# MAT 1120HF – Assignment #3

Due date: Friday, December 10, 2010.

1. Let $G_2$ be the simply connected compact Lie group corresponding to the Dynkin diagram with two vertices, connected by a directed arrow of multiplicity 3. Pretending that we don't know anything about $G_2$, show how to extract the following information from the Dynkin diagram. Give **brief** explanations.
   a) Find the Cartan matrix of $G_2$.
   b) Draw the root system $\mathfrak{R} \subset E$ of $G_2$. Indicate the simple roots and the corresponding Weyl chamber.
   c) Use the invariant invariant inner product on $E$ for which the long roots $\alpha$ satisfy $||\alpha||^2 = 2$, to identify $E^* \cong E$. Draw the co-root system $\mathfrak{R}^\vee \subset E$. Indicate the simple co-roots. (Don't draw it into the picture from (b) - it will get too messy. But use the same scale.)
   d) Show that the Weyl group of $G_2$ is the dihedral group $D_6$, i.e. the symmetry group of a regular hexagon.
   e) Find the dimension of $G_2$.

2. Let $G$ be a compact, semi-simple Lie group. Let $T \subset G$ be a given maximal torus, and $\Pi \subset \mathfrak{R}$ a given set of simple roots. A permutation $\Phi$ of $\Pi$ is called a *Dynkin diagram automorphism* if $\langle \Phi(\alpha^\vee), \Phi(\beta) \rangle = \langle \alpha^\vee, \beta \rangle$ for all $\alpha, \beta \in \Pi$. Let $\mathrm{Aut}(\Pi)$ denote the group of Dynkin diagram automorphisms. On the other hand, let $\mathrm{Aut}(G)$ be the group of automorphisms of $G$, $\mathrm{Inn}(G)$ the normal subgroup of inner automorphisms, and

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$$

   the group of *outer automorphisms*.

   a) Show that any element of $\mathrm{Out}(G)$ may be represented by an element $\phi \in \mathrm{Aut}(G)$, such that $\phi$ preserves $T$ and the induced action $\Phi$ on $X^*(T)$ preserves the set $\Pi$ of simple roots. Show that this defines a group morphism $\mathrm{Out}(G) \to \mathrm{Aut}(\Pi)$.
   b) Let $\phi \in \mathrm{Aut}(G)$ be an automorphism with $\phi(t) = t$ for all $t \in T$. Show that $\mathrm{d}\phi \in \mathrm{Aut}(\mathfrak{g})$ preserves each root space $\mathfrak{g}_\alpha$, $\alpha \in \mathfrak{R}$. Use this to define a group morphism $\mathrm{span}_{\mathbb{Z}}(\mathfrak{R}) \to \mathrm{U}(1)$. Show that any such morphism is of the form $\alpha \mapsto \alpha(h)$ for some $h \in T$, and finally that $\phi = \mathrm{Ad}_h$. This proves that the group morphism $\mathrm{Out}(G) \to \mathrm{Aut}(\Pi)$ is *injective*.
   c) Show that for $G$ simply connected, the group morphism $\mathrm{Out}(G) \to \mathrm{Aut}(\Pi)$ is also *surjective*. (Hint: use the Serre relations.)
   d) Give an example of an automorphism of $G = \mathrm{SU}(n)$, $n > 2$ that is *not* inner. Describe the corresponding Dynkin diagram automorphism.
   e) Give an example of an automorphism of $G = \mathrm{SO}(2m)$, $m > 1$ that is *not* inner. (Hint: use $\mathrm{O}(2m)$.) Describe the corresponding Dynkin diagram automorphism.