MAT 1120HF, Assignment 1

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1. Denote by $\psi : \mathrm{Mat}_n(\mathbb{H}) \hookrightarrow \mathrm{Mat}_{2n}(\mathbb{C})$ the map

$$\psi(a + bi + cj + dk) = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}.$$

This is easily checked to be an embedding of $\mathbb{R}$-algebras. Writing $a + bi + cj + dk = A + Bj$ for $A, B \in \mathrm{Mat}_n(\mathbb{C})$, we equivalently have

$$\psi(A + Bj) = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}.$$

Given an arbitrary $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{C})$, we have

$$YJ - J\bar{Y} = \begin{pmatrix} -B & A \\ -D & C \end{pmatrix} - \begin{pmatrix} \bar{C} & \bar{D} \\ -\bar{A} & -\bar{B} \end{pmatrix}$$

which is zero if and only if $C = -\bar{B}$ and $D = \bar{A}$, hence the image of ψ is

$$\mathrm{im}\ \psi = \{Y \in \mathrm{Mat}_{2n}(\mathbb{C}) : YJ = J\bar{Y}\}$$

For $X \in \mathrm{Mat}_n(\mathbb{H})$, let $X^\dagger$ denote the quaternionic conjugate transpose of X, so

$$\mathrm{Sp}(n) = \{X \in \mathrm{Mat}_n(\mathbb{H}) : X^\dagger X = I\}.$$

Given $A + Bj \in \mathrm{Mat}_n(\mathbb{H})$, we have

$$\begin{aligned} (A + Bj)^\dagger &= A^\dagger + (Bj)^\dagger \\ &= A^\dagger - jB^\dagger \\ &= A^\dagger - B^T j \end{aligned}$$

hence

$$\begin{aligned} \psi((A + Bj)^\dagger) &= \begin{pmatrix} A^\dagger & -B^T \\ B^\dagger & A^T \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}^\dagger \\ &= \psi(A + Bj)^\dagger. \end{aligned}$$

Since $U(2n) = \{Y \in \mathrm{Mat}_{2n}(\mathbb{C}) : Y^\dagger Y = I\}$, it follows that ψ gives an isomorphism

$$\psi : \mathrm{Sp}(n) \to \mathrm{im}\ \psi \cap U(2n).$$

It then suffices to show that $\text{im } \psi \cap U(2n) = \text{Sp}(2n, \mathbb{C}) \cap U(2n)$. Let $Y \in U(2n)$. Then

$$
\begin{aligned}
Y \in \text{im } \psi \quad &\Leftrightarrow \quad YJ = J\bar{Y} \\
&\Leftrightarrow \quad JY^T = Y^\dagger J \text{ (transpose both sides, use } J^T = -J) \\
&\Leftrightarrow \quad YJY^T = J \text{ (} Y \text{ is unitary)} \\
&\Leftrightarrow \quad Y^T \in \text{Sp}(2n, \mathbb{C}) \\
&\Leftrightarrow \quad Y^\dagger \in \text{Sp}(2n, \mathbb{C}) \text{ (since Sp}(2n, \mathbb{C}) \text{ is clearly closed under complex conjugation)} \\
&\Leftrightarrow \quad Y^{-1} \in \text{Sp}(2n, \mathbb{C}) \text{ (} Y \text{ is unitary)} \\
&\Leftrightarrow \quad Y \in \text{Sp}(2n, \mathbb{C})
\end{aligned}
$$

hence $\text{im } \psi \cap U(2n) = \text{Sp}(2n, \mathbb{C}) \cap U(2n)$ as desired.

2. (a) Let $U^N = \{g_1 \cdots g_N : g_i \in U\}$, and let $H = \bigcup_{N=0}^{\infty} U^N$. Our goal is then to show that $H = G$. First, we assume without loss of generality that U is closed under inversion of elements, by otherwise intersecting it with $U^{-1} = \{g^{-1} : g \in U\}$. (The set U^{-1} is an open neighbourhood of e since it is the image of U under the diffeomorphism $G \to G : g \mapsto g^{-1}$.) Then H is precisely the subgroup of G generated by U. Now, for each N, we have

$$U^N = \bigcup_{a \in U} aU^{N-1}.$$

As left-multiplication by a is a diffeomorphism of G, and $U = U^1$ is open, by induction we have that each U^N is open in G, hence H is open in G. On the other hand, we can write the complement of H in G as a union of cosets

$$G - H = \bigcup_{a \in G - H} aH.$$

Each of these cosets is open in G since H is, and so $G - H$ is open. It follows that $H = G$ since G is connected.

(b) By the inverse function theorem, there is a neighbourhood $U \subset G$ of the identity element e_G which is mapped diffeomorphically by φ to $\varphi(U) \subset H$. In particular, the image of φ contains this open neighbourhood $\varphi(U)$ of e_H; since φ is a group homomorphism and H is connected, it follows immediately by the result of part a) that φ is surjective.

Let $h \in H$ and let $K = \ker \varphi$. Choose any $g \in G$ with $\varphi(g) = h$. We claim that $h\varphi(U)$ is evenly covered by the sheets zgU for $z \in K$; that is,

i. $\varphi^{-1}(h\varphi(U)) = \bigcup_{z \in K} zgU$
ii. $wgU \cap zgU = \emptyset$ for $w, z \in K$ with $w \neq z$
iii. each zgU is mapped diffeomorphically by φ to $h\varphi(U)$.

First,

$$\begin{aligned}
\varphi^{-1}(h\varphi(U)) &= \{x \in G : \varphi(x) = \varphi(gu) \quad \exists u \in U\} \\
&= \{x \in G : Kx = Kgu \quad \exists u \in U\} \\
&= \{x \in G : x = zgu \quad \exists z \in K, u \in U\} \\
&= \bigcup_{z \in K} zgU.
\end{aligned}$$

Next, suppose $w, z \in K$ with $wgU \cap zgU \neq \emptyset$. Then for some $u, v \in U$, we have $wgu = zgv$; applying φ we have $\varphi(u) = \varphi(v)$ since $w, z \in K$, hence $u = v$ since φ is injective on U, and so $w = v$. So the sets zgU are indeed disjoint.

The last of the three claims follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
U & \xrightarrow{L_{zg}} & zgU \\
\Big\downarrow \varphi & & \Big\downarrow \varphi \\
\varphi(U) & \xrightarrow{L_h} & h\varphi(U)
\end{array}$$

Hence $h\varphi(U)$ is indeed evenly covered by the sheets zgU. Since these $h\varphi(U)$ cover H, we conclude that φ is a covering map.

3. If $A, B \in \mathrm{Mat}_2(\mathbb{C})$ with $A = \exp(B)$, then the eigenvalues of A are $\{\exp \lambda : \ \lambda \text{ an eigenvalue of } B\}$, since this is clearly true for B in Jordan canonical form and exp commutes with conjugation of matrices. Assume for contradiction that $A = \exp(B)$ with A as given in the problem, and $B \in \mathfrak{sl}_2(\mathbb{R})$. As $\mathrm{tr}B = 0$, the eigenvalues of B are of the form $\lambda, -\lambda$, and since $A = \exp(B)$ we have $e^{\lambda} = e^{-\lambda} = 1$. In particular, $\lambda \neq 0$. Then the eigenvalues of B are distinct, so B is diagonalizable (over $\mathbb{C}$), and hence A is diagonalizable: if $b = PDP^{-1}$ for diagonal D, then $A = P\exp(D)P^{-1}$ and $\exp(D)$ is diagonal. The given A is not diagonalizable, however, as it is a nontrivial Jordan block. This gives the desired contradiction.

4. We use the embedding $\psi : \mathbb{H} \hookrightarrow \mathrm{Mat}_2(\mathbb{C})$ from problem 1:

$$z + wj \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

The norm on $\mathbb{H}$ is then given by the determinant, and we have $\mathrm{SU}(2)$ identified with the unit ball in $\mathbb{H}$ under this identification. Consider the action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{H}$ given by $(A_1, A_2)x = A_1 x A_2^{-1}$. Clearly this action is smooth. Since this action preserves the determinant, it defines a map of Lie groups $\varphi : \mathrm{SU}(2) \times \mathrm{SU}(2) \to O(\mathbb{H})$. Indeed, since $\mathrm{SU}(2)$ is connected, so is $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and hence φ sends $\mathrm{SU}(2) \times \mathrm{SU}(2)$ into the identity component $\mathrm{SO}(\mathbb{H}) \cong \mathrm{SO}(4)$.

Let $(A_1, A_2) \in \ker \varphi$. Then, in particular, we have $A_1 I A_2^{-1} = I$, hence $A_1 = A_2$. Moreover, A_1 must be in the centre of $\mathrm{SU}(2)$, hence a scalar multiple of the identity, hence $\pm I$. Then $\ker \varphi$ is cyclic of order 2. In particular it is zero-dimensional, so $\ker d_e\varphi = 0$, i.e. $d_e\varphi$ is injective. We have $\dim(\mathrm{SU}(2) \times \mathrm{SU}(2)) = 3 + 3 = 6$ and $\dim(\mathrm{SO}(4)) = \binom{4}{2} = 6$, so $d_e\varphi$ is an isomorphism. Then by the result of problem 2.b), φ is a covering map. It is a double cover since $\ker \varphi$ is of order 2.