MAT 1120HF – Assignment #2

Due date: Wednesday, November 3, 2010 in class.

1. Let $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R})$ be the Lie algebra with basis e, f, h and brackets $[e,f] = h,\ [h,e] = 2e,\ [h,f] = -2f$. The adjoint representation of $\mathfrak{g}$ on itself extends to an adjoint representation on the symmetric algebra $S(\mathfrak{g})$:

$$\mathrm{ad}(\eta)(\xi_1 \cdots \xi_k) := \sum_{i=1}^{k} \xi_1 \cdots [\eta, \xi_i] \cdots \xi_k,$$

The same formula defines the adjoint representation of $\mathfrak{g}$ on the enveloping algebra $U(\mathfrak{g})$. (Of course, all products are then interpreted as products in the enveloping algebra).

   a) Show that the kernel of $\mathrm{ad}(h)$ on $S(\mathfrak{g})$ is spanned by polynomials in fe and h.

   b) Show that the invariant subspace for the adjoint action on $S(\mathfrak{g})$ (i.e. the subsace annihilated by all $\mathrm{ad}(\eta),\ \eta \in \mathfrak{g}$) is spanned by the powers of the element

$$2fe + \tfrac{1}{2}h^2 \in S^2(\mathfrak{g}).$$

   (Hint: Change the generators in part (a) to $2fe + \frac{1}{2}h^2,\ h$.)

   c) Show that the space of invariants for the action on $U(\mathfrak{g})$ is exactly the center of the enveloping algebra.

   d) Show that the center of the enveloping algebra $U(\mathfrak{g})$ is spanned by the powers of the element

$$2fe + \tfrac{1}{2}h^2 + h \in U^{(2)}(\mathfrak{g})$$

2. Let $V(n)$ be the irreducible $\mathfrak{sl}(2,\mathbb{C})$-representation of dimension $n+1$. Define a representation on $\tilde{\pi}\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(\tilde{V})$ on $\tilde{V} = \mathrm{End}(V(n))$ by

$$\tilde{\pi}(\xi)(B) = [\pi(\xi), B].$$

Determine the multiplicities of the irreducible representations $V(k)$ in $\mathrm{End}(V(n))$, i.e. find which $V(k)$ occur and with what multiplicity. (Hint: All $\pi(e^j)$ commute with $\pi(e)$. Combine this with a dimension count.)

3. a) Show that $\mathrm{SL}(2,\mathbb{R})$ has fundamental group $\mathbb{Z}$. (Hint: Use polar decomposition of real matrices to show that $\mathrm{SL}(2,\mathbb{R})$ retracts onto $\mathrm{SO}(2) \cong S^1$.)

   b) Show that $\mathrm{SL}(2,\mathbb{C})$ is simply connected. (Hint: Use polar decomposition of complex matrices to show that $\mathrm{SL}(2,\mathbb{C})$ retracts onto $\mathrm{SU}(2) \cong S^3$.)

   c) Show that the universal cover of $\mathrm{SL}(2,\mathbb{R})$ is *not* a matrix Lie group. That is, there does not exist an injective Lie group morphism

$$\widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}),$$

   for any choice of n. (Hint: Given a Lie algebra morphism $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$, complexify to get a Lie algebra morphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$.)