

IGA Lecture III: Twisted Spin_c structures

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Review: Spin_c -structures

- (V, B) a finite-dimensional Euclidean vector space,
- $\mathbb{C}l(V)$ complex Clifford algebra: generators $v \in V$, relations

$$vv' + v'v = 2B(v, v').$$

Then $\mathbb{C}l(V)$ is a finite-dimensional C^* -algebra.

Similarly, for a finite rank Euclidean vector bundle $V \rightarrow X$ with fiber metric B define a complex Clifford bundle $\mathbb{C}l(V) \rightarrow X$.

Let $V \rightarrow X$ be a Euclidean vector bundle, $\text{rank}(V)$ even.

Definition

A Spin_c -structure on V is a \mathbb{Z}_2 -graded Hermitian vector bundle $S \rightarrow X$ with a $*$ -isomorphism

$$\varrho: \mathbb{C}l(V) \rightarrow \text{End}(S).$$

S is called the spinor module.

Remarks

- The definition is equivalent to an orientation on V together with a lift of the structure group from $SO(n)$ to $\text{Spin}_c(n)$. (Connes, Plymen.)
- If V has odd rank, one defines a Spin_c -structure on V to be a Spin_c -structure on $V \oplus \mathbb{R}$.

Let $V \rightarrow X$ be a Euclidean vector bundle.

Example

Suppose $J \in \Gamma(O(V))$ is a complex structure, $J^2 = -\text{id}_V$. Get $V^{\mathbb{C}} = V^+ \oplus V^-$. Then

$$S = \wedge(V^+)$$

defines a Spin_c -structure on V , with $\varrho(v) = \sqrt{2}(\epsilon(v^+) + \iota(v^-))$ for $v \in V$.

Example

Suppose $\omega \in \Gamma(\wedge^2 V^*)$ is symplectic; let R_ω be the corresponding skew-adjoint endomorphism. Then

$$J_\omega = \frac{R_\omega}{|R_\omega|} \in \Gamma(O(V))$$

is a complex structure, defining a Spin_c -structure on V .

Basic properties

- Any two Spin_c-structure S, S' on V differ by a line bundle:

$$S' = S \otimes L \leftrightarrow L = \text{Hom}_{\mathbb{C}l}(S, S').$$

- Obstructions to existence of Spin_c-structure:

$$W_3(V) \in H^3(X, \mathbb{Z}), \quad w_1(V) \in H^1(X, \mathbb{Z}_2).$$

Example

The dual S^* of a spinor module is again a spinor module. Get a line bundle

$$K_S = \text{Hom}_{\mathbb{C}l}(S, S^*)$$

called the **canonical line bundle** for S . Note

$$K_{S \otimes L} = K_S \otimes L^{-2}.$$

If M is a manifold with a smooth Spin_c-structure S , one defines the **Spin_c-Dirac operator**

$$\not{D}: \Gamma(S) \xrightarrow{\nabla} \Gamma(TM \otimes S) \xrightarrow{e} \Gamma(S).$$

If $L \rightarrow M$ is a line bundle, denote by \not{D}_L the Spin_c-Dirac operator for $S \otimes L$.

Quantization of Hamiltonian G -spaces (in a nutshell)

Hamiltonian G -space $\Phi: M \rightarrow \mathfrak{g}^*$

- 1 $\iota(\xi_M)\omega = -d\langle\Phi, \xi\rangle,$
- 2 $d\omega = 0,$
- 3 $\ker(\omega) = 0.$

1. Pick G -invariant Riemannian metric on $M \Rightarrow \omega$ determines a Spin_c -structure.
2. Assume (M, ω, Φ) pre-quantizable; pick a pre-quantum line bundle $L \rightarrow M.$
3. Define

$$\mathcal{Q}(M) := \text{index}_G(\not{\partial}_L) \in R(G).$$

For **q-Hamiltonian spaces** already Step 1 fails, since ω may be degenerate.

Let G be a compact Lie group, and \cdot an invariant inner product on $\mathfrak{g} = \text{Lie}(G)$.

Definition

A **q-Hamiltonian G -space** (M, ω, Φ) is a G -manifold M , with $\omega \in \Omega^2(M)^G$ and $\Phi \in C^\infty(M, \mathfrak{g})^G$, satisfying

- 1 $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi$,
- 2 $d\omega = -\Phi^*\eta$,
- 3 $\ker(\omega) \cap \ker(d\Phi) = 0$.

For **q-Hamiltonian spaces** already Step 1 fails:

Problems:

- There is no notion of ‘compatible almost complex structure’
- In general, q-Hamiltonian G -spaces need not even admit Spin_c -structures.

Example

- $G = \text{Spin}(5)$ has a conjugacy class $\mathcal{C} \cong S^4$ (does not admit almost complex structure).
- $G = \text{Spin}(2k + 1)$, $k > 2$ has a conjugacy class not admitting a Spin_c -structure.

However, we will show that q -Hamiltonian spaces carry '*twisted*' Spin_c -structures.

The definition of the twisted Spin_c -structures involves *Dixmier-Douady bundles*

Notation:

- H separable complex Hilbert space, possibly $\dim H < \infty$,
- $\mathbb{B}(H)$ bounded linear operators,
- $\mathbb{K}(H)$ compact operators ($= \overline{\mathbb{B}_{\text{fin}}(H)}$)

Fact: $\text{Aut}(\mathbb{K}(H)) = \text{PU}(H)$ (strong topology).

Definition

A **DD-bundle** $\mathcal{A} \rightarrow X$ is a \mathbb{Z}_2 -graded bundle of $*$ -algebras modeled on $\mathbb{K}(H)$, (for H a \mathbb{Z}_2 -graded Hilbert space), with structure group the even part of $\text{PU}(H)$.

Theorem (Dixmier-Douady)

The obstruction to writing $\mathcal{A} = \mathbb{K}(\mathcal{E})$, with \mathcal{E} a \mathbb{Z}_2 -graded bundle of Hilbert spaces, is a class

$$\text{DD}(\mathcal{A}) \in H^3(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}_2).$$



Hence, the trivially graded DD bundles give a 'realization' of $H^3(X, \mathbb{Z})$, similar to line bundles 'realizing' $H^2(X, \mathbb{Z})$.

Remark

*This framework is not convenient for a Chern-Weil theory. A more differential-geometric realization is given by the theory of **bundle gerbes**.*

Definition

Let $\mathcal{A}_1 \rightarrow X_1$, $\mathcal{A}_2 \rightarrow X_2$ be *DD*-bundles. A Morita morphism

$$(\Phi, \mathcal{E}): \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$$

is a map $\Phi: X_1 \rightarrow X_2$ together with a \mathbb{Z}_2 -graded bundle $\mathcal{E} \rightarrow X_1$ of bimodules

$$\Phi^* \mathcal{A}_2 \circlearrowleft \mathcal{E} \circlearrowright \mathcal{A}_1,$$

locally modeled on $\mathbb{K}(H_2) \circlearrowleft \mathbb{K}(H_1, H_2) \circlearrowright \mathbb{K}(H_1)$.

Remark

- $(\Phi, \mathcal{E}): \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ exists if and only if $\text{DD}(\mathcal{A}_1) = \Phi^* \text{DD}(\mathcal{A}_2)$.
- Any two Morita bimodules $\mathcal{E}, \mathcal{E}'$ differ by a line bundle:

$$\mathcal{E}' = \mathcal{E} \otimes L \leftrightarrow L = \text{Hom}_{\Phi^* \mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}, \mathcal{E}').$$

Example

$V \rightarrow X$ Euclidean vector bundle of even rank $\Rightarrow \mathbb{C}I(V)$ is a DD-bundle. A Morita trivialization

$$(p, S^{\text{op}}): \mathbb{C}I(V) \dashrightarrow \mathbb{C}$$

is a Spin_c -structure. The DD-class is given by

$$\text{DD}(S) = (W^3(V), w_1(V)) \in H^3(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}_2).$$

Review of linear Dirac structures

- A **Dirac structure** on vector space V is a Lagrangian subspace $E \subset \mathbb{V} = V \oplus V^*$.
- For $\Theta: V_1 \rightarrow V_2$ and $\omega \in \wedge^2 V_1^*$ write

$$v_1 + \mu_1 \sim_{(\Theta, \omega)} v_2 + \mu_2 \Leftrightarrow \begin{cases} v_2 = \Theta(v_1) \\ \mu_1 = \Theta^*(\mu_2) + \omega(v_1, \cdot) \end{cases}$$

- It defines a **Dirac morphism** $(\Theta, \omega): (\mathbb{V}_1, E_1) \dashrightarrow (\mathbb{V}_2, E_2)$ if every element of E_2 is related to a unique element of E_1 .
- The definitions extend to vector bundles $V \rightarrow X$.

Example

- Hamiltonian G -spaces are described as G -equivariant Dirac morphisms

$$(\Phi, \omega): (\mathbb{T}M, TM) \dashrightarrow (\mathcal{T}\mathfrak{g}^*, E_{\mathfrak{g}^*}).$$

- q-Hamiltonian G -spaces are described as G -equivariant Dirac morphisms

$$(\Phi, \omega): (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}G_{\eta}, E_G).$$

- There is a multiplication morphism

$$(\text{Mult}_G, \varsigma): (\mathbb{T}G_{\eta}, E_G) \times (\mathbb{T}G_{\eta}, E_G) \dashrightarrow (\mathbb{T}G_{\eta}, E_G).$$

The Dirac-Dixmier-Douady functor

Theorem (Alekseev-M, 2010)

There is a functor from Dirac structures on vector bundles $V \rightarrow X$ to DD-bundles:

$$E \mapsto \mathcal{A}_E.$$

Furthermore, there are canonical Morita isomorphisms

$$\mathbb{C}l(V) \dashrightarrow \mathcal{A}_V, \quad \mathbb{C} \dashrightarrow \mathcal{A}_{V^*}$$

N.B.: We identify two Morita morphisms $\mathcal{E}, \mathcal{E}' : \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ if they are related by a trivial line bundle.

Example

The Cartan Dirac structure $(\mathbb{T}G_\eta, E_G)$ defines a DD-bundle $\mathcal{A}^{\text{Spin}} := \mathcal{A}_{E_G} \rightarrow G$. The 'multiplication morphism' for the Cartan Dirac structure gives a morphism

$$\text{Mult}_* : \mathcal{A}^{\text{Spin}} \times \mathcal{A}^{\text{Spin}} \dashrightarrow \mathcal{A}^{\text{Spin}}.$$

Example

A q-Hamiltonian G -space (M, ω, Φ) defines a Dirac morphism

$$(d\Phi, \omega) : (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}G_\eta, E_G).$$

Hence we get a Morita morphism

$$\mathbb{C}l(TM) \dashrightarrow \mathcal{A}_{TM} \dashrightarrow \mathcal{A}_{E_G} = \mathcal{A}^{\text{Spin}},$$

a 'twisted Spin_c -structure'.

Outline

- 1 From $E \subset \mathbb{V}$, construct family of skew-adjoint operators D_x , $x \in X$ acting on real Hilbert spaces \mathcal{V}_x .
- 2 From $D = \{D_x\}$, construct family of 'polarizations' of \mathcal{V}_x .
- 3 From the polarization, construct DD -bundle $\mathcal{A} \rightarrow X$.

Inspired by and/or similar to:

Carey-Mickelsson-Murray 1997, Lott 2002, Atiyah-Segal 2004,
Freed-Hopkins-Teleman 2005, Bouwknegt-Mathai-Wu 2011.

Step 1: Constructing $\{D_x, x \in X\}$

Assume $X = \text{pt}$, so V is a vector space.

Choice of Euclidean metric B identifies

$$\text{Lag}(\mathbb{V}) \cong \text{O}(V).$$

Here $A \in \text{O}(V)$ corresponds to

$$E = \left\{ \left((A - I)v, \frac{1}{2}(A + I)v \right) \in \mathbb{V} = V \oplus V^* \mid v \in V \right\}.$$

Define skew-adjoint operator $D_E = \frac{\partial}{\partial t}$ on $\mathcal{V} = L^2([0, 1], V)$, with domain

$$\text{dom}(D_E) = \{f : f(1) = -Af(0)\}.$$

Step 1: Constructing $\{D_x, x \in X\}$

Example

$E = V^*$ corresponds to $A = I$, and $f(1) = -Af(0)$ are anti-periodic boundary conditions. Note $\ker(D_E) = 0$.

Example

$E = V$ corresponds to $A = -I$, and $f(1) = -Af(0)$ are periodic boundary conditions. Note $\ker(D_E) = V$.

Note that in general, $\ker(D_E) = \ker(A + I) = E \cap V$.

Thus, if $V \rightarrow X$ is a vector bundle, the choice of a Euclidean metric takes us from Dirac structures (\mathbb{V}, E) to skew-adjoint Fredholm families

$$D_E = \{(D_E)_x, x \in X\},$$

where $(D_E)_x$ is $\frac{\partial}{\partial t}$ on $\mathcal{V}_x = L^2([0, 1], V_x)$, with boundary conditions determined by E_x .

Step 2: Polarizations

Let \mathcal{V} be a real Hilbert space. Recall $A \in \mathbb{B}(\mathcal{V})$ is Hilbert-Schmidt if $\text{tr}(A^*A) < \infty$.

Definition

An **even polarization on \mathcal{V}** is an equivalence class of orthogonal complex structures $J \in \mathcal{O}(\mathcal{V})$, where

$$J \sim J' \Leftrightarrow J - J' \text{ is Hilbert-Schmidt.}$$

An **odd polarization on \mathcal{V}** is an even polarization on $\mathcal{V} \oplus \mathbb{R}$.

Fact: Every skew-adjoint Fredholm operator D on \mathcal{V} determines a polarization, of parity depending on the parity of $\dim \ker(D)$.

If $\dim \ker(D)$ even, choose $S = -S^* \in \mathbb{B}_{\text{fin}}(\mathcal{V})$ with $\ker(D + S) = 0$.

Lemma

The even polarization defined by $J = \frac{D+S}{|D+S|}$ does not depend on choice of S .

If $\dim \ker(D)$ odd, replace \mathcal{V} with $\mathcal{V} \oplus \mathbb{R}$, and obtain odd polarization.

Step 3: The Dixmier-Douady bundle

- \mathcal{V} a real Hilbert space.
- $\mathbb{C}l(\mathcal{V})$ its complex Clifford algebra.
- $S_J = \overline{\wedge \mathcal{V}_+}$ spinor module defined by $J \in O(\mathcal{V})$, $J^2 = -\text{id}_{\mathcal{V}}$ (Hilbert space completion).

Theorem (Shale-Stinespring, 1965)

For orthogonal complex structures J, J' on \mathcal{V} ,

$$\dim \text{Hom}_{\mathbb{C}l} (S_J, S_{J'}) = \begin{cases} 1 & \text{if } J \sim J' \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{K}(S_J) = \mathbb{K}(S_{J'})$ **canonically** if $J \sim J'$.

Step 3: The Dixmier-Douady bundle

From (\mathbb{V}, E) we constructed the family D_E of skew-adjoint Fredholm operators on $\mathcal{V} = \bigcup_{x \in X} \mathcal{V}_x$, $\mathcal{V}_x = L^2([0, 1], V)$, which in turn defines a polarization on \mathcal{V} .

Use fiberwise representatives J_x to define

$$\mathcal{A}_x = \mathbb{K}(S_{J_x}).$$

Then $\mathcal{A} = \bigcup_x \mathcal{A}_x$ is a well-defined *DD*-bundle.

Remark

- $\ker(D_E) \cong E \cap V$.
- Hence if $E = V^*$, then $\ker(D_E) = 0$, and $\mathcal{A} = \mathbb{K}(S_J)$ for $J = \frac{D_E}{|D_E|}$.
- If $E = V$, then $\ker(D_E) = V$, and $\mathcal{V} = V \oplus V^\perp$. This explains $\mathbb{C}l(V) \dashrightarrow \mathcal{A}$.

Example

For the Cartan-Dirac structure $\mathbb{T}G, E$, get family

$$D_g = \frac{\partial}{\partial t}, \quad \text{dom}(D_g) = \{f \in L^2([0, 1], \mathfrak{g}) \mid f(1) = -\text{Ad}_g f(0)\}.$$

Let $\mathcal{A}^{\text{Spin}} := \mathcal{A}_{E_G}$. If G is connected, then

$$\text{DD}(\mathcal{A}^{\text{Spin}}) \in H^3(G, \mathbb{Z}) \times H^1(G, \mathbb{Z}_2)$$

is the pull-back of the generators of $H^3(\text{SO}(\mathfrak{g}), \mathbb{Z}) = \mathbb{Z}$ resp. $H^1(\text{SO}(\mathfrak{g}), \mathbb{Z}_2) = \mathbb{Z}_2$ under $\text{Ad}: G \rightarrow \text{SO}(\mathfrak{g})$. (See Atiyah-Segal.)

In particular, if G simple and simply connected, then

$$DD(\mathcal{A}^{\text{Spin}}) = h^\vee x$$

where $x \in H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ is the generator, and h^\vee is the dual Coxeter number.

Corollary

Suppose (M, ω, Φ) is a q -Hamiltonian G -space. Then

$$W_3(M) = h^\vee \Phi^* x, \quad w_1(M) = 0.$$

This follows from existence of $\mathbb{C}I(TM) \dashrightarrow \mathcal{A}^{\text{Spin}}$.

In particular, this result applies to the conjugacy classes of G .