Quantization of group-valued moment maps I

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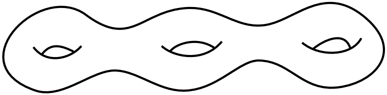
June 2, 2011

Motivation: Moduli spaces of flat connections

- G a compact simply connected Lie group,
- $\cdot$ invariant inner product on $\mathfrak{g} = \mathrm{Lie}(G)$.

$\Sigma =$ 

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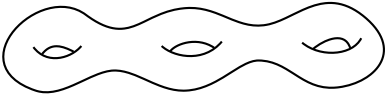
$\Sigma =$ 

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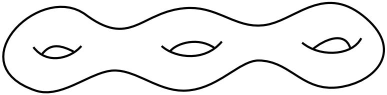
$\Sigma =$ 

$$M(\Sigma) = \frac{\{A \in \Omega^1(\Sigma, \mathfrak{g}) |\ \mathrm{d}A + \frac{1}{2}[A, A] = 0\}}{\text{gauge transformations}}$$

This is a symplectic manifold ! (with singularities).

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- Moduli space is symplectic quotient

$$M(\Sigma) = \mathrm{curv}^{-1}(0)/C^\infty(\Sigma, G).$$

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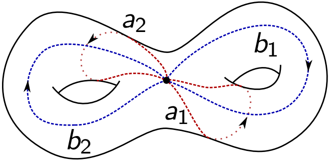
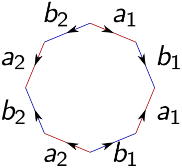
Holonomy description of the moduli space

$$M(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), G)/G = \Phi^{-1}(e)/G$$

where $\Phi\colon G^{2g} \to G$ (with g the genus of Σ) is the map

$$\Phi(a_1, b_1, \ldots, a_g, b_g) = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}.$$





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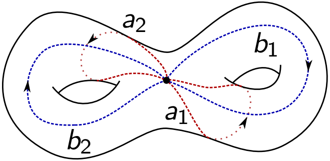

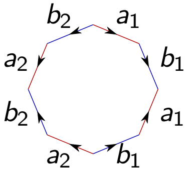

We'd like to view Φ as a moment map, and $\Phi^{-1}(e)/G$ as a symplectic quotient!

Group-valued moment maps

- $\theta^L = g^{-1}\mathrm{d}g \in \Omega^1(G, \mathfrak{g})$ left-Maurer-Cartan form
- $\theta^R = \mathrm{d}g g^{-1} \in \Omega^1(G, \mathfrak{g})$ right Maurer-Cartan form
- $\eta = \frac{1}{12}[\theta^L, \theta^L] \cdot \theta^L \in \Omega^3(G)$ Cartan 3-form

Definition (Alekseev-Malkin-M.)

A q-Hamiltonian G-space (M, ω, Φ) is a G-manifold M, with $\omega \in \Omega^2(M)^G$ and $\Phi \in C^\infty(M, G)^G$, satisfying

1. $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi$,
2. $\mathrm{d}\omega = -\Phi^*\eta$,
3. $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$.

Comparison

Hamiltonian G-space $\Phi\colon M \to \mathfrak{g}^*$

1. $\iota(\xi_M)\omega = -\mathrm{d}\langle \Phi, \xi\rangle$,
2. $\mathrm{d}\omega = 0$,
3. $\ker(\omega) = 0$.

q-Hamiltonian G-space $\Phi\colon M \to G$

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Examples: Coadjoint orbits, conjugacy classes

Example

Co-adjoint orbits $\Phi\colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$ are Hamiltonian G-spaces

$$\omega(\xi_{\mathcal{O}}, \xi'_{\mathcal{O}})_\mu = \langle \mu, [\xi, \xi'] \rangle$$

Example

Conjugacy classes $\Phi\colon \mathcal{C} \hookrightarrow G$ are q-Hamiltonian G-spaces

$$\omega(\xi_{\mathcal{C}}, \xi'_{\mathcal{C}})_a = \frac{1}{2}(\mathrm{Ad}_a - \mathrm{Ad}_{a^{-1}})\xi \cdot \xi'$$

Examples; Cotangent bundle, double

Example

Cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ (with cotangent lift of conjugation action) is Hamiltonian G-space with

$$\Phi(g,\mu) = \mathrm{Ad}_g(\mu) - \mu$$

Example

The double $D(G) = G \times G$ is a q-Hamiltonian G-space with

$$\Phi(a,b) = aba^{-1}b^{-1}$$

Examples: Planes and spheres

Example

Even-dimensional plane $\mathbb{C}^n = \mathbb{R}^{2n}$ is Hamiltonian $\mathrm{U}(n)$-space.

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Even-dimensional sphere S^{2n} is a q-Hamiltonian $\mathrm{U}(n)$-space (Hurtubise-Jeffrey-Sjamaar).

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Similar examples with $G = \mathrm{Sp}(n)$, and $M = \mathbb{H}P(n)$ resp. $\mathbb{H}^n$ (Eshmatov).

Basic constructions: Products

Products: If $(M_1, \omega_1, \Phi_1), (M_2, \omega_2, \Phi_2)$ are q-Hamiltonian G-spaces then so is

$$(M_1 \times M_2,\ \omega_1 + \omega_2 + \tfrac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R,\ \Phi_1\Phi_2).$$

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Example

For instance, $D(G)^g = G^{2g}$ is a q-Hamiltonian G-space with moment map

$$\Phi(a_1, b_1, \ldots, a_g, b_g) = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}.$$

Basic constructions: Reduction

Reduction: If (M, ω, Φ) is a q-Hamiltonian G-space then the symplectic quotient

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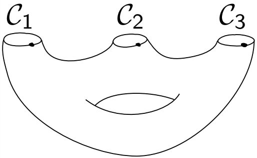


Example (and Theorem)

The symplectic quotient

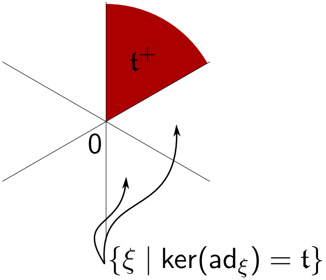
$$G^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r/\!\!/G = \mathcal{M}(\Sigma_g^r; \mathcal{C}_1, \ldots, \mathcal{C}_r)$$

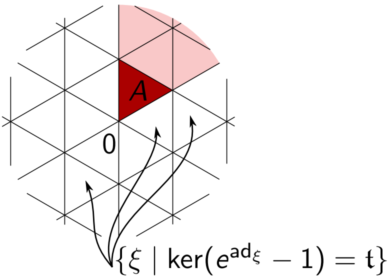
is the moduli space of flat G-bundles over a surface with boundary, with boundary holonomies in prescribed conjugacy classes.

Notation: Weyl chambers and Weyl alcoves

Notation

- G compact and simply connected (e.g. $G = \mathrm{SU}(n)$),
- T a maximal torus in G, $\mathfrak{t} = \mathrm{Lie}(T)$,
- $\mathfrak{t}_+ \cong \mathfrak{t}$ fundamental Weyl chamber,
- $A \subset \mathfrak{t}_+ \subset \mathfrak{t}$ fundamental Weyl alcove





Moment polytope

For every $\nu \in \mathfrak{g}^*$ there is a unique $\mu \in \mathfrak{t}^*_+$ with $\nu \in G.\mu$.

Theorem (Atiyah, Guillemin-Sternberg, Kirwan)

For a compact connected Hamiltonan G-space (M, ω, Φ), the set

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For every $g \in G$ there is a unique $\xi \in A$ with $g \in G.\exp(\xi)$.

Theorem (M-Woodward)

For any connected q-Hamiltonian G-space (M, ω, Φ), the set

$$\Delta(M) = \{\xi \in A|\ \exp(\xi) \in \Phi(M)\}$$

is a convex polytope.

Application to eigenvalue problems

The Hamiltonian convexity theorem gives eigenvalue inequalities for sums of Hermitian matrices with prescribed eigenvalues. (Schur-Horn problem).

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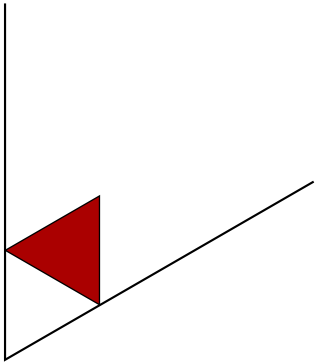
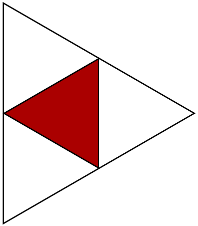
The q-Hamiltonian convexity theorem gives eigenvalue inequalities for products of unitary matrices with prescribed eigenvalues.

Examples of moment polytopes (due to C. Woodward)



A multiplicity-free Hamiltonian SU(3)-space



A multiplicity-free q-Hamiltonian SU(3)-space

Volume forms

Let

$$\Upsilon = \det{}^{1/2}\Big(\frac{1+\mathrm{Ad}_g}{2}\Big)\exp\Big(\frac{1}{4}\frac{\mathrm{Ad}_g-1}{\mathrm{Ad}_g+1}\theta^L\cdot\theta^L\Big).$$

It turns out that $\Upsilon\in\Omega(G)$ is a well-defined smooth differential form.

Theorem (Alekseev-M-Woodward)

For any q-Hamiltonian G-space (M,ω,Φ), the form

$$(\Phi^*\Upsilon\exp\omega)_{[top]}\in\Omega(M)$$

is non-vanishing, i.e. a volume form.

This is the analogue of the Liouville form $(\exp\omega)_{[top]}\in\Omega(M)$

Further parallels between Hamiltonian / q-Hamiltonian theories:

1. Liouville volumes are computable by localization
2. Duistermaat-Heckman theory
3. Intersection pairings on symplectic quotients via localization
4. Connectivity of fibers of the moment map
5. Cross-section theorems
6. Kirwan surjectivity theorems (Bott-Tolman-Weitsman)
7. ...

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In these lectures, we will focus on the *quantization* of q-Hamiltonian G-spaces.