Linear Algebra Notes

Lecture Notes, University of Toronto, Fall 2016

1. Dual spaces

Given a vector space V, one can consider the space of linear maps $\phi\colon V \to F$. Typical examples include:

- For the vector space $V = \mathcal{F}(X, F)$ of functions from a set X to F, and any given $c \in X$, the evaluation
$$\mathrm{ev}_c\colon \mathcal{F}(X,F) \to F, \quad f \mapsto f(c).$$
- The *trace* of a matrix,
$$\mathrm{tr}\colon V = \mathrm{Mat}_{n\times n}(F) \to F,\ A \mapsto A_{11} + A_{22} + \ldots + A_{nn}.$$
More generally, for a fixed matrix $B \in \mathrm{Mat}_{n\times n}(F)$, there is a linear functional
$$A \mapsto \mathrm{tr}(BA).$$
- For the vector space F^n, written as column vectors, the i-th coordinate function
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto x_i.$$
More generally, any given $b_1, \ldots, b_n$ defines a linear functional
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto x_1 b_1 + \ldots + x_n b_n.$$
(Note that this can also be written as matrix multiplication with the row vector $(b_1, \ldots, b_n)$.)
- This generalizes to the space F^∞ of infinite sequences: We have maps
$$F^\infty \to F, \quad x = (x_1, x_2, \ldots) \mapsto a_i.$$
More generally, letting $F^\infty_{\mathrm{fin}} \subset F^\infty$ denote the subspace of finite sequences, every $y = (y_1, y_2, \ldots) \in F^\infty_{\mathrm{fin}}$ defines a linear map
$$F^\infty \to F, \quad x = (x_1, x_2, \ldots) \mapsto \sum_{i=1}^\infty x_i y_i;$$
similarly every $x = (x_1, x_2, \ldots) \in F^\infty$ defines a linear map
$$F^\infty_{\mathrm{fin}} \to F, \quad y \mapsto \sum_{i=1}^\infty x_i y_i.$$

Definition 1.1. For any vector space V over a field F, we denote by

$$V^* = \mathcal{L}(V, F)$$

the *dual space* of V.

Proposition 1.2.

$$\dim(V^*) = \dim(V).$$

Proof.

$$\dim(V^*) = \dim \mathcal{L}(V, F) = \dim(V)\dim(F) = \dim(V).$$

□

If V is finite-dimensional, this means that V and V^* are isomorphic. But this is false if $\dim V = \infty$. For instance, if V has an infinite, but countable basis (such as the space $V = \mathcal{P}(F)$), one can show that V^* does *not* have a countable basis, and hence cannot be isomorphic to V. In a homework problem, you were asked to show that the map associating to $x \in F^\infty$ a linear functional on F^∞_{fin} defines an isomorphism

$$F^\infty \cong (F^\infty_{\text{fin}})^*;$$

thus $(F^\infty_{\text{fin}})^*$ is much bigger than F^∞_{fin} (the latter has a countable basis, the former does not).

Definition 1.3. Suppose V, W are vector spaces, and V^*, W^* their dual spaces. Given a linear map

$$T\colon V \to W$$

one defines a *dual map* (or *transpose map*)

$$T^*\colon W^* \to V^*, \quad \psi \mapsto \psi \circ T.$$

The composition of $\psi \in W^* = \mathcal{L}(W, F)$ with $T \in \mathcal{L}(V, W)$.

Note that T^* is a linear map from W^* to V^* since

$$T^*(\psi_1 + \psi_2) = (\psi_1 + \psi_2) \circ T = \psi_1 \circ T + \psi_2 \circ T;$$

similarly for scalar multiplication. Note that the dual map goes in the 'opposite direction'. In fact, under composition of $T \in \mathcal{L}(V, W),\ T \in \mathcal{L}(U, V)$,

$$(T \circ S)^* = S^* \circ T^*.$$

(We leave this as an Exercise.)

Bases Let us now see what all this means in terms of bases. We will take all the vector spaces involves to be finite-dimensional.

Thus let $\dim V < \infty$, and let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V.

Lemma 1.4. *The dual space V^* has a unique basis $\beta^* = \{v_1^*, \ldots, v_n^*\}$ with the property*

$$v_j^*(v_i) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

One calls $\beta^* = \{v_1^*, \ldots, v_n^*\}$ the *dual basis* to β.

Proof. Any linear map, and in particular every linear functional, is uniquely determined by its action on basis vectors. Hence, the formulas above define n linear functionals $v_1^*, \ldots, v_n^*$. To check that they are linearly independent, let $a_1 v_1^* + \ldots + a_n v_n^* = 0$. Evaluating on v_i, we obtain $a_i = 0$, for all $i = 1, \ldots, n$. □

Theorem 1.5. *Let β, γ be ordered bases for V, W respectively, and $T \in \mathcal{L}(V, W)$. Then the matrix of $T^* \in \mathcal{L}(W^*, V^*)$ relative to the dual bases γ^*, β^* is the transpose of the matrix $[T]_\beta^\gamma$:*

$$[T^*]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t.$$

Proof. Write

$$\beta = \{v_1, \ldots, v_n\},\ \gamma = \{w_1, \ldots, w_m\},\ \beta^* = \{v_1^*, \ldots, v_n^*\},\ \gamma = \{w_1^*, \ldots, w_m^*\}.$$

Let $A = [T]_\beta^\gamma, \quad B = [T^*]_{\gamma^*}^{\beta^*}$. By definition,

$$T(v_j) = \sum_k A_{kj} w_k, \quad (T^*)(w_i^*) = \sum_l B_{li} v_l^*.$$

Applying w_i^* to the first equation, we get

$$w_i^*(T(v_j)) = \sum_k A_{kj} w_i^*(w_k) = A_{ij}.$$

On the other hand,

$$w_i^*(T(v_j)) = (T^*(w_i^*))(v_j) = \sum_l B_{li} v_l^*(v_j) = B_{ji}.$$

This shows $A_{ij} = B_{ji}$. □

This shows that the dual map is the 'coordinate-free' version of the transpose of a matrix. For this reason, one also calls the dual map the 'transpose map', denoted T^t.

Given a subspace $S \subset V$, one can consider the space of all linear functional vanishing on S. This space is denoted S^0 (also $\mathrm{ann}(S)$), and is called the *annihilator of* S:

$$S^0 = \{\phi \in V^* |\ \phi(v) = 0\ \forall\ v \in S\} \subset V^*.$$

Given a linear functional $[\phi]$ on the quotient space V/S, one obtains a linear functional ϕ on V by composition: $V \to V/S \to F$. Note that ϕ obtained in this way vanishes on S. Thus, we have a map

$$(V/S)^* \to S^0.$$

Lemma 1.6. *This map is an isomorphism. In particular, if $\dim V < \infty$, then $\dim S^0 = \dim V - \dim S$.*

Proof. Exercise. □

Consider a linear map $T \in \mathcal{L}(V, W)$, with dual map $T^* \in \mathcal{L}(W^*, V^*)$.

Lemma 1.7. *The null space of T^* is the annhilator of the range of T:*

$$N(T^*) = R(T)^0.$$

Proof.

$$\begin{aligned}\psi \in N(T^*) &\Leftrightarrow T^*(\psi) = 0 \\ &\Leftrightarrow T^*(\psi)(v) = 0 \text{ for all } v \in V \\ &\Leftrightarrow \psi(T(v)) = 0 \text{ for all } v \in V \\ &\Leftrightarrow \psi(w) = 0 \text{ for all } w \in R(T) \\ &\Leftrightarrow \psi \in R(T)^0\end{aligned}$$

□

As a consequence, we see that

$$\dim N(T^*) = \dim R(T)^0 = \dim W - \dim R(T),$$

hence

$$\dim R(T^*) = \dim W - \dim N(T^*) = \dim R(T).$$

This is the basis-free proof of the fact that a matrix and its transpose have the same rank. We also see (for finite-dimensional spaces) that T is injective (resp. surjective) if and only if T^* is surjective (resp. injective).

More on dual spaces. (not covered in class)

0. Every element $v \in V$ defines a linear functional on V^*, by $\phi \mapsto \phi(v)$. This gives a map $V \to (V^*)^*$, which for $\dim V < \infty$ is an isomorphism. (For $\dim(V) = \infty$, it is not.)
1. The physicist Dirac invented the 'bra-ket' notation, where elements of a vector spaces V are denoted by 'ket' $|v\rangle$ etc, and the elements of the dual space V^* by 'bra' $\langle\phi|$. The pairing is then a bracket (*bra-ket*) $\langle\phi|\, v\rangle$. The linear map

$$T\colon V \to V,\ x \mapsto \phi(x)v$$

defined by v, ϕ is denoted $T = |v\rangle\langle\phi|$; this works nicely since

$$|x\rangle \mapsto |v\rangle\langle\phi|\, x\rangle.$$

Given a basis $v_i \in V$ as above, one write $v_i = |v_i\rangle$ and $v_i^* = \langle v_i|$ (now omitting the star). The definition of dual basis now reads as $\langle v_i|\, v_j\rangle = \delta_{ij}$. One has the following expression for the identity map:

$$I_V = \sum_i |v_i\rangle\langle v_i|.$$

The matrix elements of a linear map $T\colon V \to W$ are

$$\langle w_j|\, T|\, v_i\rangle,$$

and T itself can be written

$$T = \sum_{ij} |w_j\rangle\langle w_j|\, T|\, v\rangle\langle v_i|.$$

The claim that the matrices for T, T^* are transposes of each other becomes the statement

$$\langle w_j|\, T|\, v_i\rangle = \langle v_i|\, T^*|\, w_j\rangle$$