psi.a = a^{-1}$. Taking the fusion product $G \circledast G$, the moment map for the diagonal action becomes

$$(a_1, a_2) \mapsto (a_1a_2^{-1}, a_1^{-1}a_2).$$

Let us re-name coordinates so that $a = a_1,\ b = a_2^{-1}$. Then the action becomes

$$(g_1, g_2).(a, b) = (g_1ag_2^{-1}, g_2bg_1^{-1}),$$

the moment map reads

$$(\Phi_1, \Phi_2):\ (a, b) \mapsto (ab, a^{-1}b^{-1}),$$

and the 2-form (obtained by fusion) is

$$\omega = \tfrac{1}{2}(a^*\theta^L \cdot b^*\theta^R + a^*\theta^R \cdot b^*\theta^L).$$

We call $D(G) = G \times G$ with this structure of a Hamiltonian $G \times G$-space the *double* of G. Finally, we can fuse the two G-factors to obtain a 2-form on $\tilde{D}(G)$, where the moment map is now,

$$(a, b) \mapsto aba^{-1}b^{-1} = [a, b].$$