Quantization of group-valued moment maps II

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Representation ring

For a finite-dimensional complex G-representation $\pi\colon G \to \mathrm{Aut}(V)$, let $\chi_V \in C^\infty(G)$ be its character

$$\chi_V(g) = \mathrm{tr}(\pi(g)).$$

Properties

- $\chi_{V\oplus W} = \chi_V + \chi_W$,
- $\chi_{V^*} = \chi_V^*$,
- $\chi_{V\otimes W} = \chi_V \chi_W$.

Definition

The representation ring $R(G) \subset C^\infty(G)$ is the subring generated by characters χ_V.

Additively, $R(G)$ has basis the irreducible characters.

Representation ring

Let $T \subset G$ be a maximal torus, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$.

Definition

The *weight lattice* $P \subset \mathfrak{t}^*$ consists of $\mu \in \mathfrak{t}^*$ such that $2\pi\sqrt{-1}\mu \colon \mathfrak{t} \to 2\pi\sqrt{-1}\mathbb{R}$ exponentiates to $\mathrm{e}_\mu \colon T \to \mathrm{U}(1)$.

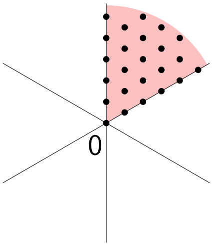
For any G-representation $\pi \colon G \to \mathrm{Aut}(V)$, one can consider its weights

$$P(V) = \{\mu \in P \mid \exists v \in V \backslash \{0\},\ \pi(t)v = \mathrm{e}_\mu(t)v\}.$$

Representation ring

Definition

$P_+ = \mathfrak{t}_+^* \cap P$ are the *dominant weights* of G.



Weyl's theorem: P_+ labels irreducible G-representations, by taking V to its unique highest weight $\mu \in P(V)$. Thus

$$R(G) = \mathbb{Z}[P_+]$$

additively.

Quantization of Hamiltonian G-spaces

Let (M, ω, Φ) be a Hamiltonian G-space, $\Phi\colon M \to \mathfrak{g}^*$.

Definition

A pre-quantum line bundle $L \to M$ is a G-equivariant Hermitian line bundle with connection ∇, such that

1. $\mathrm{curv}(\nabla) = \omega$,
2. The $\mathfrak{g}$-action on L is given by Kostant's formula

$$\xi_L = \mathrm{Lift}_\nabla(\xi_M) + \langle \Phi, \xi \rangle \partial_\theta$$

where $\partial_\theta \in \mathfrak{X}(L)$ generates the S^1-action on L.

If G is simply connected, the existence of the lift is automatic.

Choose a G-invariant compatible almost complex structure $J\colon TM \to TM$, i.e. $g(v, w) = \omega(Jv, w)$ is a Riemannian metric. Then

$$\mathsf{S} = \wedge T^{0,1}M \otimes L$$

is a spinor module; let $\partial\!\!\!/$ be its Spin_c-Dirac operator. Write

$$\partial\!\!\!/^{\pm}\colon \Gamma(\mathsf{S}^{\pm}) \to \Gamma(\mathsf{S}^{\mp}).$$

Definition

The quantization $\mathcal{Q}(M) \in R(G)$ of the pre-quantized Hamiltonian G-space (M, ω, Φ) is the G-index

$$\mathcal{Q}(M) = \mathrm{index}_G(\partial\!\!\!/) = \chi_{\ker(\partial\!\!\!/^{+})} - \chi_{\ker(\partial\!\!\!/^{-})}.$$

Quantization of Hamiltonian G-spaces

$\mathcal{Q}(M) \in R(G)$ is independent of the choices made.

Basic Properties:

- $\mathcal{Q}(M_1 \cup M_2) = \mathcal{Q}(M_1) + \mathcal{Q}(M_2)$,
- $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$,
- $\mathcal{Q}(M^*) = \mathcal{Q}(M)^*$,
- Borel-Weil-Bott (weak version): $G.\mu,\ \mu \in \mathfrak{t}^*_+$ is pre-quantized if and only if $\mu \in P_+$. In this case,

$$\mathcal{Q}(G.\mu) = \chi_\mu.$$

Quantization of Hamiltonian G-spaces

Let $R(G) \to \mathbb{Z}$, $\chi \mapsto \chi^G$ be the map defined by $\chi^G_\mu = \delta_{\mu,0}$.

Theorem (Quantization commutes with reduction)

$$\mathcal{Q}(M)^G = \mathcal{Q}(M/\!\!/G).$$

This was conjectured (and proved in many cases) by Guillemin-Sternberg.

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One has to take care of the singularities of $M/\!\!/G$ (M-Sjamaar).

On the other hand, $\mathcal{Q}(M) = \mathrm{index}_G(\partial\!\!\!/)$ is given by

Theorem (Atiyah-Segal-Singer)

$$\mathcal{Q}(M)(g) = \sum_{F \subset M^g} \int_F \frac{\widehat{A}(F)\,\mathrm{Ch}(\mathcal{L}|_F, g)^{1/2}}{D_{\mathbb{R}}(\nu_F, g)}$$

a sum over fixed point manifolds $F \subset M^g$.

Here $\mathcal{L}$ is the 'Spin$_c$-line bundle' $\mathcal{L} = L^2 \otimes K^{-1}$, and ν_F is the normal bundle to F.

Quantization of Hamiltonian G-spaces

Remark

One can also write this in 'Riemann-Roch form',

$$\mathcal{Q}(M) = \sum_{F \subset M^g} \int_F \frac{\mathrm{Td}(F)\,\mathrm{Ch}(L|_F, g)}{D_{\mathbb{C}}(\nu_F, g)}$$

But the 'Spin$_c$-form' will be more convenient for our discussion.

Quantization of q-Hamiltonian G-spaces ?

Recall axioms of q-Hamiltonian G-spaces, $\Phi \colon M \to G$:

1. $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi$,
2. $\mathrm{d}\omega = -\Phi^*\eta$,
3. $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$.

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Questions / Problems

- Where should $\mathcal{Q}(M)$ take values in ??
- ω is not closed, hence 'pre-quantum line bundle' does not make sense.
- ω could be degenerate, so 'compatible almost complex structure' does not make sense.
- No suitable 'Dirac operator' in sight.

Answer to first question:

For q-Hamiltonian spaces, $\mathcal{Q}(M)$ should take values in the fusion ring (Verlinde algebra).

The level k fusion ring (Verlinde algebra)

Assume G compact, simple and simply connected, $P_+ \subset P \subset \mathfrak{t}_+^*$ its dominant weights.

Notation

- $\theta \in P_+$ adjoint representation (i.e. $\chi_\theta = \mathrm{tr}(\mathrm{Ad}_g)$),
- $\rho \in P_+$ shortest weight in $P \cap \mathrm{int}(\mathfrak{t}_+^*)$.
- The basic inner product $\cdot$ on $\mathfrak{g} \cong \mathfrak{g}^*$ is the unique invariant inner product with $\theta \cdot \theta = 2$.
- The dual Coxeter number is defined by

$$\mathsf{h}^\vee = 1 + \rho \cdot \theta \in \mathbb{N}.$$

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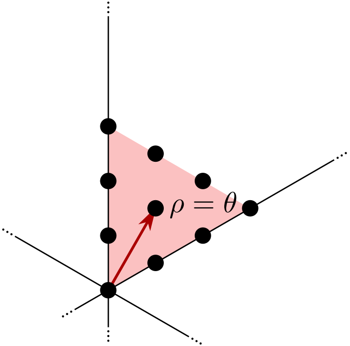
For $G = \mathrm{SU}(n)$, one has $\mathrm{h}^\vee = n$.

The level k fusion ring (Verlinde algebra)

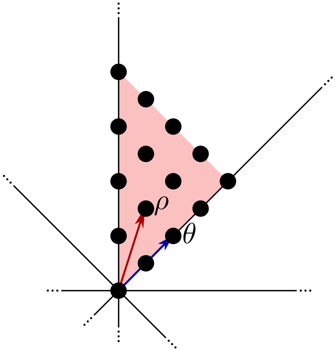
$A = \{\xi \in \mathfrak{t}_+ |\ \theta \cdot \xi \leq 1\}$ is the fundamental alcove.

Definition

The level k weights are elements of $P_k = P \cap kA$.



$$\begin{aligned} G &= \mathrm{SU}(3) \\ k &= 3 \end{aligned}$$



$$\begin{aligned} G &= \mathrm{Spin}(5) \\ k &= 4 \end{aligned}$$

For $\lambda \in P_k$ define the special element

$$t_\lambda = \exp(\frac{\lambda+\rho}{k+\mathsf{h}^\vee}) \in T.$$

Definition

The level k fusion ring (Verlinde algebra) is the quotient

$$R_k(G) = R(G)/I_k(G)$$

where

$$I_k(G) = \{\chi \in R(G) |\ \chi(t_\lambda) = 0 \ \ \forall\ \lambda \in P_k\}.$$

The level k fusion ring (Verlinde algebra)

Remark

$R_k(G)$ is the fusion ring of level k projective representations of the loop group LG. (But we don't need that here.)

The level k fusion ring (Verlinde algebra)

Some properties of $R_k(G) = R(G)/I_k(G)$:

- $R_k(G)$ is unital ring with involution.
- $R_k(G)$ has finite $\mathbb{Z}$-basis the images τ_μ of $\chi_\mu, \mu \in P_k$. Thus
$$R_k(G) = \mathbb{Z}[P_k].$$
- $R_k(G)$ has a trace,
$$R_k(G) \to \mathbb{Z},\ \tau \mapsto \tau^G$$
where $\tau_\mu^G = \delta_{\mu,0}$.

The level k fusion ring (Verlinde algebra)

Notation

- Tensor coefficents

$$N_{\mu_1\mu_2\mu_3} = (\chi_{\mu_1}\chi_{\mu_2}\chi_{\mu_3})^G, \quad \mu_i \in P_+$$

- Level k fusion coefficents

$$N^{(k)}_{\mu_1\mu_2\mu_3} = (\tau_{\mu_1}\tau_{\mu_2}\tau_{\mu_3})^G, \quad \mu_i \in P_k.$$

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Then

$$N^{(k)}_{\mu_1\mu_2\mu_3} = N_{\mu_1\mu_2\mu_3}, \quad k >> 0.$$

Example: SU(2)

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Example

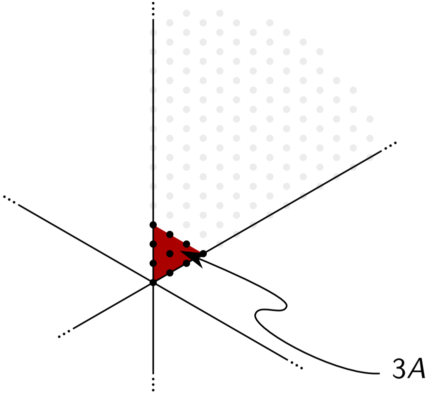
Calculation of $\tau_3 \tau_4$ in $R_5(\mathrm{SU}(2))$:

$$\chi_3 \chi_4 = \chi_7 + \chi_5 + \chi_3 + \chi_1 \Rightarrow \tau_3 \tau_4 = \tau_3 + \tau_1$$

since $\chi_7 \mapsto -\tau_5,\ \chi_5 \mapsto \tau_5$.

The level k fusion ring (Verlinde algebra)

For general G the quotient map is 'signed reflection' for a shifted Stiefel diagram.



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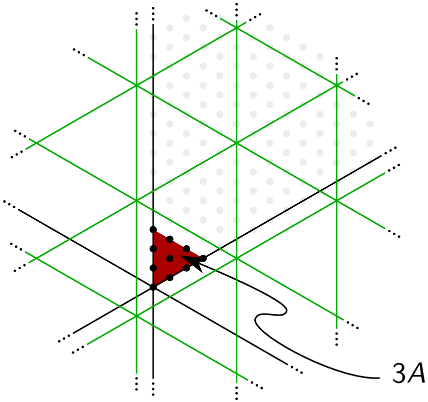

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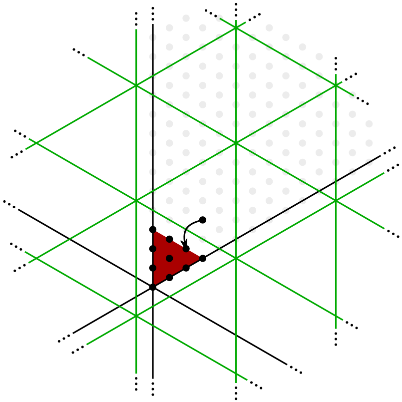

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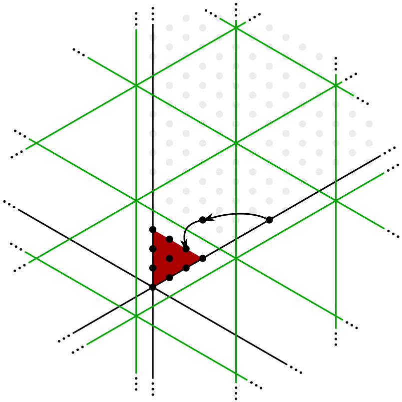

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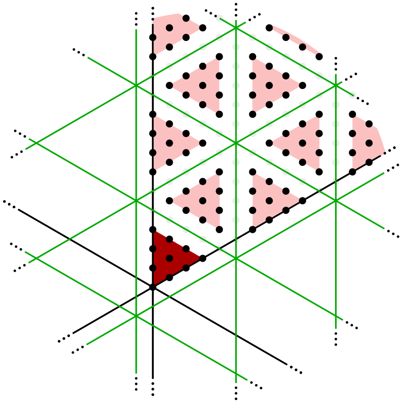

Shifted affine Weyl action at level $k = 3$, $G = \mathrm{SU}(3)$

S-matrix

Evaluation of characters at $t_\lambda = \exp(\frac{\lambda+\rho}{k+\mathsf{h}^\vee})$ descends to the fusion ring:

$$R_k(G) \to \mathbb{C}, \quad \tau \mapsto \tau(t_\lambda).$$

$R_k(G) \otimes \mathbb{C}$ has another basis $\tilde{\tau}_\mu$ s.t. $\tilde{\tau}_\mu(t_\lambda) = \delta_{\lambda,\mu}$. In the new basis,

$$\tilde{\tau}_\mu \tilde{\tau}_\nu = \delta_{\mu,\nu} \tilde{\tau}_\nu.$$

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The bases are related by the S-matrix:

$$\tau_\mu = \sum_{\nu \in P_k} S_{0,\nu}^{-1} \, S_{\mu,\nu}^* \tilde{\tau}_\nu;$$

here S is a symmetric, unitary matrix with $S_{0,\nu} > 0$.

S-matrix

$\Rightarrow$ Verlinde formula for fusion coefficients:

$$N^{(k)}_{\mu_1\mu_2\mu_3} = \sum_{\nu\in P_k} \frac{S_{\mu_1,\nu}S_{\mu_2,\nu}S_{\mu_3,\nu}}{S_{0,\nu}}.$$

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Tomorrow, we'll take $R_k(G)$ as the target for quantization of q-Hamiltonian spaces.