MAT 1120HF – Assignment #1

Due date: Wednesday, October 6, 2010 in class.

1. Let J be the $2n \times 2n$-matrix, given as
$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$
One defines the *complex symplectic group* $\mathrm{Sp}(2n, \mathbb{C})$ as follows,
$$\mathrm{Sp}(2n, \mathbb{C}) = \{A \in \mathrm{GL}(2n, \mathbb{C}) |\ A^T J A = J\}.$$
In class, we had defined $\mathrm{Sp}(n)$ as the subgroup of $\mathrm{GL}(n, \mathbb{H})$ preserving the norm on $\mathbb{H}^n \cong \mathbb{R}^{4n}$. Show that
$$\mathrm{Sp}(n) \cong \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n).$$
Hint: View $\mathrm{Mat}_n(\mathbb{H})$ as the subalgebra of $\mathrm{Mat}_{2n}(\mathbb{C})$ of matrices of block form
$$\begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix},$$
with $a, b, c, d \in \mathrm{Mat}_n(\mathbb{R})$.
Remark: The non-compact group $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{GL}(2n, \mathbb{R})$ (defined similarly to $\mathrm{Sp}(2n, \mathbb{C})$) is the group of transformations preserving the symplectic form on $\mathbb{R}^{2n}$. Both $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{Sp}(n)$ are real forms of the complex Lie group $\mathrm{Sp}(2n, \mathbb{C})$, in the sense that their complexified Lie algebras are $\mathfrak{sp}(2n, \mathbb{C})$.

2. a) Let G be a connected Lie group, and U an open neighborhood of the group unit e. Show that any $g \in G$ can be written as a product $g = g_1 \cdots g_N$ of elements $g_i \in U$.
b) Let $\phi \colon G \to H$ be a morphism of connected Lie groups, and assume that the differential $\mathrm{d}_e\phi \colon T_e G \to T_e H$ is bijective. Show that ϕ is a (surjective) covering.

3. Give an explicit construction of a double covering of $\mathrm{SO}(4)$ by $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Hint: Represent the quaternion algebra $\mathbb{H}$ as an algebra of matrices $\mathbb{H} \subset \mathrm{Mat}_2(\mathbb{C})$, as in problem 1 above. Use this to define an action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{H}$ preserving the norm.

4. Show that $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ does not lie in the image of the exponential map for $\mathrm{SL}(2, \mathbb{R})$. Hence $\exp \colon \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not surjective. Hint: Assuming $A = \exp(B)$, what would the eigenvalues of B have to be?
Encores (do not hand in): 1) Find a parametrization of the set of conjugacy classes of $\mathrm{SL}(2, \mathbb{R})$. Can you find all conjugacy classes of elements that are not in the image of $\exp$? 2) Show that the exponential map for $\mathrm{GL}(2, \mathbb{C})$ is surjective.