Lie groups and Lie algebras

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1. TERMINOLOGY AND NOTATION

1.1. Lie groups.

Definition 1.1. A Lie group is a group $G$, equipped with a manifold structure such that the group operations

\[ \text{Mult} : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 g_2 \]

\[ \text{Inv} : G \to G, \quad g \mapsto g^{-1} \]

are smooth. A morphism of Lie groups $G, G'$ is a morphism of groups $\phi : G \to G'$ that is smooth.

Remark 1.2. Using the implicit function theorem, one can show that smoothness of Inv is in fact automatic. (Exercise)

The first example of a Lie group is the general linear group

\[ \text{GL}(n, \mathbb{R}) = \{ A \in \text{Mat}_n(\mathbb{R}) \mid \det(A) \neq 0 \} \]

of invertible $n \times n$ matrices. It is an open subset of $\text{Mat}_n(\mathbb{R})$, hence a submanifold, and the smoothness of group multiplication follows since the product map for $\text{Mat}_n(\mathbb{R})$ is obviously smooth.

Our next example is the orthogonal group

\[ \text{O}(n) = \{ A \in \text{Mat}_n(\mathbb{R}) \mid A^T A = I \} \]

To see that it is a Lie group, it suffices to show that $\text{O}(n)$ is an embedded submanifold of $\text{Mat}_n(\mathbb{R})$. In order to construct submanifold charts, we use the exponential map of matrices

\[ \exp : \text{Mat}_n(\mathbb{R}) \to \text{Mat}_n(\mathbb{R}), \quad B \mapsto \exp(B) = \sum_{n=0}^{\infty} \frac{1}{n!} B^n \]

(an absolutely convergent series). One has $\frac{d}{dt}|_{t=0} \exp(tB) = B$, hence the differential of $\exp$ at 0 is the identity $\text{id}_{\text{Mat}_n(\mathbb{R})}$. By the inverse function theorem, this means that there is $\epsilon > 0$ such that $\exp$ restricts to a diffeomorphism from the open neighborhood $U = \{ B : \|B\| < \epsilon \}$ of 0 onto an open neighborhood $\exp(U)$ of $I$. Let

\[ \sigma(n) = \{ B \in \text{Mat}_n(\mathbb{R}) \mid B + B^T = 0 \} \]
We claim that
\[ \exp(\mathfrak{o}(n) \cap U) = O(n) \cap \exp(U), \]
so that \( \exp \) gives a submanifold chart for \( O(n) \) over \( \exp(U) \). To prove the claim, let \( B \in U \). Then
\[ \exp(B) \in O(n) \iff \exp(B)^T = \exp(B)^{-1} \]
\[ \iff \exp(B^T) = \exp(-B) \]
\[ \iff B^T = -B \]
\[ \iff B \in \mathfrak{o}(n). \]

For a more general \( A \in O(n) \), we use that the map \( \text{Mat}_n(\mathbb{R}) \to \text{Mat}_n(\mathbb{R}) \) given by left multiplication is a diffeomorphism. Hence, \( A \exp(U) \) is an open neighborhood of \( A \), and we have
\[ A \exp(U) \cap O(n) = A(\exp(U) \cap O(n)) = A \exp(U \cap \mathfrak{o}(n)). \]
Thus, we also get a submanifold chart near \( A \). This proves that \( O(n) \) is a submanifold. Hence its group operations are induced from those of \( \text{GL}(n, \mathbb{R}) \), they are smooth. Hence \( O(n) \) is a Lie group. Notice that \( O(n) \) is compact (the column vectors of an orthogonal matrix are an orthonormal basis of \( \mathbb{R}^n \); hence \( O(n) \) is a subset of \( S^{n-1} \times \cdots S^{n-1} \subset \mathbb{R}^n \times \cdots \mathbb{R}^n \)).

A similar argument shows that the special linear group
\[ \text{SL}(n, \mathbb{R}) = \{ A \in \text{Mat}_n(\mathbb{R}) | \det(A) = 1 \} \]
is an embedded submanifold of \( \text{GL}(n, \mathbb{R}) \), and hence is a Lie group. The submanifold charts are obtained by exponentiating the subspace
\[ \mathfrak{sl}(n, \mathbb{R}) = \{ B \in \text{Mat}_n(\mathbb{R}) | \text{tr}(B) = 0 \}, \]
using the identity \( \det(\exp(B)) = \exp(\text{tr}(B)) \).

Actually, we could have saved most of this work with \( O(n) \), \( \text{SL}(n, \mathbb{R}) \) once we have the following beautiful result of E. Cartan:

**Fact:** Every closed subgroup of a Lie group is an embedded submanifold, hence is again a Lie group.

We will prove this very soon, once we have developed some more basics of Lie group theory. A closed subgroup of \( \text{GL}(n, \mathbb{R}) \) (for suitable \( n \)) is called a **matrix Lie group**. Let us now give a few more examples of Lie groups, without detailed justifications.

**Examples 1.3.**

(a) Any finite-dimensional vector space \( V \) over \( \mathbb{R} \) is a Lie group, with product \( \text{Mult} \) given by addition.

(b) Let \( \mathcal{A} \) be a finite-dimensional associative algebra over \( \mathbb{R} \), with unit \( 1_{\mathcal{A}} \). Then the group \( \mathcal{A}^\times \) of invertible elements is a Lie group. More generally, if \( n \in \mathbb{N} \) we can create the algebra \( \text{Mat}_n(\mathcal{A}) \) of matrices with entries in \( \mathcal{A} \), and the general linear group
\[ \text{GL}(n, \mathcal{A}) := \text{Mat}_n(\mathcal{A})^\times \]
is a Lie group. If \( \mathcal{A} \) is commutative, one has a determinant map \( \det: \text{Mat}_n(\mathcal{A}) \to \mathcal{A} \), and \( \text{GL}(n, \mathcal{A}) \) is the pre-image of \( \mathcal{A}^\times \). One may then define a **special linear group**
\[ \text{SL}(n, \mathcal{A}) = \{ g \in \text{GL}(n, \mathcal{A}) | \det(g) = 1 \}. \]
We mostly have in mind the cases \( A = \mathbb{R}, \mathbb{C}, \mathbb{H} \). Here \( \mathbb{H} \) is the algebra of *quaternions* (due to Hamilton). Recall that \( \mathbb{H} = \mathbb{R}^4 \) as a vector space, with elements \((a, b, c, d) \in \mathbb{R}^4\) written as \( x = a + ib + jc + kd \) with imaginary units \( i, j, k \). The algebra structure is determined by \( i^2 = j^2 = k^2 = -1, \ ij = k, \ jk = i, \ ki = j \).

Note that \( \mathbb{H} \) is non-commutative (e.g. \( ji = -ij \)), hence \( \text{SL}(n, \mathbb{H}) \) is not defined. On the other hand, one can define complex conjugates \( x = a - ib - jc - kd \) and \( \|x\|^2 := x\overline{x} = a^2 + b^2 + c^2 + d^2 \).

defines a norm \( x \mapsto \|x\| \), with \( \|x_1 x_2\| = \|x_1\| \|x_2\| \) just as for complex or real numbers. The spaces \( \mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n \) inherit norms, by putting \( \|x\|^2 = \sum_{i=1}^{n} |x_i|^2, \ x = (x_1, \ldots, x_n) \).

The subgroups of \( \text{GL}(n, \mathbb{R}), \text{GL}(n, \mathbb{C}), \text{GL}(n, \mathbb{H}) \) preserving this norm (in the sense that \( \|Ax\| = \|x\| \) for all \( x \)) are denoted

\[ \text{O}(n), \quad \text{U}(n), \quad \text{Sp}(n) \]

and are called the *orthogonal, unitary, and symplectic group*, respectively. Since the norms of \( \mathbb{C}, \mathbb{H} \) coincide with those of \( \mathbb{C} \cong \mathbb{R}^2, \mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4 \), we have

\[ \text{U}(n) = \text{GL}(n, \mathbb{C}) \cap \text{O}(2n), \quad \text{Sp}(n) = \text{GL}(n, \mathbb{H}) \cap \text{O}(4n). \]

In particular, all of these groups are compact. One can also define

\[ \text{SO}(n) = \text{O}(n) \cap \text{SL}(n, \mathbb{R}), \quad \text{SU}(n) = \text{U}(n) \cap \text{SL}(n, \mathbb{C}), \]

these are called the *special orthogonal* and *special unitary groups*. The groups \( \text{SO}(n), \text{SU}(n), \text{Sp}(n) \) are often called the *classical groups* (but this term is used a bit loosely).

(d) For any Lie group \( G \), its universal cover \( \widetilde{G} \) is again a Lie group. The universal cover \( \widetilde{\text{SL}(2, \mathbb{R})} \) is an example of a Lie group that is not isomorphic to a matrix Lie group.

1.2. Lie algebras.

**Definition 1.4.** A Lie algebra is a vector space \( \mathfrak{g} \), together with a bilinear map \( [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) satisfying *anti-symmetry*

\[ [\xi, \eta] = -[\eta, \xi] \text{ for all } \xi, \eta \in \mathfrak{g}, \]

and the *Jacobi identity*,

\[ [\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0 \text{ for all } \xi, \eta, \zeta \in \mathfrak{g}. \]

The map \( [\cdot, \cdot] \) is called the Lie bracket. A morphism of Lie algebras \( \mathfrak{g}_1, \mathfrak{g}_2 \) is a linear map \( \phi : \mathfrak{g}_1 \to \mathfrak{g}_2 \) preserving brackets.
The space \( \mathfrak{gl}(n, \mathbb{R}) = \text{Mat}_n(\mathbb{R}) \)
is a Lie algebra, with bracket the commutator of matrices. (The notation indicates that we think of \( \text{Mat}_n(\mathbb{R}) \) as a Lie algebra, not as an algebra.)

A Lie subalgebra of \( \mathfrak{gl}(n, \mathbb{R}) \), i.e. a subspace preserved under commutators, is called a matrix Lie algebra. For instance,
\[
\mathfrak{sl}(n, \mathbb{R}) = \{ B \in \text{Mat}_n(\mathbb{R}) : \text{tr}(B) = 0 \}
\]

and
\[
\mathfrak{o}(n) = \{ B \in \text{Mat}_n(\mathbb{R}) : B^T = -B \}
\]
are matrix Lie algebras (as one easily verifies). It turns out that every finite-dimensional real Lie algebra is isomorphic to a matrix Lie algebra (Ado’s theorem), but the proof is not easy.

The following examples of finite-dimensional Lie algebras correspond to our examples for Lie groups. The origin of this correspondence will soon become clear.

**Examples 1.5.** (a) Any vector space \( V \) is a Lie algebra for the zero bracket.
(b) Any associative algebra \( \mathcal{A} \) can be viewed as a Lie algebra under commutator. Replacing \( \mathcal{A} \) with matrix algebras over \( \mathcal{A} \), it follows that \( \mathfrak{gl}(n, \mathcal{A}) = \text{Mat}_n(\mathcal{A}) \), is a Lie algebra, with bracket the commutator. If \( \mathcal{A} \) is commutative, then the subspace \( \mathfrak{sl}(n, \mathcal{A}) \subset \mathfrak{gl}(n, \mathcal{A}) \) of matrices of trace 0 is a Lie subalgebra.
(c) We are mainly interested in the cases \( \mathcal{A} = \mathbb{R}, \mathbb{C}, \mathbb{H} \). Define an inner product on \( \mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n \) by putting
\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,
\]
and define \( \mathfrak{o}(n), \mathfrak{u}(n), \mathfrak{sp}(n) \) as the matrices in \( \mathfrak{gl}(n, \mathbb{R}), \mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{H}) \) satisfying
\[
\langle Bx, y \rangle = -\langle x, By \rangle
\]
for all \( x, y \). These are all Lie algebras called the (infinitesimal) orthogonal, unitary, and symplectic Lie algebras. For \( \mathbb{R}, \mathbb{C} \) one can impose the additional condition \( \text{tr}(B) = 0 \), thus defining the special orthogonal and special unitary Lie algebras \( \mathfrak{so}(n), \mathfrak{su}(n) \).
Actually,
\[
\mathfrak{so}(n) = \mathfrak{o}(n)
\]
since \( B^T = -B \) already implies \( \text{tr}(B) = 0 \).

**Exercise 1.6.** Show that \( \mathfrak{Sp}(n) \) can be characterized as follows. Let \( J \in U(2n) \) be the unitary matrix
\[
\begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}
\]
where $I_n$ is the $n \times n$ identity matrix. Then

$$\text{Sp}(n) = \{ A \in U(2n) | \overline{A} = JAJ^{-1} \}.$$ 

Here $\overline{A}$ is the componentwise complex conjugate of $A$.

**Exercise 1.7.** Let $R(\theta)$ denote the $2 \times 2$ rotation matrix

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ 

Show that for all $A \in \text{SO}(2m)$ there exists $O \in \text{SO}(2m)$ such that $OAO^{-1}$ is of the block diagonal form

$$\begin{pmatrix} R(\theta_1) & 0 & \cdots & 0 \\ 0 & R(\theta_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R(\theta_m) \end{pmatrix}.$$ 

For $A \in \text{SO}(2m+1)$ one has a similar block diagonal presentation, with $m$ $2 \times 2$ blocks $R(\theta_i)$ and an extra 1 in the lower right corner. Conclude that $\text{SO}(n)$ is connected.

**Exercise 1.8.** Let $G$ be a connected Lie group, and $U$ an open neighborhood of the group unit $e$. Show that any $g \in G$ can be written as a product $g = g_1 \cdots g_N$ of elements $g_i \in U$.

**Exercise 1.9.** Let $\phi : G \to H$ be a morphism of connected Lie groups, and assume that the differential $d_e \phi : T_eG \to T_eH$ is bijective (resp. surjective). Show that $\phi$ is a covering (resp. surjective). Hint: Use Exercise 1.8.

### 2. The Covering SU(2) → SO(3)

The Lie group $\text{SO}(3)$ consists of rotations in 3-dimensional space. Let $D \subset \mathbb{R}^3$ be the closed ball of radius $\pi$. Any element $x \in D$ represents a rotation by an angle $||x||$ in the direction of $x$. This is a 1-1 correspondence for points in the interior of $D$, but if $x \in \partial D$ is a boundary point then $x, -x$ represent the same rotation. Letting $\sim$ be the equivalence relation on $D$, given by antipodal identification on the boundary, we have $D^3/\sim = \mathbb{RP}(3)$. Thus

$$\text{SO}(3) = \mathbb{RP}(3)$$

(at least, topologically). With a little extra effort (which we’ll make below) one can make this into a diffeomorphism of manifolds.

By definition

$$\text{SU}(2) = \{ A \in \text{Mat}_2(\mathbb{C}) | A^\dagger = A^{-1}, \det(A) = 1 \}.$$ 

Using the formula for the inverse matrix, we see that $\text{SU}(2)$ consists of matrices of the form

$$\text{SU}(2) = \left\{ \begin{pmatrix} z & \overline{w} \\ w & \overline{z} \end{pmatrix} | |w|^2 + |z|^2 = 1 \right\}.$$ 

That is, $\text{SU}(2) = S^3$ as a manifold. Similarly,

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} it & -\overline{u} \\ u & -it \end{pmatrix} | t \in \mathbb{R}, u \in \mathbb{C} \right\}$$

gives an identification $\mathfrak{su}(2) = \mathbb{R} \oplus \mathbb{C} = \mathbb{R}^3$. Note that for a matrix $B$ of this form, $\det(B) = t^2 + |u|^2$, so that $\det$ corresponds to $|| \cdot ||^2$ under this identification.
The group SU(2) acts linearly on the vector space $\mathfrak{su}(2)$, by matrix conjugation: $B \mapsto ABA^{-1}$. Since the conjugation action preserves det, we obtain a linear action on $\mathbb{R}^3$; preserving the norm. This defines a Lie group morphism from SU(2) into O(3). Since SU(2) is connected, this must take values in the identity component:

$$\phi: SU(2) \rightarrow SO(3).$$

The kernel of this map consists of matrices $A \in SU(2)$ such that $ABA^{-1} = B$ for all $B \in \mathfrak{su}(2)$. Thus, $A$ commutes with all skew-adjoint matrices of trace 0. Since $A$ commutes with multiples of the identity, it then commutes with all skew-adjoint matrices. But since $\text{Mat}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$ (the decomposition into skew-adjoint and self-adjoint parts), it then follows that $A$ is a multiple of the identity matrix. Thus $\ker(\phi) = \{I, -I\}$ is discrete. Since $d_v\phi$ is an isomorphism, it follows that the map $\phi$ is a double covering. This exhibits SU(2) = $S^3$ as the double cover of SO(3).

**Exercise 2.1.** Give an explicit construction of a double covering of SO(4) by SU(2) × SU(2).

Hint: Represent the quaternion algebra $\mathbb{H}$ as an algebra of matrices $\mathbb{H} \subset \text{Mat}_2(\mathbb{C})$, by

$$x = a + ib + jc + kd \mapsto x = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}.$$

Note that $|x|^2 = \det(x)$, and that SU(2) = $\{x \in \mathbb{H} | \det(x) = 1\}$. Use this to define an action of SU(2) × SU(2) on $\mathbb{H}$ preserving the norm.

### 3. The Lie algebra of a Lie group

#### 3.1. Review: Tangent vectors and vector fields

We begin with a quick reminder of some manifold theory, partly just to set up our notational conventions.

Let $M$ be a manifold, and $C^\infty(M)$ its algebra of smooth real-valued functions. For $m \in M$, we define the tangent space $T_mM$ to be the space of directional derivatives:

$$T_mM = \{v \in \text{Hom}(C^\infty(M), \mathbb{R}) | v(fg) = v(f)g + v(g)f\}.$$  

Here $v(f)$ is local, in the sense that $v(f) = v(f')$ if $f' - f$ vanishes on a neighborhood of $m$.

**Example 3.1.** If $\gamma: J \rightarrow M$, $J \subset \mathbb{R}$ is a smooth curve we obtain tangent vectors to the curve,

$$\dot{\gamma}(t) \in T_{\gamma(t)}M, \quad \dot{\gamma}(t)(f) = \frac{\partial}{\partial t}|_{t=0}f(\gamma(t)).$$

**Example 3.2.** We have $T_x\mathbb{R}^n = \mathbb{R}^n$, where the isomorphism takes $a \in \mathbb{R}^n$ to the corresponding velocity vector of the curve $x + ta$. That is,

$$v(f) = \frac{\partial}{\partial t}|_{t=0}f(x + ta) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}.$$

A smooth map of manifolds $\phi: M \rightarrow M'$ defines a tangent map:

$$d_m\phi: T_mM \rightarrow T_{\phi(m)}M', \quad (d_m\phi(v))(f) = v(f \circ \phi).$$

The locality property ensures that for an open neighborhood $U \subset M$, the inclusion identifies $T_mU = T_mM$. In particular, a coordinate chart $\phi: U \rightarrow \phi(U) \subset \mathbb{R}^n$ gives an isomorphism

$$d_m\phi: T_mM = T_mU \rightarrow T_{\phi(m)}\phi(U) = T_{\phi(m)}\mathbb{R}^n = \mathbb{R}^n.$$
Hence $T_m M$ is a vector space of dimension $n = \dim M$. The union $TM = \bigcup_{m \in M} T_m M$ is a vector bundle over $M$, called the tangent bundle. Coordinate charts for $M$ give vector bundle charts for $TM$. For a smooth map of manifolds $\phi: M \to M'$, the entirety of all maps $d_m \phi$ defines a smooth vector bundle map

$$d\phi: TM \to TM'.$$

A vector field on $M$ is a derivation $X: C^\infty(M) \to C^\infty(M)$. That is, it is a linear map satisfying

$$X(fg) = X(f)g + fX(g).$$

The space of vector fields is denoted $\mathfrak{X}(M) = \text{Der}(C^\infty(M))$. Vector fields are local, in the sense that for any open subset $U$ there is a well-defined restriction $X|_U \in \mathfrak{X}(U)$ such that $X|_U(f|_U) = (X(f))|_U$. For any vector field, one obtains tangent vectors $X_m \in T_m M$ by $X_m(f) = X(f)|_m$. One can think of a vector field as an assignment of tangent vectors, depending smoothly on $m$. More precisely, a vector field is a smooth section of the tangent bundle $TM$. In local coordinates, vector fields are of the form $\sum a_i \frac{\partial}{\partial x_i}$ where the $a_i$ are smooth functions.

It is a general fact that the commutator of derivations of an algebra is again a derivation. Thus, $\mathfrak{X}(M)$ is a Lie algebra for the bracket

$$[X, Y] = X \circ Y - Y \circ X.$$

In general, smooth maps $\phi: M \to M'$ of manifolds do not induce maps of the Lie algebras of vector fields (unless $\phi$ is a diffeomorphism). One makes the following definition.

**Definition 3.3.** Let $\phi: M \to N$ be a smooth map. Vector fields $X, Y$ on $M, N$ are called $\phi$-related, written $X \sim_\phi Y$, if

$$X(f \circ \phi) = Y(f) \circ \phi$$

for all $f \in C^\infty(M')$.

In short, $X \circ \phi^* = \phi^* \circ Y$ where $\phi^*: C^\infty(N) \to C^\infty(M), \ f \mapsto f \circ \phi$.

One has $X \sim_\phi Y$ if and only if $Y_{\phi(m)} = d_m \phi(X_m)$. From the definitions, one checks

$$X_1 \sim_\phi Y_1, \ X_2 \sim_\phi Y_2 \Rightarrow [X_1, X_2] \sim_\phi [Y_1, Y_2].$$

**Example 3.4.** Let $j: S \hookrightarrow M$ be an embedded submanifold. We say that a vector field $X$ is tangent to $S$ if $X_m \in T_m S \subset T_m M$ for all $m \in S$. We claim that if two vector fields are tangent to $S$ then so is their Lie bracket. That is, the vector fields on $M$ that are tangent to $S$ form a Lie subalgebra.

Indeed, the definition means that there exists a vector field $X_S \in \mathfrak{X}(S)$ such that $X_S \sim_j X$. Hence, if $X, Y$ are tangent to $S$, then $[X_S, Y_S] \sim_j [X, Y]$, so $[X_S, Y_S]$ is tangent.

Similarly, the vector fields vanishing on $S$ are a Lie subalgebra.

Let $X \in \mathfrak{X}(M)$. A curve $\gamma(t), \ t \in J \subset \mathbb{R}$ is called an integral curve of $X$ if for all $t \in J$,

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

In local coordinates, this is an ODE $\frac{dx}{dt} = a_i(x(t))$. The existence and uniqueness theorem for ODE’s (applied in coordinate charts, and then patching the local solutions) shows that for any $m \in M$, there is a unique maximal integral curve $\gamma(t), \ t \in J_m$ with $\gamma(0) = m$. 


Definition 3.5. A vector field $X$ is complete if for all $m \in M$, the maximal integral curve with \( \gamma(0) = m \) is defined for all $t \in \mathbb{R}$.

In this case, one obtains a smooth map
\[
\Phi : \mathbb{R} \times M \to M, \quad (t, m) \mapsto \Phi_t(m)
\]
such that $\gamma(t) = \Phi_{-t}(m)$ is the integral curve through $m$. The uniqueness property gives
\[
\Phi_0 = \text{Id}, \quad \Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}
\]
i.e. $t \mapsto \Phi_t$ is a group homomorphism. Conversely, given such a group homomorphism such that the map $\Phi$ is smooth, one obtains a vector field $X$ by setting
\[
X = \frac{\partial}{\partial t}|_{t=0} \Phi^{-1}_t,
\]
as operators on functions. That is, $X(f)(m) = \frac{\partial}{\partial t}|_{t=0} f(\Phi^{-1}_t(m))$. \(^1\)

The Lie bracket of vector fields measure the non-commutativity of their flows. In particular, if $X,Y$ are complete vector fields, with flows $\Phi^X_t$, $\Phi^Y_s$, then $[X,Y] = 0$ if and only if
\[
\Phi^X_t \circ \Phi^Y_s = \Phi^Y_s \circ \Phi^X_t.
\]
In this case, $X + Y$ is again a complete vector field with flow $\Phi^{X+Y}_t = \Phi^X_t \circ \Phi^Y_t$. (The right hand side defines a flow since the flows of $X,Y$ commute, and the corresponding vector field is identified by taking a derivative at $t = 0$.)

3.2. The Lie algebra of a Lie group. Let $G$ be a Lie group, and $TG$ its tangent bundle. For all $a \in G$, the left,right translations
\[
L_a : G \to G, \quad g \mapsto ag
\]
\[
R_a : G \to G, \quad g \mapsto ga
\]
are smooth maps. Their differentials at $e$ define isomorphisms $d_a L_a : T_a G \to T_a G$, and similarly for $R_a$. Let
\[
g = T_e G
\]
be the tangent space to the group unit.

A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if
\[
X \sim_{L_a} X
\]
for all $a \in G$, i.e. if it commutes with $L_a^*$. The space $\mathfrak{X}^L(G)$ of left-invariant vector fields is thus a Lie subalgebra of $\mathfrak{X}(G)$. Similarly the space of right-invariant vector fields $\mathfrak{X}^R(G)$ is a Lie subalgebra.

\(^1\)The minus sign is convention, but it is motivated as follows. Let $\text{Diff}(M)$ be the infinite-dimensional group of diffeomorphisms of $M$. It acts on $C^\infty(M)$ by $\Phi.f = f \circ \Phi^{-1} = (\Phi^{-1})^* f$. Here, the inverse is needed so that $\Phi_1.\Phi_2.f = (\Phi_1 \Phi_2).f$. We think of vector fields as ‘infinitesimal flows’, i.e. informally as the tangent space at id to $\text{Diff}(M)$. Hence, given a curve $t \mapsto \Phi_t$ through $\Phi_0 = \text{id}$, smooth in the sense that the map $\mathbb{R} \times M \to M$, $(t, m) \mapsto \Phi_t(m)$ is smooth, we define the corresponding vector field $X = \frac{\partial}{\partial t}|_{t=0} \Phi_t$ in terms of the action on functions: as
\[
X.f = \frac{\partial}{\partial t}|_{t=0} \Phi_t.f = \frac{\partial}{\partial t}|_{t=0}(\Phi_t^{-1})^* f.
\]
If $\Phi_t$ is a flow, we have $\Phi_t^{-1} = \Phi_{-t}$.
Lemma 3.6. The map 
\[ X^L(G) \rightarrow \mathfrak{g}, \; X \mapsto X_e \]
is an isomorphism of vector spaces. (Similarly for \( X^R(G) \).)

Proof. For a left-invariant vector field, \( X_a = (d_e L_a) X_e \), hence the map is injective. To show that it is surjective, let \( \xi \in \mathfrak{g} \), and put \( X_a = (d_e L_a) \xi \in T_a G \). We have to show that the map \( G \rightarrow TG, \; a \mapsto X_a \) is smooth. It is the composition of the map \( G \rightarrow \mathfrak{g} \times \mathfrak{g}, \; g \mapsto (g, \xi) \) (which is obviously smooth) with the map \( \mathfrak{g} \times \mathfrak{g} \rightarrow TG, \; (g, \xi) \mapsto d_e L_g(\xi) \). The latter map is the restriction of \( d\text{Mult} : TG \times TG \rightarrow TG \rightarrow G \times \mathfrak{g} \subset TG \times TG \), and hence is smooth. \( \square \)

We denote by \( \xi^L \in X^L(G), \; \xi^R \in X^R(G) \) the left, right invariant vector fields defined by \( \xi \in \mathfrak{g} \). Thus \( \xi^L|_e = \xi^R|_e = \xi \).

Definition 3.7. The Lie algebra of a Lie group \( G \) is the vector space \( \mathfrak{g} = T_e G \), equipped with the unique bracket such that \( [\xi, \eta]^L = [\xi^L, \eta^L], \; \xi \in \mathfrak{g} \).

Remark 3.8. If you use the right-invariant vector fields to define the bracket on \( \mathfrak{g} \), we get a minus sign. Indeed, note that \( \text{Inv} : G \rightarrow G \) takes left translations to right translations. Thus, \( \xi^R \) is \( \text{Inv} \)-related to some left invariant vector field. Since \( d_e \text{Inv} = -\text{Id} \), we see \( \xi^R \sim_{\text{Inv}} -\xi^L \). Consequently, \( [\xi^R, \eta^R] \sim_{\text{Inv}} [-\xi^L, -\eta^L] = [\xi, \eta]^L \).

But also \( -[\xi, \eta]^L \sim_{\text{Inv}} [\xi, \eta]^L \), hence we get \( [\xi^R, \xi^R] = -[\xi, \xi]^L \).

The construction of a Lie algebra is compatible with morphisms. That is, we have a functor from Lie groups to finite-dimensional Lie algebras.

Theorem 3.9. For any morphism of Lie groups \( \phi : G \rightarrow G' \), the tangent map \( d_e \phi : \mathfrak{g} \rightarrow \mathfrak{g}' \) is a morphism of Lie algebras. For all \( \xi \in \mathfrak{g}, \; \xi' = d_e \phi(\xi) \) one has \( \xi^L \sim_{\phi} (\xi')^L, \; \xi^R \sim_{\phi} (\xi')^R \).

Proof. Suppose \( \xi \in \mathfrak{g} \), and let \( \xi' = d_e \phi(\xi) \in \mathfrak{g}' \). The property \( \phi(ab) = \phi(a)\phi(b) \) shows that \( L_{\phi(a)} \circ \phi = \phi \circ L_a \). Taking the differential at \( e \), and applying to \( \xi \) we find \( (d_e L_{\phi(a)}) \xi' = (d_a \phi)(d_a L_a(\xi)) \) hence \( (\xi')^L_{\phi(a)} = (d_a \phi)(\xi^L_a) \). That is \( \xi^L \sim_{\phi} (\xi')^L \). The proof for right-invariant vector fields is similar. Since the Lie brackets of two pairs of \( \phi \)-related vector fields are again \( \phi \)-related, it follows that \( d_e \phi \) is a Lie algebra morphism. \( \square \)

Remark 3.10. Two special cases are worth pointing out.

(a) Let \( V \) be a finite-dimensional (real) vector space. A representation of a Lie group \( G \) on \( V \) is a Lie group morphism \( G \rightarrow \text{GL}(V) \). A representation of a Lie algebra \( \mathfrak{g} \) on \( V \) is a Lie algebra morphism \( \mathfrak{g} \rightarrow \text{gl}(V) \). The Theorem shows that the differential of any Lie group representation is a representation of its a Lie algebra.
(b) An automorphism of a Lie group $G$ is a Lie group morphism $\phi: G \to G$ from $G$ to itself, with $\phi$ a diffeomorphism. An automorphism of a Lie algebra is an invertible morphism from $\mathfrak{g}$ to itself. By the Theorem, the differential of any Lie group automorphism is an automorphism of its Lie algebra. As an example, $SU(n)$ has a Lie group automorphism given by complex conjugation of matrices; its differential is a Lie algebra automorphism of $\mathfrak{su}(n)$ given again by complex conjugation.

**Exercise 3.11.** Let $\phi: G \to G$ be a Lie group automorphism. Show that its fixed point set is a closed subgroup of $G$, hence a Lie subgroup. Similarly for Lie algebra automorphisms. What is the fixed point set for the complex conjugation automorphism of $SU(n)$?

4. The exponential map

**Theorem 4.1.** The left-invariant vector fields $\xi^L$ are complete, i.e. they define a flow $\Phi^\xi_t$ such that

$$\xi^L = \frac{\partial}{\partial t}|_{t=0}(\Phi^{-\xi}_{-t})^*.$$

Letting $\phi^\xi(t)$ denote the unique integral curve with $\phi^\xi(0) = e$. It has the property

$$\phi^\xi(t_1 + t_2) = \phi^\xi(t_1)\phi^\xi(t_2),$$

and the flow of $\xi^L$ is given by right translations:

$$\Phi^\xi_t(g) = g\phi^\xi(-t).$$

Similarly, the right-invariant vector fields $\xi^R$ are complete. $\phi^\xi(t)$ is an integral curve for $\xi^R$ as well, and the flow of $\xi^R$ is given by left translations, $g \mapsto \phi^\xi(-t)g$.

**Proof.** If $\gamma(t)$, $t \in J \subset \mathbb{R}$ is an integral curve of a left-invariant vector field $\xi^L$, then its left translates $a\gamma(t)$ are again integral curves. In particular, for $t_0 \in J$ the curve $t \mapsto \gamma(t_0)\gamma(t)$ is again an integral curve. Hence it coincides with $\gamma(t_0 + t)$ for all $t \in J \cap (J - t_0)$. In this way, an integral curve defined for small $|t|$ can be extended to an integral curve for all $t$, i.e. $\xi^L$ is complete.

Since $\xi^L$ is left-invariant, so is its flow $\Phi^\xi_t$. Hence

$$\Phi^\xi_t(g) = \Phi^\xi_t \circ L_g(e) = L_g \circ \Phi^\xi_t(e) = g\Phi^\xi_t(e) = g\phi^\xi(-t).$$

The property $\Phi^\xi_{t_1 + t_2} = \Phi^\xi_{t_1} \Phi^\xi_{t_2}$ shows that $\phi^\xi(t_1 + t_2) = \phi^\xi(t_1)\phi^\xi(t_2)$. Finally, since $\xi^L \sim_{\text{Inv }} -\xi^R$, the image

$$\text{Inv}(\phi^\xi(t)) = \phi^\xi(t)^{-1} = \phi^\xi(-t)$$

is an integral curve of $-\xi^R$. Equivalently, $\phi^\xi(t)$ is an integral curve of $\xi^R$. \hfill \Box

Since left and right translations commute, it follows in particular that

$$[\xi^L, \eta^R] = 0.$$

**Definition 4.2.** A 1-parameter subgroup of $G$ is a group homomorphism $\phi: \mathbb{R} \to G$.

We have seen that every $\xi \in \mathfrak{g}$ defines a 1-parameter group, by taking the integral curve through $e$ of the left-invariant vector field $\xi^L$. Every 1-parameter group arises in this way:
Proposition 4.3. If $\phi$ is a 1-parameter subgroup of $G$, then $\phi = \phi^\xi$ where $\xi = \dot{\phi}(0)$. One has

$$\phi^\xi(t) = \phi^{\xi(st)}.$$ 

The map

$$\mathbb{R} \times \mathfrak{g} \rightarrow G, \ (t, \xi) \mapsto \phi^\xi(t)$$ 

is smooth.

Proof. Let $\phi(t)$ be a 1-parameter group. Then $\Phi_t(g) := g\phi(-t)$ defines a flow. Since this flow commutes with left translations, it is the flow of a left-invariant vector field, $\xi^L$. Here $\xi$ is determined by taking the derivative of $\Phi_{-t}(e) = \phi(t)$ at $t = 0$: $\xi = \dot{\phi}(0)$. This shows $\phi = \phi^\xi$. As an application, since $\psi(t) = \phi^\xi(st)$ is a 1-parameter group with $\dot{\psi}(0) = s\dot{\phi}(0) = s\xi$, we have $\phi^\xi(st) = \phi^{\xi(t)}$. Smoothness of the map $(t, \xi) \mapsto \phi^\xi(t)$ follows from the smooth dependence of solutions of ODE’s on parameters. □

Definition 4.4. The exponential map for the Lie group $G$ is the smooth map defined by

$$\exp: \mathfrak{g} \rightarrow G, \ \xi \mapsto \phi^\xi(1),$$

where $\phi^\xi(t)$ is the 1-parameter subgroup with $\dot{\phi}(0) = \xi$.

Proposition 4.5. We have

$$\phi^\xi(t) = \exp(t\xi).$$

If $[\xi, \eta] = 0$ then $\exp(\xi + \eta) = \exp(\xi) \exp(\eta)$.

Proof. By the previous Proposition, $\phi^\xi(t) = \phi^{\xi(1)} = \exp(t\xi)$. For the second claim, note that $[\xi, \eta] = 0$ implies that $\xi^L, \eta^L$ commute. Hence their flows $\Phi^\xi, \Phi^\eta$, and $\Phi^\xi \circ \Phi^\eta$ is the flow of $\xi^L + \eta^L$. Hence it coincides with $\Phi^{\xi+\eta}$. Applying to $e$, we get $\phi^\xi(t)\phi^\eta(t) = \phi^{\xi+\eta}(t)$. Now put $t = 1$. □

In terms of the exponential map, we may now write the flow of $\xi^L$ as $\Phi^\xi_t(g) = g\exp(-t\xi)$, and similarly for the flow of $\xi^R$. That is,

$$\xi^L = \frac{\partial}{\partial t} |_{t=0} R^*_{\exp(t\xi)} \xi^R = \frac{\partial}{\partial t} |_{t=0} L^*_{\exp(t\xi)}.$$ 

Proposition 4.6. The exponential map is natural with respect to Lie group homomorphisms $\phi: G \rightarrow H$. That is,

$$\phi(\exp(\xi)) = \exp((d_e\phi)(\xi)), \ \xi \in \mathfrak{g}.$$ 

Proof. $t \mapsto \phi(\exp(t\xi))$ is a 1-parameter subgroup of $H$, with differential at $e$ given by

$$\frac{d}{dt} |_{t=0} \phi(\exp(t\xi)) = d_e\phi(\xi).$$

Hence $\phi(\exp(t\xi)) = \exp(td_e\phi(t))$. Now put $t = 1$. □

Proposition 4.7. Let $G \subset \text{GL}(n, \mathbb{R})$ be a matrix Lie group, and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ its Lie algebra. Then $\exp: \mathfrak{g} \rightarrow G$ is just the exponential map for matrices,

$$\exp(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n.$$ 

Furthermore, the Lie bracket on $\mathfrak{g}$ is just the commutator of matrices.
Proof. By the previous Proposition, applied to the inclusion of $G$ in $GL(n, \mathbb{R})$, the exponential map for $G$ is just the restriction of that for $GL(n, \mathbb{R})$. Hence it suffices to prove the claim for $G = GL(n, \mathbb{R})$. The function $\sum_{n=0}^{\infty} \frac{L^t}{n!} \xi^n$ is a 1-parameter group in $GL(n, \mathbb{R})$, with derivative at 0 equal to $\xi \in gl(n, \mathbb{R})$. Hence it coincides with $\exp(t\xi)$. Now put $t = 1$. \hfill \Box

Proposition 4.8. For a matrix Lie group $G \subset GL(n, \mathbb{R})$, the Lie bracket on $g = T_1G$ is just the commutator of matrices.

Proof. It suffices to prove for $G = GL(n, \mathbb{R})$. Using $\xi^L = \frac{\partial}{\partial t} |_{t=0} R^*_\exp(t\xi)$ we have

$$\frac{\partial}{\partial s} \bigg|_{s=0} \frac{\partial}{\partial t} \bigg|_{t=0} (R^*_\exp(-t\xi) R^*_\exp(-s\eta) R^*_\exp(t\xi) R^*_\exp(s\eta))$$

$$= \frac{\partial}{\partial s} \bigg|_{s=0} (R^*_\exp(-s\eta) \xi^L R^*_\exp(s\eta) - \xi^L)$$

$$= \xi^L \eta^L - \eta^L \xi^L$$

$$= [\xi, \eta]^L.$$ 

On the other hand, write

$$R^*_\exp(-s\eta) R^*_\exp(t\xi) R^*_\exp(s\eta) = R^*_\exp(-s\eta) \exp(t\xi) \exp(s\eta).$$

Since the Lie group exponential map for $GL(n, \mathbb{R})$ coincides with the exponential map for matrices, we may use Taylor’s expansion,

$$\exp(-t\xi) \exp(-s\eta) \exp(t\xi) \exp(s\eta) = I + st(\xi \eta - \eta \xi) + \ldots = \exp(st(\xi \eta - \eta \xi)) + \ldots$$

where $\ldots$ denotes terms that are cubic or higher in $s, t$. Hence

$$R^*_\exp(-s\eta) \exp(t\xi) \exp(s\eta) = R^*_\exp(st(\xi \eta - \eta \xi)) + \ldots$$

and consequently

$$\frac{\partial}{\partial s} \bigg|_{s=0} \frac{\partial}{\partial t} \bigg|_{t=0} R^*_\exp(-s\eta) \exp(t\xi) \exp(s\eta) = \frac{\partial}{\partial s} \bigg|_{s=0} \frac{\partial}{\partial t} \bigg|_{t=0} R^*_\exp(st(\xi \eta - \eta \xi)) = (\xi \eta - \eta \xi)^L.$$ 

We conclude that $[\xi, \eta] = \xi \eta - \eta \xi$. \hfill \Box

Remark 4.9. Had we defined the Lie algebra using right-invariant vector fields, we would have obtained minus the commutator of matrices. Nonetheless, some authors use that convention. The exponential map gives local coordinates for the group $G$ on a neighborhood of $e$.

Proposition 4.10. The differential of the exponential map at the origin is $d_0 \exp = \text{id}$. As a consequence, there is an open neighborhood $U$ of 0 $\in g$ such that the exponential map restricts to a diffeomorphism $U \to \exp(U)$.

Proof. Let $\gamma(t) = t\xi$. Then $\dot{\gamma}(0) = \xi$ since $\exp(\gamma(t)) = \exp(t\xi)$ is the 1-parameter group, we have

$$(d_0 \exp)(\xi) = \frac{\partial}{\partial t} |_{t=0} \exp(t\xi) = \xi.$$

\hfill \Box

Exercise 4.11. Show hat the exponential map for $SU(n)$, $SO(n)$ $U(n)$ are surjective. (We will soon see that the exponential map for any compact, connected Lie group is surjective.)
Exercise 4.12. A matrix Lie group $G \subset \text{GL}(n, \mathbb{R})$ is called unipotent if for all $A \in G$, the matrix $A - I$ is nilpotent (i.e. $(A - I)^r = 0$ for some $r$). The prototype of such a group are the upper triangular matrices with 1’s down the diagonal. Show that for a connected unipotent matrix Lie group, the exponential map is a diffeomorphism.

Exercise 4.13. Show that $\exp : \mathfrak{gl}(2, \mathbb{C}) \to \text{GL}(2, \mathbb{C})$ is surjective. More generally, show that the exponential map for $\text{GL}(n, \mathbb{C})$ is surjective. (Hint: First conjugate the given matrix into Jordan normal form).

Exercise 4.14. Show that $\exp : \mathfrak{sl}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R})$ is not surjective, by proving that the matrices
\[
\begin{pmatrix}
-1 & \pm 1 \\
0 & -1
\end{pmatrix}
\]
are not in the image. (Hint: Assuming these matrices are of the form $\exp(B)$, what would the eigenvalues of $B$ have to be?) Show that these two matrices represent all conjugacy classes of elements that are not in the image of $\exp$. (Hint: Find a classification of the conjugacy classes of $\text{SL}(2, \mathbb{R})$, e.g. in terms of eigenvalues.)

5. Cartan’s theorem on closed subgroups

Using the exponential map, we are now in position to prove Cartan’s theorem on closed subgroups.

Theorem 5.1. Let $H$ be a closed subgroup of a Lie group $G$. Then $H$ is an embedded submanifold, and hence is a Lie subgroup.

We first need a Lemma. Let $V$ be a Euclidean vector space, and $S(V)$ its unit sphere. For $v \in V \setminus \{0\}$, let $[v] = \frac{v}{||v||} \in S(V)$.

Lemma 5.2. Let $v_n, v \in V \setminus \{0\}$ with $\lim_{n \to \infty} v_n = 0$. Then
\[
\lim_{n \to \infty} [v_n] = [v] \iff \exists a_n \in \mathbb{N}: \lim_{n \to \infty} a_n v_n = v.
\]

Proof. The implication $\Leftarrow$ is obvious. For the opposite direction, suppose $\lim_{n \to \infty} [v_n] = [v]$. Let $a_n \in \mathbb{N}$ be defined by $a_n - 1 < \frac{||v||}{||v_n||} \leq a_n$. Since $v_n \to 0$, we have $\lim_{n \to \infty} a_n \frac{||v_n||}{||v||} = 1$, and
\[
a_n v_n = \left(a_n \frac{||v_n||}{||v||}\right) [v_n] ||v|| \to [v] ||v|| = v.
\]

Proof of E. Cartan’s theorem. It suffices to construct a submanifold chart near $e \in H$. (By left translation, one then obtains submanifold charts near arbitrary $a \in H$.) Choose an inner product on $\mathfrak{g}$.

We begin with a candidate for the Lie algebra of $H$. Let $W \subset \mathfrak{g}$ be the subset such that $\xi \in W$ if and only if either $\xi = 0$, or $\xi \neq 0$ and there exists $\xi_n \neq 0$ with
\[
\exp(\xi_n) \in H, \quad \xi_n \to 0, \quad [\xi_n] \to [\xi].
\]

We will now show the following:

(i) $\exp(W) \subset H$,

(ii) $W$ is a subspace of $\mathfrak{g}$,
(iii) There is an open neighborhood \( U \) of 0 and a diffeomorphism \( \phi: U \to \phi(U) \subset G \) with 
\[ \phi(0) = e \]
(Thus \( \phi \) defines a submanifold chart near \( e \).)

Step (i). Let \( \xi \in W\setminus\{0\} \), with sequence \( \xi_n \) as in the definition of \( W \). By the Lemma, there are \( a_n \in \mathbb{N} \) with \( a_n \xi_n \to \xi \). Since \( \exp(a_n \xi_n) = \exp(\xi_n)^{a_n} \in H \), and \( H \) is closed, it follows that 
\[ \exp(\xi) = \lim_{n \to \infty} \exp(a_n \xi_n) \in H. \]

Step (ii). Since the subset \( W \) is invariant under scalar multiplication, we just have to show that it is closed under addition. Suppose \( \xi, \eta \in W \). To show that \( \xi + \eta \in W \), we may assume that \( \xi, \eta, \xi + \eta \) are all non-zero. For \( t \) sufficiently small, we have 
\[ \exp(t \xi) \exp(t \eta) = \exp(u(t)) \]
for some smooth curve \( t \mapsto u(t) \in g \) with \( u(0) = 0 \). Then \( \exp(u(t)) \in H \) and 
\[ \lim_{n \to \infty} n u(\frac{1}{n}) = \lim_{h \to 0} \frac{u(h)}{h} = \dot{u}(0) = \xi + \eta. \]
hence \( u(\frac{1}{n}) \to 0, \exp(u(\frac{1}{n})) \in H, \left[ u(\frac{1}{n}) \right] \to [\xi + \eta] \). This shows \( [\xi + \eta] \in W \), proving (ii).

Step (iii). Let \( W' \) be a complement to \( W \) in \( g \), and define 
\[ \phi: g \cong W \oplus W' \to G, \quad \phi(\xi + \xi') = \exp(\xi) \exp(\xi'). \]

Since \( d\phi \) is the identity, there is an open neighborhood \( U \subset g \) of 0 such that \( \phi: U \to \phi(U) \) is a diffeomorphism. It is automatic that \( \phi(W \cap U) \subset \phi(W) \cap \phi(U) \subset H \cap \phi(U) \). We want to show that we can take \( U \) sufficiently small so that we also have the opposite inclusion 
\[ H \cap \phi(U) \subset \phi(W \cap U). \]

Suppose not. Then, any neighborhood \( U_n \subset g = W \oplus W' \) of 0 contains an element \( (\eta_n, \eta'_n) \) such that 
\[ \phi(\eta_n, \eta'_n) = \exp(\eta_n) \exp(\eta'_n) \in H \]
(i.e. \( \exp(\eta'_n) \in H \)) but \( (\eta_n, \eta'_n) \notin W \) (i.e. \( \eta'_n \neq 0 \)). Thus, taking \( U_n \) to be a nested sequence of neighborhoods with intersection \( \{0\} \), we could construct a sequence \( \eta'_n \in W' - \{0\} \) with 
\( \eta'_n \to 0 \) and \( \exp(\eta'_n) \in H \). Passing to a subsequence we may assume that \( [\eta'_n] \to [\eta] \) for some \( \eta \in W'\setminus\{0\} \). On the other hand, such a convergence would mean \( \eta \in W \), by definition of \( W \).

Contradiction. \( \square \)

As remarked earlier, Cartan’s theorem is very useful in practice. For a given Lie group \( G \), the term “closed subgroup” is often used as synonymous to “embedded Lie subgroup.”

**Examples 5.3.**

(a) The matrix groups \( G = O(n), \text{Sp}(n), \text{SL}(n, \mathbb{R}), \ldots \) are all closed subgroups of some \( \text{GL}(N, \mathbb{R}) \), and hence are Lie groups.

(b) Suppose that \( \phi: G \to H \) is a morphism of Lie groups. Then 
\[ \ker(\phi) = \phi^{-1}(e) \subset G \]
is a closed subgroup. Hence it is an embedded Lie subgroup of \( G \).

(c) The center \( Z(G) \) of a Lie group \( G \) is the set of all \( a \in G \) such that 
\( ag = ga \) for all \( a \in G \).

It is a closed subgroup, and hence an embedded Lie subgroup.

(d) The group of automorphisms of a Lie algebra \( g \) is closed in the group \( \text{End}(g) \times \) of vector space automorphisms, hence it is a Lie group.
6. The adjoint representations

6.1. The adjoint representation of $G$. Recall that an automorphism of a Lie group $G$ is an invertible morphism from $G$ to itself. The automorphisms form a group $\text{Aut}(G)$. Any $a \in G$ defines an ‘inner’ automorphism $\text{Ad}_a \in \text{Aut}(G)$ by conjugation:

$$\text{Ad}_a(g) = aga^{-1}$$

Indeed, $\text{Ad}_a$ is an automorphism since $\text{Ad}_a^{-1} = \text{Ad}_{a^{-1}}$ and

$$\text{Ad}_a(g_1g_2) = ag_1g_2a^{-1} = ag_1a^{-1}ag_2a^{-1} = \text{Ad}_a(g_1)\text{Ad}_a(g_2).$$

Note also that $\text{Ad}_{a_1a_2} = \text{Ad}_{a_1}\text{Ad}_{a_2}$, thus $\text{Ad}$ defines a group morphism $G \to \text{Aut}(G)$ into the group of automorphisms.

Its differential at the identity is a $G$-representation $G \to \text{Aut}(g)$ by automorphisms of the Lie algebra $g$. This is the adjoint representation of $G$, and it is common to denote it by the same symbol $\text{Ad}_a := d_e \text{Ad}_a$:

$$\text{Ad}_a: g \to g, \xi \mapsto \text{Ad}_a \xi.$$

Since the $\text{Ad}_a$ are Lie algebra/group morphisms, they are compatible with the exponential map,

$$\exp(\text{Ad}_a \xi) = \text{Ad}_a \exp(\xi).$$

Remark 6.1. If $G \subset \text{GL}(n, \mathbb{R})$ is a matrix Lie group, then $\text{Ad}_a \in \text{Aut}(g)$ is the conjugation of matrices

$$\text{Ad}_a(\xi) = a\xi a^{-1}.$$

This follows by taking the derivative of $\text{Ad}_a(\exp(t\xi)) = a\exp(t\xi)a^{-1}$, using that $\exp$ is just the exponential series for matrices.

6.2. The adjoint representation of $g$. A derivation of a Lie algebra $g$ is a linear map $D \in \text{End}(g)$ such that $D[\xi, \zeta] = [D\xi, \zeta] + [\xi, D\zeta]$. Derivations of $g$ form a Lie algebra under commutator. For instance, Lie bracket $[\xi, \cdot]$ with a given element of $g$ is a derivation (by Jacobi’s identity); derivations of this type are called inner.

For any Lie algebra $g$, one defines the adjoint representation

$$\text{ad}: g \to \text{Der}(g) \subset \text{End}(g)$$

by

$$\text{ad}_\xi = [\xi, \cdot].$$

The fact that this is a representation is again a consequence of the Jacobi identity.

Suppose now that $G$ is a Lie group, with Lie algebra $g$. Recall that the differential of any $G$-representation is a $g$-representation.

**Theorem 6.2.** If $g$ is the Lie algebra of $G$, then the adjoint representation $\text{ad}$ of $g$ is the differential of the adjoint representation of $G$. One has the equality of operators

$$\exp(\text{ad}_\xi) = \text{Ad}(\exp \xi)$$

for all $\xi \in g$. 
Proof. We have \( \exp(s \text{Ad}_{\exp(t\xi)} \eta) = \text{Ad}_{\exp(t\xi)} \exp(s\eta) = \exp(t\xi) \exp(s\eta) \exp(-t\xi) \). Hence

\[
\frac{\partial}{\partial t} \bigg|_{t=0} (\text{Ad}_{\exp(t\xi)} \eta)^L = \frac{\partial}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial s} \bigg|_{s=0} R^*_{\exp(s \text{Ad}_{\exp(t\xi)} \eta)}
\]

\[
= \frac{\partial}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial s} \bigg|_{s=0} R^*_{\exp(t\xi)} \exp(s\eta) \exp(-t\xi)
\]

\[
= \frac{\partial}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial s} \bigg|_{s=0} R^*_{\exp(t\xi)} R^*_{\exp(s\eta)} R^*_{\exp(-t\xi)}
\]

\[
= \frac{\partial}{\partial t} \bigg|_{t=0} R^*_{\exp(t\xi)} \eta^L R^*_{\exp(-t\xi)}
\]

\[
= [\xi^L, \eta^L]
\]

\[
= [\xi, \eta]^L = (\text{ad}_\xi \eta)^L,
\]

proving \( \frac{\partial}{\partial t} \bigg|_{t=0} \text{Ad}_{\exp(t\xi)} \eta = \text{ad}_\xi \eta \). The last part follows, since the exponential map is functorial with respect to Lie group morphisms (in this case \( \text{Ad}: G \to \text{End}(g)^\times \)).\( \square \)

Remark 6.3. As a special case, this formula holds for matrices. That is, for \( B, C \in \text{Mat}_n(\mathbb{R}) \),

\[
e^B C e^{-B} = \sum_{n=0}^\infty \frac{1}{n!} [B, [B, \cdots [B, C] \cdots]].
\]

The formula also holds in some other contexts, e.g. if \( B, C \) are elements of an algebra with \( B \) nilpotent (i.e. \( B^N = 0 \) for some \( N \)). In this case, both the exponential series for \( e^B \) and the series on the right hand side are finite. (Indeed, \( [B, [B, \cdots [B, C] \cdots]] \) with \( n \) \( B \)'s is a sum of terms \( B^j CB^{n-j} \), and hence must vanish if \( n \geq 2N \).)