IGA Lecture II: Dirac Geometry

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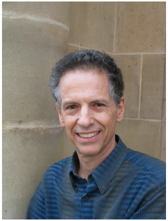
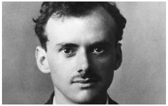
Dirac geometry

Dirac geometry was introduced by T. Courant and A. Weinstein as a common geometric framework for

- Poisson structures
  $\pi \in \Gamma(\wedge^2 TM),\ [\pi, \pi] = 0,$
- closed 2-forms
  $\omega \in \Gamma(\wedge^2 T^*M),\ \mathrm{d}\omega = 0.$



The name comes from relation with Dirac theory of constraints.



Linear Dirac geometry

- V vector space, $\mathbb{V} = V \oplus V^*$,
- $\langle v_1 + \alpha_1, v_2 + \alpha_2 \rangle = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle$.

Definition

$E \subset \mathbb{V}$ is Lagrangian if $E = E^\perp$. The pair $(\mathbb{V}, E)$ is called a linear Dirac structure.

Linear Dirac geometry

Examples of Lagrangian subspaces:

1. $\omega \in \wedge^2(V^*) \Rightarrow \mathrm{Gr}(\omega) = \{v + \iota_v\omega | v \in V\} \in \mathrm{Lag}(\mathbb{V})$.
2. $\pi \in \wedge^2(V) \Rightarrow \mathrm{Gr}(\pi) = \{\iota_\alpha\pi + \alpha | \alpha \in V^*\} \in \mathrm{Lag}(\mathbb{V})$.
3. $S \subseteq V \Rightarrow S + \mathrm{ann}(S) \in \mathrm{Lag}(\mathbb{V})$.

Most general $E \subset \mathbb{V}$ given by $S \subset V,\ \omega \in \wedge^2(S^*)$:

$$E = \{v + \alpha |\ v \in V,\ \iota_v\omega = \alpha|_S\}.$$

Linear Dirac geometry

Let V, V' be vector spaces.

Definition

A morphism $R\colon \mathbb{V} \dashrightarrow \mathbb{V}'$ is a Lagrangian subspaces $R \subset \mathbb{V}' \times \overline{\mathbb{V}}$ whose projection to $V' \times V$ is the graph of a map $A\colon V \to V'$.

A morphism defines a relation $x \sim_R x'$; composition of morphisms is composition of relations.

Definition

A morphism of Dirac structures $R\colon (\mathbb{V}, E) \dashrightarrow (\mathbb{V}', E')$ is a morphism $R\colon \mathbb{V} \dashrightarrow \mathbb{V}'$ such that

$$E' = R \circ E, \qquad E \cap \ker(R) = 0.$$

Here $\ker(R) = \{x \in \mathbb{V} | \; x \sim_R 0\}$.

Linear Dirac geometry

Equivalently, a morphism $R\colon \mathbb{V} \dashrightarrow \mathbb{V}'$ is given by a linear map $A\colon V \to V'$ together with a 2-form $\omega \in \wedge^2(V^*)$, where

$$v + \alpha \sim_R v' + \alpha' \Leftrightarrow \begin{cases} v' = A(v) \\ \alpha = A^*(\alpha') + \iota_v \omega \end{cases}$$

Hence, we will also refer to such pairs (A, ω) as morphisms. Composition of morphisms reads:

$$(A', \omega') \circ (A, \omega) = (A' \circ A, \omega + A^* \omega').$$

The conditions $E' = R \circ E,\ \ker(R) \cap E = 0$ for Dirac morphisms mean that

$$\forall x' \in E' \ \ \exists^! x \in E\colon x \sim_R x'.$$

This defines a map $E' \to E,\ x' \mapsto x$.

Dirac structures on manifolds

Let M be a manifold, $\mathbb{T}M = TM \oplus T^*M$.

Definition

The Courant bracket on $\Gamma(\mathbb{T}M)$ is

$$[\![v_1 + \alpha_1, v_2 + \alpha_2]\!] = [v_1, v_2] + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2} \mathrm{d}\alpha_1.$$

Definition

A Dirac structure on M is a sub-bundle $E \subseteq \mathbb{T}M$ such that

- $E = E^\perp$,
- $\Gamma(E)$ is closed under $[\![\cdot, \cdot]\!]$.

Dirac structures on manifolds

Examples of Dirac structures:

1. For $\omega \in \Gamma(\wedge^2 T^*M)$, $\mathrm{Gr}(\omega)$ is a Dirac structure $\Leftrightarrow \mathrm{d}\omega = 0$.
2. For $\pi \in \Gamma(\wedge^2 TM)$, $\mathrm{Gr}(\pi)$ is a Dirac structure $\Leftrightarrow [\pi, \pi] = 0$.
3. For $S \subset TM$ a distribution, $S + \mathrm{ann}(S)$ is a Dirac structure $\Leftrightarrow S$ is integrable.

Dirac structures on manifolds

More generally, one can twist by a closed 3-form $\eta \in \Omega^3(M)$. Put $\mathbb{T}M_\eta = TM \oplus T^*M$.

Definition

The Courant bracket on $\Gamma(\mathbb{T}M_\eta)$ is

$$[\![v_1 + \alpha_1, v_2 + \alpha_2]\!] = [v_1, v_2] + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2} \mathrm{d}\alpha_1 + \iota_{v_1}\iota_{v_2}\eta.$$

Definition

A Dirac structure on M is a sub-bundle $E \subseteq \mathbb{T}M_\eta$ such that

- $E = E^\perp$,
- $\Gamma(E)$ is closed under $[\![\cdot,\cdot]\!]$.

Dirac structures on manifolds

Examples of Dirac structures in $\mathbb{T}M_\eta$:

1. For $\omega \in \Gamma(\wedge^2 T^*M)$, $\mathrm{Gr}(\omega)$ is a Dirac structure $\Leftrightarrow \mathrm{d}\omega = \eta$.
2. For $\pi \in \Gamma(\wedge^2 TM)$, $\mathrm{Gr}(\pi)$ is a Dirac structure
$\Leftrightarrow \frac{1}{2}[\pi, \pi] = -\pi^\sharp(\eta)$.
3. For $S \subset TM$ a distribution, $S + \mathrm{ann}(S)$ is a Dirac structure
$\Leftrightarrow S$ is integrable and $\eta|\wedge^3 S = 0$.

Dirac structures on manifolds

Definition

A map $\Phi\colon M \to M'$ together with $\omega \in \Omega^2(M)$ is called a Courant morphism $(\Phi,\omega)\colon \mathbb{T}M_\eta \dashrightarrow \mathbb{T}M'_{\eta'}$ if

$$\eta = \Phi^*\eta' + \mathrm{d}\omega.$$

Definition

A Dirac morphism $(\Phi,\omega)\colon (\mathbb{T}M, E) \dashrightarrow (\mathbb{T}M', E')$ is a Courant morphism such that $(\mathrm{d}\Phi,\omega)$ defines linear Dirac morphisms fiberwise.

Application to Hamiltonian geometry

- $G \circlearrowright \mathfrak{g}^*$ coadjoint action
- $\mathrm{d}\mu \in \Omega^1(\mathfrak{g}^*, \mathfrak{g}^*)$ tautological 1-form

For $\xi \in \mathfrak{g}$ put $e(\xi) = \xi_{\mathfrak{g}^*} + \langle \mathrm{d}\mu, \xi \rangle \in \Gamma(\mathbb{T}\mathfrak{g}^*)$. These satisfy

$$[\![e(\xi_1), e(\xi_2)]\!] = e([\xi_1, \xi_2]),$$

hence span a Dirac structure $E_{\mathfrak{g}^*} \subset \mathbb{T}\mathfrak{g}^*$.

Remark

Equivalently, $E_{\mathfrak{g}^*}$ is the graph of the Kirillov-Poisson bivector on $\mathfrak{g}^*$.

Application to Hamiltonian geometry

A Dirac morphism

$$(\Phi, \omega) \colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}\mathfrak{g}^*, E_{\mathfrak{g}^*})$$

is a Hamiltonian $\mathfrak{g}$-space. That is, $\mathfrak{g}$ acts on M, ω, Φ are invariant, and

$$\omega(\xi_{\mathfrak{g}^*}, \cdot) + \langle \mathrm{d}\Phi, \xi \rangle = 0, \quad \mathrm{d}\omega = 0, \quad \ker(\omega) = 0.$$

Application to q-Hamiltonian geometry

- $G \circlearrowright G$ conjugation action,
- $\cdot$ invariant metric on $\mathfrak{g} = \mathrm{Lie}(G)$,
- $\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L]$ Cartan 3-form,

For $\xi \in \mathfrak{g}$ put $e(\xi) = \xi_G + \frac{1}{2}(\theta^L + \theta^R) \cdot \xi \in \Gamma(\mathbb{T}G_\eta)$. These satisfy

$$[\![e(\xi_1), e(\xi_2)]\!] = e([\xi_1, \xi_2]),$$

hence span a Dirac structure $E_G \subset \mathbb{T}G_\eta$.

Application to q-Hamiltonian geometry

Theorem (Bursztyn-Crainic)

A q-Hamiltonian $\mathfrak{g}$-space is a Dirac morphism

$$(\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}G_\eta, E_G).$$

This new viewpoint is extremely useful.

Application to q-Hamiltonian geometry

Lemma

Let $\varsigma = \frac{1}{2}\,\mathrm{pr}_1^*\,\theta^L \cdot \mathrm{pr}_2^*\,\theta^R \in \Omega^2(G \times G)$. *Then* $(\mathrm{Mult}_G, \varsigma)$ *defines a Dirac morphism*

$$(\mathrm{Mult}_G, \varsigma)\colon (\mathbb{T}G_\eta, E_G) \times (\mathbb{T}G_\eta, E_G) \dashrightarrow (\mathbb{T}G_\eta, E_G).$$

Hence, given two q-Hamiltonian G-spaces (M_i, ω_i, Φ_i), one can define their fusion product by composition

$$(\mathrm{Mult}_G, \varsigma) \circ \big((\Phi_1, \omega_1) \times (\Phi_2, \omega_2)\big).$$

Application to q-Hamiltonian geometry

Use $\cdot$ to identify $\mathfrak{g} \cong \mathfrak{g}^*$.

Lemma

Let $\varpi \in \Omega^2(\mathfrak{g})$ be the standard primitive of $\exp^* \eta$. Then $(\exp, \varpi)$ defines a Dirac morphism

$$(\exp, \varpi)\colon (\mathbb{T}\mathfrak{g}, E_{\mathfrak{g}}) \dashrightarrow (\mathbb{T}G_\eta, E_G)$$

over the subset of $\mathfrak{g}$ where $\exp$ is regular.

Hence, if (M, ω_0, Φ_0) is a Hamiltonian G-space, such that $\exp$ regular over $\Phi_0(M)$, then

$$(\Phi, \omega) := (\exp, \varpi) \circ (\Phi_0, \omega_0)$$

defines a q-Hamiltonian G-space.

Application to q-Hamiltonian geometry

We will use the Dirac geometry viewpoint to explain the following fact. Suppose G is compact and simply connected.

Fact: q-Hamiltonian G-spaces (M, ω, Φ) carry distinguished invariant volume forms.

These are the analogues of the 'Liouville forms' of symplectic manifolds.

We will need the concept of 'pure spinors'.

Pure spinors

Return to the linear algebra set-up: $\mathbb{V} = V \oplus V^*$, $\langle \cdot, \cdot \rangle$.

Definition

- The Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{V})$ is the unital algebra with generators $x \in \mathbb{V}$ and relations

$$x_1 x_2 + x_2 x_1 = \langle x_1, x_2 \rangle.$$

- The spinor module over $\mathbb{C}\mathrm{l}(\mathbb{V})$ is given by

$$\varrho \colon \mathbb{C}\mathrm{l}(\mathbb{V}) \to \mathrm{End}(\wedge V^*), \quad \varrho(v + \alpha)\phi = \iota_v \phi + \alpha \wedge \phi.$$

Pure spinors

For $\phi \in \wedge V^*$ let

$$N(\phi) = \{x \in \mathbb{V} |\ \varrho(x)\phi = 0\}.$$

Lemma

For $\phi \neq 0$, the space $N(\phi) \subseteq \mathbb{V}$ is isotropic.

(Exercise!)

Definition (E. Cartan)

$\phi \in \wedge V^*$ is a pure spinor if $N(\phi)$ is Lagrangian.

Fact: Every $E \in \mathrm{Lag}(\mathbb{V})$ is given by a pure spinor, unique up to scalar.

Pure spinors

Example

- $\mathrm{Gr}(\omega) = N(\phi)$ for $\phi = e^{-\omega}$.
- $\mathrm{Gr}(\pi) = N(\phi)$ for $\phi = e^{-\iota(\pi)}\Lambda$, where $\Lambda \in \wedge^{top} V^* - \{0\}$.
- $S + \mathrm{ann}(S) = N(\phi)$ for $\phi \in \wedge^{top}(\mathrm{ann}(S)) - \{0\}$.

Lemma

Suppose $\phi \in \wedge(V^*)$ is a pure spinor. Then

$$\phi^{[top]} \neq 0 \Leftrightarrow N(\phi) \cap V = 0.$$

(Exercise!)

Example

Let $\phi = e^{-\omega}$. Then $N(\phi) \cap V = \mathrm{Gr}(\omega) \cap V = \ker(\omega)$ is trivial if and only if $(e^{-\omega})^{[top]} \neq 0$.

Pure spinors

Lemma

Suppose $(A, \omega)\colon (\mathbb{V}, E) \dashrightarrow (\mathbb{V}', E')$ *is a Dirac morphism. If* $\phi' \in \wedge(V')^*$ *is a pure spinor with* $E' \cap N(\phi') = 0$*, then*

$$\phi = e^{-\omega} A^* \phi'$$

is a pure spinor with $E \cap N(\phi) = 0$.

Exercise!

In particular if $E = V$ then $(e^{-\omega} A^* \phi')^{[top]}$ is a volume form.

The q-Hamiltonian volume form

Back to q-Hamiltonian G-spaces, viewed as Morita morphisms

$$(\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}G_\eta, E_G)$$

If we can find $\psi \in \Gamma(G, \wedge T^*G)$ with $E \cap N(\psi) = 0$, then $(e^{-\omega}\Phi^*\psi)^{[\mathrm{top}]}$ is a volume form on M.

The q-Hamiltonian volume form

Recall: E_G is spanned by sections

$$e(\xi) = (\xi^L - \xi^R) + \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi.$$

Let F_G be spanned by sections

$$f(\xi) = \tfrac{1}{2}(\xi^L + \xi^R) + \frac{1}{4}(\theta^L - \theta^R) \cdot \xi.$$

Then $\mathbb{T}G_\eta = E_G \oplus F_G$ is a Lagrangian splitting.

The q-Hamiltonian volume form

Suppose G is 1-connected. (Actually, it suffices that $\mathrm{Ad}\colon G \to \mathrm{SO}(\mathfrak{g})$ lifts to $\mathrm{Spin}(\mathfrak{g})$.)

Fact: $F_G = N(\psi)$ is given by a distinguished pure spinor:

$$\psi = \det{}^{1/2}\big(\frac{1+\mathrm{Ad}_g}{2}\big)\exp\Big(\frac{1}{4}\big(\frac{1-\mathrm{Ad}_g}{1+\mathrm{Ad}_g}\big)\theta^L\cdot\theta^L\Big) \in \Omega(G).$$

Putting all together:

Theorem

For any q-Hamiltonian G-space (M,ω,Φ), the top degree part of

$$e^{-\omega}\Phi^*\psi$$

defines an invariant volume form on M.

The q-Hamiltonian volume form

Remark

Assuming only the existence of the invariant metric (=non-degenerate symmetric bilinear form) $\cdot$, one still gets an invariant measure on G.

The q-Hamiltonian volume form

This result applies in particular to conjugacy classes in G.

Example

If G is a simply connected semi-simple Lie group, then the conjugacy classes $\mathcal{C} \subseteq G$ carry distinguished volume forms. (Take $\cdot$ the Killing form.)

Example

$G = \mathrm{SO}(3)$ has a non-orientable conjugacy class $\mathcal{C} \cong \mathbb{R}P(2)$.

Example

Let G be the 2-dimensional group $\mathbb{R}_{>0} \ltimes \mathbb{R}$ (acting on $\mathbb{R}$ by dilations and translations). Then G has conjugacy classes not admitting invariant measures. Here $\mathfrak{g}$ does not admit an invariant metric $\cdot$.

A pure spinor defining F_G

We should still explain how ψ is obtained.

Explanation: TG carries a Riemannian metric B (from inner product on $\mathfrak{g}$). There is an isometric isomorphism

$$TG \oplus \overline{TG} \to \mathbb{T}G.$$

$\Rightarrow$ get embedding $\kappa\colon \mathrm{SO}(TG) \hookrightarrow \mathrm{SO}(\mathbb{T}G)$.

$\mathrm{SO}(TG) \cong G \times \mathrm{SO}(\mathfrak{g})$ has distinguished section $g \mapsto \mathrm{Ad}_g$. We have

$$E_G = \kappa(\mathrm{Ad})(T^*G), \quad F_G = \kappa(\mathrm{Ad})(TG).$$

A pure spinor defining F_G

$$F_G = \kappa(\mathrm{Ad})(TG).$$

Suppose G simply connected. Then the section $\kappa(\mathrm{Ad})$ of $SO(\mathbb{T}G)$ lifts to a section $\widetilde{\kappa(\mathrm{Ad})}$ of $\mathrm{Spin}(\mathbb{T}G) \subset \mathbb{C}\mathrm{l}(\mathbb{T}G)$.

Since TG is given by the pure spinor $1 \in \Gamma(\wedge T^*G) = \Omega(G)$, the bundle F_G is given by a pure spinor

$$\psi = \widetilde{\kappa(\mathrm{Ad})}.1 \in \Gamma(\wedge T^*G).$$

One can calculate this.

Properties of q-Hamiltonian volume forms

Some basic properties of the q-Hamiltonian volume form Γ:

- Suppose (M, ω, Φ) is the 'exponential' of a Hamiltonian G-space (M, ω_0, Φ_0). Then

$$\Gamma = \Phi_0^* J^{1/2} \Gamma_0$$

  where $\Gamma_0 = (\exp(-\omega_0))^{[top]}$ is the Liouville form, and J is the Jacobian determinant of exp.
- The volume form for a fusion product of q-Hamiltonian spaces (M_i, ω_i, Φ_i) is the product of the volume forms.
- The volume form for $D(G) = G \times G$ is given by the canonical orentation and Haar measure.

Properties of q-Hamiltonian volume forms

- Let $\mathfrak{m} = \Phi_*|\Gamma| \in \mathcal{D}'(G)$ be the q-Hamiltonian Duistermaat-Heckman measure. $\mathfrak{m}$ is continuous, and

$$\mathfrak{m}|_e = c\,\mathrm{Vol}(M/\!\!/G)$$

  where c is the number of elements in a generic stabilizer.
- Recall $\mathcal{M}(\Sigma_h^0) = D(G)^h/\!\!/G$. Hence we get a formula for the symplectic volume $\mathrm{Vol}(\mathcal{M}(\Sigma_h^0))$: Push-forward Haar measure on G^{2h} under the map

$$\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod a_i b_i a_i^{-1} b_i^{-1}$$

  and evaluate at e. The result gives Witten's volume formula for $\mathcal{M}(\Sigma_h^0)$.