# Differentiable manifolds – Assignment #2

(Revised version)

## Due in class on December 5

You may assume the cohomology of $\mathbb{R}^n$ and $S^n$ as known.

**Problem #1:** [*15 points*] The complex projective space $\mathbb{C}P(m)$ is the quotient of $\mathbb{C}^{m+1}\backslash\{0\}$ by the equivalence relation, $z \sim cz$ for $c \in \mathbb{C}\backslash\{0\}$.

a) Show that $\mathbb{C}P(m)$ is a compact manifold of dimension $2m$, and that $\mathbb{C}P(1) = S^2$.

b) For all $m \geq 1$, the inclusion $\mathbb{C}^m\backslash\{0\} \hookrightarrow \mathbb{C}^{m+1}\backslash\{0\}$ induces an embedding $\mathbb{C}P(m-1) \to \mathbb{C}P(m)$. Show that the complement of $\mathbb{C}P(m)\backslash\mathbb{C}P(m-1)$ is naturally diffeomorphic to $\mathbb{C}^m$.

c) Calculate the cohomology $H^p(\mathbb{C}P(m))$ for all $p, m$. Hint: Write $\mathbb{C}P(m) = \mathbb{C}^m \cup \mathbb{C}P(m-1)$ as in (b). Apply Mayer-Vietoris to the cover $\{U, V\}$ given by $U = \mathbb{C}^m \subset \mathbb{C}P(m)$ and $V = (\mathbb{C}^m\backslash\{0\}) \cup \mathbb{C}P(m-1)$.

**Problem #2:** [*15 points*] Let $M$ be a manifold, and $S \subset M$ an embedded submanifold. Let $\iota : S \to M$ be the inclusion map. Define

$$\Omega^p(M, S) := \Omega^p(M) \oplus \Omega^{p-1}(S),$$

and let

$$\mathrm{d} : \Omega^p(M, S) \to \Omega^{p+1}(M, S), \quad \mathrm{d}(\omega, \beta) = (\mathrm{d}\omega, \iota^*\omega - \mathrm{d}\beta).$$

a) Show that $\Omega(M, S) = \bigoplus_p \Omega^p(M, S)$ is a differential complex.

b) Show that the sequence,

$$0 \to \Omega^{p-1}(S) \xrightarrow{j} \Omega^p(M, S) \xrightarrow{k} \Omega^p(M) \to 0$$

where $j(\beta) = (-1)^p\,(0, \beta)$ and $k(\omega, \beta) = \omega$, defines an exact sequence of differential complexes if we redefine the grading for the first term as $E^p = \Omega^{p-1}(S)$. Write down the corresponding long exact sequence in cohomology and give an explicit description of the connecting homomorphism on the level of cocycles.

c) Let $M = S^2$, and $S \cong S^1$ the "equator". Find all cohomology groups $H^p(M, S)$.

**Problem #3:** [*10 points*] Let $M$ be a manifold, and

$$R = \big\{[\alpha] \in H(M) \mid\, [\alpha \wedge \omega] = 0 \text{ for all } [\omega] \in H(M)\big\}$$

Let $[\alpha], [\beta], [\gamma] \in R$ be of degree $p, q, r$, and write $\alpha \wedge \beta = \mathrm{d}\phi, \quad \beta \wedge \gamma = \mathrm{d}\psi$. Show that

$$\alpha \wedge \psi - (-1)^p \phi \wedge \gamma$$

is closed, and that its cohomology class depends only on the classes of $\alpha, \beta, \gamma$.

**Problem #4:** [*15 points*] Let $M, N$ be compact, connected, oriented manifolds of the same dimension. Their *connected sum* $M\#N$ is defined by removing small open balls around $p \in M$, $q \in N$, and gluing the resulting manifolds $M', N'$ with boundary $\partial M' \cong \partial N' \cong S^{m-1}$ by an orientation-reversing diffeomorphism of the boundaries.

a) By working with coordinate charts around $p, q$, define the structure of a manifold on $M\#N$. (It can be shown that $M\#N$ is independent, up to diffeomorphism, from the choices involved in its construction.)

b) Calculate the cohomology of $M\#N$ in terms of those of $M, N$.

**Problem #5:** [*10 points*] Let $M, N$ be compact, oriented manifolds of dimensions $m < n$, and let $F \in C^\infty(N, M)$. For each $p$, let $K^p \subset H^p(M)$ be the kernel of the map $F^* : H^p(M) \to H^p(N)$.

a) Suppose $[\alpha] \in K^{p+1}$, $[\beta] \in K^{n-p}$ are represented by $\alpha \in Z^{p+1}(M)$, $\beta \in Z^{n-p}(M)$. Write $F^*\alpha = \mathrm{d}\omega$ where $\omega \in \Omega^p(N)$. Show that

$$\int_N \omega \wedge F^*\beta$$

depends only on the cohomology classes of $\alpha, \beta$. Thus we obtain a bilinear map

$$K^{p+1} \times K^{n-p} \to \mathbb{R}, \ ([\alpha], [\beta]) \mapsto \langle [\alpha], [\beta] \rangle_F := \int_N \omega \wedge F^*\beta$$

b) Show that

$$\langle [\alpha], [\beta] \rangle_F = (-1)^{n(p+1)} \langle [\beta], [\alpha] \rangle_F.$$

c) Show that $\langle [\alpha], [\beta] \rangle_F$ is unchanged under smooth homotopies of $F$.

**Problem #6:** [*5 points*] Let $M$ be a compact, oriented $m$-dimensional manifold with boundary $\partial M$, and $N$ a compact connected oriented manifold (without boundary) of dimension $n = m - 1$. Let $\phi : \partial M \to N$ be a smooth map. Show that if $\phi$ extends to a smooth map $\psi : M \to N$ then the mapping degree $\deg(\phi) = 0$. (If $\partial M$ is disconnected, define the mapping degree of $\phi$ to be the sum of the mapping degrees of its connected components.)