3. Review of Hamiltonian group actions

A symplectic manifold is a pair (M, ω) consisting of a manifold M with a symplectic (i.e. closed and non-degenerate) 2-form ω. An action of a Lie group G on M is called symplectic if $g^*\omega = \omega$ for all $g \in G$. This implies that for all $\xi \in \mathfrak{g}$, the contraction of the generating vector field $\xi_M,\ \ \xi \in \mathfrak{g}$ with ω is a closed 1-form:

$$\mathrm{d}\iota(\xi_M)\omega = \mathrm{d}\iota(\xi_M)\omega + \iota(\xi_M)\mathrm{d}\omega = L(\xi_M)\omega = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(-t\xi)^*\omega = 0.$$

A symplectic action is is called *weakly Hamiltonian* if all these 1-forms are exact. Equivalently, the action is Hamiltonian if there exists a smooth map

$$\Phi :\ M \to \mathfrak{g}^*$$

such

$$\iota(\xi_M)\omega + \mathrm{d}\langle \Phi, \xi\rangle = 0$$

for all $\xi \in \mathfrak{g}$. The action is called Hamiltonian if the moment map Φ can be chosen G-equivariant with respect to the co-adjoint action; the triple (M, ω, Φ) is a Hamiltonian G-manifold. Examples include co-adjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$, and cotangent bundles $M = T^*X$ of G-manifolds X. One of the consequences of the equivariance condition is that the map $\mathfrak{g} \to C^\infty(M),\ \xi \mapsto \langle \Phi, \xi\rangle$ becomes a Lie algebra homomorphism, for the Poisson bracket on $C^\infty(M)$. Suppose (M, ω, Φ) is a Hamiltonian G-manifold. It is an easy consequence of the moment map condition that for all $m \in M$, the image of the tangent map to Φ is equal to the annihilator of the stabilizer algebra $\mathfrak{g}_m = \mathrm{Lie}(G_m)$, and the kernel consist of vectors that are ω-orthogonal to the orbit through m:

$$\mathrm{im}(\mathrm{d}_m\Phi) = \mathrm{ann}(\mathfrak{g}_m), \quad \ker(\mathrm{d}_m\Phi) = T_m(G.m)^\omega.$$

As a consequence of the first equality, $\mu \in \mathfrak{g}^*$ is a regular value of Φ if and only if the action of G_μ on $\Phi^{-1}(\mu)$ has finite stabilizers. In particular, the quotient $M_\mu = \Phi^{-1}(\mu)/G_\mu$ has the structure of an orbifold (i.e. looks locally like the quotient of a manifold by a finite group.) From the second equality, one finds that ω descends to a closed, non-degenerate 2-form ω_μ on M_μ. One calls (M_μ, ω_μ) the symplectic quotient. M_μ only depends on the co-adjoint orbit $\mathcal{O}$ of μ since

$$M_\mu = \Phi^{-1}(\mu)/G_\mu = \Phi^{-1}(\mathcal{O})/G.$$

for this reason, one sometimes writes $M_{\mathcal{O}}$ rather than M_μ.

Remarks 3.1. (a) If μ is not a regular value, the symplectic quotient acquires more serious singularities. One can show, however, that M_μ has a canonical finite decomposition into smooth symplectic manifolds.

(b) If (M, ω, Φ) carries a compatible Kähler metric (i.e. a metric of the form $(v, w) = \omega(v, Jw)$ for an integrable complex structure J on M) then the induced Riemannian metric on M_μ is also Kähler. (See Guillemin-Sternberg. "multiplicities of group representations and geometric quantization" (1982).)

(c) Sometimes, one also considers *affine*-equivariant moment maps: These are moment maps that are equivariant with respect to an affine action of G on $\mathfrak{g}^*$, with the coadjoint action as its linear part. The Marsden-Weinstein theorem extends to this setting. For a detailed discussion see the book by Liberman-Marle (1981), p. 200.

4. The symplectic geometry of the moduli space of flat connections

4.1. Symplectic structure on the space of connections. Let Σ be a compact surface, possibly with boundary.

Any choice of a Riemannian metric on Σ defines an inner product on the vector space $\Omega^1(\Sigma)$ by

$$\bar{g}(a,b) = \int_\Sigma g(a,b)\ |\Lambda_g|,$$

where $|\Lambda_g|$ is the Riemannian measure corresponding to g. This inner product $\bar{g}$ depends only on the conformal class of g, since multiplying g by positive function λ^2 scales the length of covectors by λ^{-1} and the measure by λ^2.

If Σ is orientable, then any choice of an orientation on Σ defines a symplectic structure on the vector space $\Omega^1(\Sigma)$ of real-valued 1-forms, given by wedge product and integration,

$$\omega(a,b) = \int_\Sigma a \wedge b.$$

Prescribing a conformal structure on Σ together with an orientation is equivalent to prescribing a complex structure on Σ. Let $* : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ [1] be the star operator defined by g and the orientation. Then $*^2 = -1$ (on 1-forms on 2-manifolds), so $J := *$ is a complex structure on $\Omega^1(\Sigma)$. The symplectic and complex structure and the inner product on $\Omega^1(\Sigma)$ are compatible in the sense that

$$\bar{g}(a,b) = \omega(a, J(b))$$

Indeed, by definition of the star operator,

$$\omega(a, J(b)) = \int_\Sigma a \wedge *b = \int_\Sigma \mathrm{g}(a,b)\ \Lambda_g.$$

Thus $\Omega^1(\Sigma)$ becomes a Hermitian vector space. To summarize, we have:

(1)	Conformal structure on Σ	$\mapsto$	Inner product on $\Omega^1(\Sigma)$
(2)	Orientation on Σ	$\mapsto$	Symplectic structure on $\Omega^1(\Sigma)$
(3)	Complex structure on Σ	$\mapsto$	Hermitian structure on $\Omega^1(\Sigma)$.

More generally, given a real vector bundle $E \to \Sigma$ with a fiberwise inner product on E, the orientation on Σ defines a natural symplectic structure on the space $\Omega^1(\Sigma, E)$ of

[1] We use the following sign convention: If (M, g) is a Riemannian manifold with Riemannian volume form Λ_g, the $(\alpha,\beta)\,\Lambda_\mathrm{g} = \alpha \wedge *\beta$. With this convention, the $*$ operator on $\mathbb{R}^2 = \mathbb{C}$ is $*\mathrm{d}x = \mathrm{d}y$ and $*1 = \mathrm{d}x \wedge \mathrm{d}y$.

E-valued 1-forms, and a complex structure on Σ makes $\Omega^1(\Sigma, E)$ into a Hermitian vector space. We are interested in the special case $E = \mathfrak{g}(\mathcal{P})$, where $\mathcal{P} \to \Sigma$ is a principal G-bundle for a compact connected Lie group G, and the inner product on $\mathfrak{g}(\mathcal{P})$ is induced by an invariant inner product on the Lie algebra $\mathfrak{g}$. We also assume that Σ comes equipped with an orientation, so $\Omega^1(\Sigma, \mathfrak{g}(\mathcal{P}))$ is a symplectic vector space. Recall that the space $\mathcal{A}(P)$ of connections on $\mathcal{P}$ is an affine space with $\Omega^1(\Sigma, \mathfrak{g}(\mathcal{P}))$ as its space of translations. ω can be viewed as a translation invariant symplectic 2-form on $\mathcal{A}(\mathcal{P})$, making it into an infinite-dimensional symplectic manifold. Given furthermore a conformal structure on Σ, the corresponding J is a translation invariant complex structure, and $\mathcal{A}(\mathcal{P})$ becomes an infinite-dimensional Kähler manifold. We will not make any use of the complex structure in this course, and focus mainly on symplectic aspects.

4.2. A moment map for the gauge action. It is immediate from the definition that if $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ is an isomorphism of principal G-bundles (not necessarily inducing the identity on the base), then the induced map

$$\phi^* : \mathcal{A}(\mathcal{P}_2) \to \mathcal{A}(\mathcal{P}_1)$$

is a symplectomorphism if ϕ induces an orientation-preserving map on the base, and an anti-symplectomorphism otherwise.

In particular, the action of the gauge group $\mathrm{Gau}(\mathcal{P})$ on $\mathcal{A}(\mathcal{P})$ is by symplectomorphisms. We will show that this action is generated by a moment map. Earlier we have calculated the generating vector field for the gauge action, and found that for $\xi \in \mathrm{gau}(\mathcal{P})$, $\xi_{\mathcal{A}}(\theta) = \mathrm{d}^\theta \xi$. Therefore,

$$\begin{aligned} \omega(\xi_{\mathcal{A}}(\theta), b) &= \int_\Sigma B(\mathrm{d}^\theta \xi, b) \\ &= -\int_\Sigma B(\xi, \mathrm{d}^\theta b) - \int_{\partial\Sigma} B(\xi, b) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Big(\int_\Sigma B(\xi, F^{\theta+tb}) + \int_{\partial\Sigma} B(\xi, \iota_{\partial\Sigma}^*(\theta + tb) - \theta_0) \Big) \end{aligned}$$

where θ_0 is some fixed reference connection on $\mathcal{P}|_{\partial\Sigma} \to \partial\Sigma$ It follows that the map

$$\mathcal{A}(\mathcal{P}) \to \Omega^2(\Sigma, \mathfrak{g}(\mathcal{P})) \oplus \Omega^1(\partial\Sigma, \mathfrak{g}(\mathcal{P})), \quad \theta \mapsto (F^\theta, \, \iota_{\partial\Sigma}^*\theta - \theta_0) \tag{4}$$

satisfies the moment map condition. The first component is gauge-equivariant. However, the second term involves the gauge action on connections, and is only affine-equivariant.

If Σ has no boundary, the moment map is simply the curvature, and we see that the symplectic quotient $\mathcal{M}_G(\mathcal{P}) = \Phi^{-1}(0)/\,\mathrm{Gau}(\mathcal{P})$ is exactly the moduli space of flat connections. The choice of a conformal structure on Σ induces a Kähler metric on $\mathcal{M}_G(\mathcal{P})$.

4.3. The case with non-empty boundary. In the case with boundary, one can consider reductions at curvature $F^\theta = 0$, but for a given non-trivial pull-back to the boundary $\iota_{\partial\Sigma}^*\theta = \mu \in \Omega^1(\partial\Sigma, \mathfrak{g}(\mathcal{P}))$. The corresponding moduli space corresponds to flat connections on $\mathcal{P}$ for which the holonomy around the boundary components lies in prescribed conjugacy classes.

Let $\mathrm{Gau}(\mathcal{P}, \partial\Sigma)$ be the gauge transformations that are trivial over the boundary $\partial\Sigma$. That is, $\mathrm{Gau}(\mathcal{P}, \partial\Sigma) \subset \mathrm{Gau}(\mathcal{P})$ is the kernel of the restriction map

$$\iota_{\partial\Sigma}^* : \ \mathrm{Gau}(\mathcal{P}) \to \mathrm{Gau}(\mathcal{P}|\partial\Sigma). \tag{5}$$

It is a normal subgroup of $\mathrm{Gau}(\mathcal{P})$, with quotient $\mathrm{Gau}(\mathcal{P})/\,\mathrm{Gau}(\mathcal{P}, \partial\Sigma)$ isomorphic to the image $\mathrm{Gau}(\mathcal{P})|\partial\Sigma$ of the restriction map. Its Lie algebra $\mathrm{gau}(\mathcal{P}, \partial\Sigma)$ consists of sections ξ of $\mathfrak{g}(\mathcal{P})$ that vanish along $\partial\Sigma$.

For based gauge transformation the boundary term disappears. That is, the action of $\mathrm{Gau}(\mathcal{P}|\partial\Sigma)$ is Hamiltonian in the usual sense, with moment map simply the curvature. The symplectic quotient

$$\mathcal{M}_G(\mathcal{P}, \partial\Sigma) = \mathcal{A}(\mathcal{P}) /\!\!/ \,\mathrm{Gau}(\mathcal{P}|\partial\Sigma) = \mathcal{A}_{\mathrm{flat}}(\mathcal{P})/\,\mathrm{Gau}(\mathcal{P}|\partial\Sigma)$$

is the moduli space of flat connections, up to gauge transformations that are trivial along $\partial\Sigma$. Since $\mathrm{Gau}(\mathcal{P}|\partial\Sigma)$ acts freely on $\mathcal{A}(\mathcal{P})$, it $\mathcal{M}_G(\mathcal{P}, \partial\Sigma)$ is a smooth symplectic manifold – however, it is still infinite-dimensional. The original action of $\mathrm{Gau}(\mathcal{P})$ descends to a Hamiltonian action of $\mathrm{Gau}(\mathcal{P})|_{\partial\Sigma}$, with moment map induced by pull-backs of connections:

$$\Psi : \ \mathcal{M}_G(\mathcal{P}, \partial\Sigma) \to \Omega^1(\partial\Sigma, \mathfrak{g}(\mathcal{P})), \quad [\theta] \mapsto \iota_{\partial\Sigma}^*\theta - \theta_0,$$

where $\theta_0 \in \mathcal{A}(\mathcal{P}|_{\partial\Sigma})$ is a fixed reference connection.

Let us choose trivializations of $\mathcal{P}$ over the boundary, i.e. a framing

$$\psi : \ \mathcal{P}_{\partial\Sigma} \cong \partial\Sigma \times G.$$

Using ψ we identify $G(\mathcal{P}|_{\partial\Sigma}) = \partial\Sigma \times G$ and therefore

$$\mathrm{Gau}(\mathcal{P}|_{\partial\Sigma}) = \Gamma^\infty(\partial\Sigma, G(\mathcal{P}|_{\partial\Sigma})) \cong C^\infty(\partial\Sigma, G) = \times_{j=1}^r C^\infty((\partial\Sigma)_j, G).$$

Choose furthermore parametrizations $(\partial\Sigma)_j \cong S^1$ of the boundary components, the group $C^\infty((\partial\Sigma)_j, G)$ gets identified with the *loop group*

$$LG = C^\infty(S^1, G).$$

4.4. Hamiltonian loop group spaces. Let us denote $L\mathfrak{g}^* := \Omega^1(S^1, \mathfrak{g})$, equipped with the "gauge action" of LG,

$$g.\mu = \mathrm{Ad}_g(\mu) - g^*\theta^R = \mathrm{Ad}_g(\mu) - \mathrm{d}g\; g^{-1}.$$

We will refer to this as the coadjoint loop group action; note that it is only an affine action. A pairing of $L\mathfrak{g}^*$ with $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$ is given by taking the inner product and integrating over S^1.

DEFINITION 4.1. A Hamiltonian loop group manifold is a triple $(\mathcal{M}, \omega, \Psi)$ where $\mathcal{M}$ is an (infinite-dimensional) LG-manifold, ω is an invariant symplectic form on $\mathcal{M}$, and $\Psi : \mathcal{M} \to L\mathfrak{g}^*$ an equivariant map satisfying the usual moment map condition,

$$\iota(\xi_M)\omega + \mathrm{d}\langle \Phi, \xi \rangle = 0, \quad \xi \in \Omega^0(S^1, \mathfrak{g}).$$

A first example of Hamiltonian loop group spaces are coadjoint orbits for the loop group. Let $\mathcal{O} \subset L\mathfrak{g}^*$ be an orbit of the loop group action. Then $\mathcal{O}$ carries a unique structure of a Hamiltonian LG-manifold such that the moment map is the inclusion. A formula for the 2-form can be worked out from the moment map condition; it is given by

$$\omega_\mu(\xi_{\mathcal{O}}(\mu), \eta_{\mathcal{O}}(\mu)) = \langle \mathrm{d}_\mu \xi, \eta \rangle = \int_{S^1} (\mathrm{d}_\mu \xi) \cdot \eta.$$

(The axioms are verified just like for coadjoint orbits of a finite dimensional Lie group; we leave this as an exercise.) For any Hamiltonian loop group manifold, the reduced space (symplectic quotient) at $\mu \in L\mathfrak{g}^*$ is defined as

$$\mathcal{M}_\mu = \Phi^{-1}(\mu)/(LG)_\mu = \Phi^{-1}(\mathcal{O})/LG.$$

If the moment map is proper, M_μ is a compact space. We will see later on that it is in fact a (singular) symplectic space.

4.5. The space $\mathcal{M}_G(\Sigma, \partial\Sigma)$ as a Hamiltonian loop group space. The moduli space $\mathcal{M}(\mathcal{P}, \partial\Sigma)$ of flat connections on $\mathcal{P} \to \Sigma$, up to gauge transformations that are trivial over $\partial\Sigma$, comes very close to being a Hamiltonian $(LG)^r = L(G^r)$-manifold, except that it does not carry an action of the full loop group LG^r but only of a subgroup. The problem is that in general, the restriction map $\mathrm{Gau}(\mathcal{P}) \to \mathrm{Gau}(\mathcal{P}|_{(\partial\Sigma)_j}) = LG$ need not be surjective. On the other hand, the restriction map on the Lie algebra level $\mathrm{gau}(\mathcal{P}) \to \mathrm{gau}(\mathcal{P}|_{(\partial\Sigma)_j}) = L\mathfrak{g}$ certainly is surjective. Thus $\mathrm{Gau}(\mathcal{P}|)_{(\partial\Sigma)_j}$ is a union of connected components of LG. The connected components of LG are labeled by $\pi_1(G)$.

It would be much preferable to work with a genuine loop group manifold. This is achieved by working with a space $\mathcal{M}_G(\Sigma, \partial\Sigma)$ of framed flat bundles, rather than flat connections on a fixed bundle.

Denote by $\mathcal{M}_G(\Sigma, \partial\Sigma)$ the moduli space of flat G-bundles $(\mathcal{P}, \theta)$ with framing $\psi : \mathcal{P}_{\partial\Sigma} \cong \partial\Sigma \times G$ over $\partial\Sigma$. The loop group $LG^r \cong C^\infty(\partial\Sigma, G)$ acts on $\mathcal{M}_G(\Sigma, \partial\Sigma)$ by changing the framing. For any flat bundle $(\mathcal{P}, \theta)$, the choice of a framing ψ over the boundary defines an inclusion

$$\iota_\psi : \mathcal{M}(\mathcal{P}, \partial\Sigma) \hookrightarrow \mathcal{M}_G(\Sigma, \partial\Sigma), [\theta] \mapsto [(\mathcal{P}, \theta, \psi)]$$

as a connected component of $\mathcal{M}_G(\Sigma, \partial\Sigma)$. If two such framings ψ, ψ' are related by some gauge transformation $\phi \in \mathrm{Gau}(\mathcal{P})$, i.e. $\psi' = \psi \circ \phi|_{\partial\Sigma}$, this identification changes by composition with that gauge transformation:

$$\iota_{\psi \circ \phi|_{\partial\Sigma}} = \iota_\psi \circ \phi.$$

In particular, there is a unique symplectic structure on $\mathcal{M}_G(\Sigma, \partial\Sigma)$ for which the maps ι_ψ are symplectic. Moreover, the inclusion ι_ψ for fixed ψ is $\mathrm{Gau}(\mathcal{P})|_{\partial\Sigma}$-equivariant.

We have

$$\mathcal{M}_G(\Sigma, \partial\Sigma) = \coprod_{[(\mathcal{P},\psi)]} \mathcal{M}(\mathcal{P}, \partial\Sigma) \times \{\psi\}.$$

To summarize:

THEOREM 4.2. *$\mathcal{M}_G(\Sigma, \partial\Sigma)$ is a Hamiltonian LG^r-manifold, with moment map*

$$\mathcal{M}_G(\Sigma, \partial\Sigma) \to (L\mathfrak{g}^r)^*, \quad [(\mathcal{P}, \theta, \psi)] \mapsto \tilde{\psi}(\iota_{\partial\Sigma}^*\theta).$$

Here $\tilde{\psi} : \mathcal{A}(\mathcal{P}|_{\partial\Sigma}) \to (L\mathfrak{g}^r)^* = \Omega^1(S^1, \mathfrak{g}^r)$ is the isomorphism defined by the framing ψ.

4.6. Reduced spaces. The reduced spaces of $\mathcal{M}_G(\Sigma, \partial\Sigma)$ have a very simple interpretation. Recall that flat bundles over S^1 are in 1-1 corresponsence with conjugacy classes:

$$\mathcal{M}_G(S^1) \cong \mathrm{Conj}(G),$$

where the isomorphism takes $[(P, \theta)]$ to the conjugacy class of the holonomy of θ around S^1. Given

$$\mu_j \in L\mathfrak{g}^*, \quad j = 1, \ldots, r$$

viewed as a connection on the jth boundary component, the level set $\Psi^{-1}(\mu_1, \ldots, \mu_r)$ consists of equivalence classes of framed flat bundles over Σ which restrict to the prescribed flat bundle over the boundary. Thus if we let $\mathcal{C}_j \subset G$ be the conjugacy class of the holonomy of μ_j;

$$\mathcal{M}_G(\Sigma, \partial\Sigma)_{(\mu_1,\ldots,\mu_r)} = \mathcal{M}_G(\Sigma; \mathcal{C}_1, \ldots, \mathcal{C}_r)$$

are equivalence classes of flat bundles over Σ which over $(\partial\Sigma)_j$ restrict to the equivalence class given by $\mathcal{C}_j$.

4.7. Cutting and gluing. The space $\mathcal{M}_G(\Sigma, \partial\Sigma)$ behave nicely under gluing of surfaces. Suppose, for example, that Σ is a closed surface, and Σ^+, Σ^- the surfaces with boundary obtained by cutting Σ along a seperating curve C. Conversely, Σ is obtained from $\Sigma^\pm$ by identifying the boundaries $\partial\Sigma^\pm$ by an **orientation reversing** diffeomorphism. Changing the orientation on one of the boundaries, say $\partial\Sigma^-$, changes the sign of the corresponding moment map for $\mathcal{M}_G(\Sigma^-, \partial\Sigma^-)$. This is because the pairing of $\Omega^1(S^1, \mathfrak{g})$ with $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$ is given by integration over S^1, which involves the orientation. After this change of orientation, the zero level set for the diagonal action on $\mathcal{M}_G(\Sigma^+, \partial\Sigma^+) \times \mathcal{M}_G(\Sigma^-, \partial\Sigma^-)$ consists of pairs $[(\mathcal{P}^\pm, \theta^\pm, \psi^\pm)]$ such that for suitable choices of equivalence classes,

$$\mu^+ = \tilde{\psi^+}(\iota_{\partial\Sigma^+}^*\theta^+) = \mu^- = \tilde{\psi^-}(\iota_{\partial\Sigma^-}^*\theta^-).$$

These are exactly pairs of flat framed bundles $(\mathcal{P}^\pm, \theta^\pm, \psi^\pm)$ which glue together under the framing. Quotienting the LG-action on the zero level set just means that we forget about the framing. That is, we have the **"Gluing equals Reduction"** principle,

$$\mathcal{M}_G(\Sigma) = (\mathcal{M}_G(\Sigma^+, \partial\Sigma^+) \times \mathcal{M}_G(\Sigma^-, \partial\Sigma^-))/\!\!/LG.$$

This principle generalizes to more boundary components in the obvious way.

4.8. Holonomy description. The moduli space $\mathcal{M}_G(\Sigma, \partial\Sigma)$ can be described in terms of holonomies. Let $x_1 \in (\partial\Sigma)_j$ correspond to the unit of $(\partial\Sigma)_j = S^1$. Let $\mathrm{Hol}: \mathcal{A}(\mathcal{P}|\partial\Sigma_j) \to G$ the corresponding holonomy map. Let $\delta_1, \ldots, \delta_r$ be paths from x_1 to $x_2, \ldots, x_r$, and let $\alpha_1, \ldots, \beta_h$ be loops based at x_1, such that

$$\prod_{j=1}^{h}[\alpha_j, \beta_j] = \prod_{i=1}^{r} \mathrm{Ad}_{\delta_i}(\partial\Sigma_i)$$

We hava a map $\mathcal{M}_G(\mathcal{P}, \partial\Sigma) \to G^{2h} \times G^{r-1} \times \mathcal{A}(\mathcal{P}|_{\partial\Sigma})$ taking $[\theta]$ to the parallel transports along the paths $\alpha_j, \beta_j, \delta_j$ and the restriction to the boundary.

LEMMA 4.3. *Suppose G is compact, connected. The moduli space $\mathcal{M}_G(\Sigma, \partial\Sigma)$ is isomorphic to the submanifold of $G^{2h} \times G^{r-1} \times (L\mathfrak{g}^*)^r$ consisting of elements*

$$(a_1, b_1, \ldots, a_h, b_h, d_2, \ldots, d_r, \mu_1, \ldots \mu_r)$$

with

$$\prod_{j=1}^{h}[a_j, b_j] = \prod_{i=1}^{r} \mathrm{Ad}_{d_i} \mathrm{Hol}(\mu_j) \tag{6}$$

where we have put $d_1 = e$.

OUTLINE OF PROOF. Given $[(P, \theta, \psi)]$, let $a_j, b_j, d_j \in G$ be the holonomies along the paths $\alpha_j, \beta_j, \delta_j$. (These are defined as elements of G, using the trivialization ψ.) Also let μ_j be the pull-back $\iota^*_{(\partial\Sigma)_j}\theta \in \mathcal{A}(\mathcal{P}|_{\partial\Sigma}) \cong L\mathfrak{g}^*$. Since $\alpha_j, \beta_j, \mathrm{Ad}_{\delta_j}(\partial\Sigma)_j$ generate the fundamental group of Σ w.r.t. x_1, the (a_j, b_j, d_j, μ_j) have to satisfy the relation described in the Lemma. Conversely, given (a_j, b_j, d_j, μ_j) satisfying the relation, we can pick a framed flat bundle $(P, \theta, \tilde{\psi})$ with holonomies $a_j, b_j, d_j, \mathrm{Hol}(\mu_j)$. Let $\tilde{\mu}_j$ be the pull-back of θ in terms of the framing $\tilde{\psi}$. Since $\mathrm{Hol}(\mu_j) = \mathrm{Hol}(\tilde{\mu}_j)$, we can change the framing by a based gauge transformation in $\mathrm{Gau}(\mathcal{P}_{(\partial\Sigma)_j}, x_j)$ to achieve $\tilde{\mu}_j = \mu_j$. $\square$

This description shows clearly that the moment map Ψ for $\mathcal{M}_G(\Sigma, \partial\Sigma)$ is *proper*: Pre-images of compact sets are compact. In particular, all reduced spaces are compact.