Quantization of group-valued moment maps III

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June 4, 2011

Pre-quantization of q-Hamiltonian spaces

Recall again the axioms of q-Hamiltonian G-spaces, $\Phi\colon M \to G$:

1. $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R)\cdot\xi$,
2. $\mathrm{d}\omega = -\Phi^*\eta$,
3. $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$.

Here $\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G)$ is a closed 3-form on G.

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Cone construction

Definition

Let $F^\bullet \colon S^\bullet \to R^\bullet$ be a cochain map between cochain complexes. The algebraic mapping cone is the cochain complex

$$\mathrm{cone}^k(F) = R^{k-1} \oplus S^k, \quad \mathrm{d}(x, y) = (F(y) - \mathrm{d}x, \mathrm{d}y).$$

Its cohomology is denoted $H^\bullet(F)$.

For a q-Hamiltonian G-space, we have $\mathrm{d}\omega = -\Phi^*\eta, \; \mathrm{d}\eta = 0$. Thus:

The pair $(\omega, -\eta) \in \Omega^3(\Phi) := \mathrm{cone}^3(\Phi^*)$ is a cocycle.

Pre-quantization

Suppose G simple, simply connected, $\cdot$ the basic inner product.

Definition

Let (M, ω, Φ) be a q-Hamiltonian G-space, $\Phi\colon M \to G$. A level k pre-quantization of (M, ω, Φ) is an integral lift of

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There is an equivariant version of this condition, but for G simply connected equivariance is automatic.

Geometric interpretation involves 'gerbes'.

Pre-quantization: Examples

Proposition

(M, ω, Φ) is pre-quantizable at level k if and only if for all $\Sigma \in Z_2(M)$, and any $X \in C_3(G)$ with $\Phi(\Sigma) = \partial X$,

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Example

The q-Hamiltonian SU(n)-space $M = S^{2n}$ is pre-quantized for all $k \in \mathbb{N}$, since $H_2(M) = 0$.

Pre-quantization of conjugacy classes

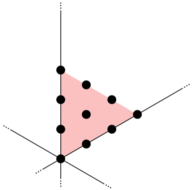
Recall:

- $G/\,\mathrm{Ad}(G) \cong A$ (the alcove), taking $\xi \in A$ to $G.\exp\xi$.
- $P_k = P \cap kA$.

Example

The level k pre-quantized conjugacy classes are those indexed by

$$\xi \in \frac{1}{k} P_k \subset A.$$



$$\begin{aligned} G &= \mathrm{SU}(3) \\ k &= 3 \end{aligned}$$

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Example

- $G = \mathrm{Spin}(5)$ has a conjugacy class $\mathcal{C} \cong S^4$ (does not admit almost complex structure).
- $G = \mathrm{Spin}(2k+1)$, $k > 2$ has a conjugacy class not admitting a Spin_c-structure.

Quantization of q-Hamiltonian G-space

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Theorem (M)

Let (M, ω, Φ) be a level k pre-quantized q-Hamiltonian G-space. Then there is a distinguished $R(G)$-module homomorphism

$$\Phi_* \colon K_0^G(M) \to R_k(G).$$

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This push-forward does not involve a Dirac operator. (There's not enough time here to explain how it is defined – sorry.)

Quantization of q-Hamiltonian G-spaces

Definition

The quantization of a level k pre-quantized q-Hamiltonian G-space (M, ω, Φ) is the element

$$\mathcal{Q}(M) = \Phi_*(1) \in R_k(G).$$

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Properties of the quantization:

- $\mathcal{Q}(M_1 \cup M_2) = \mathcal{Q}(M_1) + \mathcal{Q}(M_2)$,
- $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$,
- $\mathcal{Q}(M^*) = \mathcal{Q}(M)^*$,
- Let $\mathcal{C}$ be the conjugacy class of $\exp(\frac{1}{k}\mu)$, $\mu \in P_k$. Then

$$\mathcal{Q}(\mathcal{C}) = \tau_\mu.$$

Recall the trace $R_k(G) \to \mathbb{Z},\ \tau \mapsto \tau^G$ where $\tau^G_\mu = \delta_{\mu,0}$.

Theorem (Quantization commutes with reduction)

Let (M, ω, Φ) be a level k prequantized q-Hamiltonian G-space. Then

$$\mathcal{Q}(M)^G = \mathcal{Q}(M/\!\!/G).$$

Example

Let $\mathcal{C}_i$ be the conjugacy classes of $\exp(\frac{1}{k}\mu_i)$, $\mu_i \in P_k$. Then

$$\mathcal{Q}(\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 /\!\!/ G) = (\tau_{\mu_1}\tau_{\mu_2}\tau_{\mu_3})^G = N^{(k)}_{\mu_1\mu_2\mu_3}.$$

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Hamiltonian analogue:

Example

Let $\mathcal{O}_i$ be the coadjoint orbits of $\mu_i \in P_+$. Then

$$\mathcal{Q}(\mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 /\!\!/ G) = (\chi_{\mu_1}\chi_{\mu_2}\chi_{\mu_3})^G = N_{\mu_1\mu_2\mu_3}.$$

Examples

Example

The double $D(G) = G \times G$, $\Phi(a, b) = aba^{-1}b^{-1}$ has level k quantization

$$\mathcal{Q}(D(G)) = \sum_{\mu \in P_k} \tau_\mu \tau_\mu^*.$$

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The Hamiltonian analogue is the non-compact Hamiltonian G-space T^*G. Any reasonable quantization scheme for non-compact spaces gives

$$\mathcal{Q}(T^*G) = \sum_{\mu \in P_+} \chi_\mu \chi_\mu^*$$

(character for conjugation action on $L^2(G)$, defined in a completion of $R(G)$).

Can re-write this in terms of the basis $\tilde{\tau}_\mu$, where $\tilde{\tau}_\mu(t_\lambda) = \delta_{\lambda,\mu}$:

$$\mathcal{Q}\big(G.\exp(\frac{1}{k}\mu)\big) = \tau_\mu = \sum_{\nu \in P_k} \frac{S^*_{\mu,\nu}}{S_{0,\nu}} \tilde{\tau}_\nu .$$

$$\mathcal{Q}(D(G)) = \sum_{\nu \in P_k} \frac{1}{S^2_{0,\nu}} \tilde{\tau}_\nu$$

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Using $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$ this gives ...

Examples

Let $\mu_1,\ldots,\mu_r \in P_k$, and $\mathcal{C}_j = G.\exp(\frac{1}{k}\mu_j)$. Then

$$\mathcal{Q}\Big(D(G)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r\Big) = \sum_{\nu \in P_k} \frac{S^*_{\mu_1,\nu} \cdots S^*_{\mu_r,\nu}}{S_{0,\nu}^{2g+r}} \tilde{\tau}_\nu$$

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Using $\mathcal{Q}(M /\!\!/ G) = \mathcal{Q}(M)^G$ and $\tilde{\tau}_\nu^G = S_{0,\nu}^2$ this gives...

Verlinde formulas

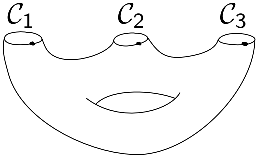
Theorem (Symplectic Verlinde formulas)

Let $\mu_1, \ldots, \mu_r \in P_k$*, and* $\mathcal{C}_j = G.\exp(\frac{1}{k}\mu_j)$*. The level* k *quantization of the moduli space*

$$\mathcal{M}(\Sigma_g^r, \mathcal{C}_1, \ldots, \mathcal{C}_r) = (D(G)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r)/\!\!/ G$$

is given by

$$\mathcal{Q}\Big(\mathcal{M}(\Sigma_g^r, \mathcal{C}_1, \ldots, \mathcal{C}_r)\Big) = \sum_{\nu \in P_k} \frac{S_{\mu_1,\nu} \cdots S_{\mu_r,\nu}}{S_{0,\nu}^{2g+r-2}}$$



Let (M, ω, Φ) be a level k pre-quantized q-Hamiltonian G-space.

Fact: For $F \subset M^{t_\lambda}$, the bundle $TM|_F$ acquires a t_λ-equivariant Spin_c-structure.

Theorem

Let (M, ω, Φ) be a level k pre-quantized q-Hamiltonian G-space. For $\lambda \in P_k$,

$$\mathcal{Q}(M)(t_\lambda) = \sum_{F \subset M^{t_\lambda}} \int_F \frac{\widehat{A}(F)\,\mathrm{Ch}(\mathcal{L}_F, t_\lambda)^{1/2}}{D_{\mathbb{R}}(\nu_F, t_\lambda)}$$

where $\mathcal{L}_F$ is the Spin_c*-line bundle for $TM|_F$.*

Remark

- *In Alekseev-M-Woodward (2000), $\mathcal{Q}(M)$ was essentially defined in terms of the localization formula, but phrased in terms of loop group actions.*
- *The $[\mathcal{Q},\mathcal{R}]=0$ theorem was proved in those terms.*
- *The more satisfactory definition of $\mathcal{Q}(M)$ as a K-homology push-forward was develped more recently (M (2010)).*