CHAPTER 1

Symmetric bilinear forms

Throughout, $\mathbb{K}$ will denote a field of characteristic $\neq 2$. We are mainly interested in the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and sometimes specialize to those two cases.

1. Quadratic vector spaces

Suppose V is a finite-dimensional vector space over $\mathbb{K}$. For any bilinear form $B\colon V \times V \to \mathbb{K}$, define a linear map

$$B^\flat\colon V \to V^*, \ v \mapsto B(v, \cdot).$$

The bilinear form B is called *symmetric* if it satisfies $B(v_1, v_2) = B(v_2, v_1)$ for all $v_1, v_2 \in V$. Since $\dim V < \infty$ this is equivalent to $(B^\flat)^* = B^\flat$. The symmetric bilinear form B is uniquely determined by the associated quadratic form, $Q_B(v) = B(v, v)$ by the *polarization identity*,

$$B(v, w) = \tfrac{1}{2}\big(Q_B(v + w) - Q_B(v) - Q_B(w)\big). \tag{1}$$

The *kernel* (also called *radical*) of B is the subspace

$$\ker(B) = \{v \in V |\ B(v, v_1) = 0 \text{ for all } v_1 \in V\},$$

i.e. the kernel of the linear map $B^\flat$. The bilinear form B is called *non-degenerate* if $\ker(B) = 0$, i.e. if and only if $B^\flat$ is an isomorphism. A vector space V together with a non-degenerate symmetric bilinear form B will be referred to as a *quadratic vector space*. Assume for the rest of this chapter that (V, B) is a quadratic vector space.

Definition 1.1. A vector $v \in V$ is called *isotropic* if $B(v, v) = 0$, and *non-isotropic* if $B(v, v) \neq 0$.

For instance, if $V = \mathbb{C}^n$ over $\mathbb{K} = \mathbb{C}$, with the standard bilinear form $B(z, w) = \sum_{i=1}^n z_i w_i$, then $v = (1, i, 0, \ldots, 0)$ is an isotropic vector. If $V = \mathbb{R}^2$ over $\mathbb{K} = \mathbb{R}$, with bilinear form $B(x, y) = x_1 y_1 - x_2 y_2$, then the set of isotropic vectors $x = (x_1, x_2)$ is given by the 'light cone' $x_1 = \pm x_2$.

The *orthogonal group* $\mathrm{O}(V)$ is the group

$$\mathrm{O}(V) = \{A \in \mathrm{GL}(V) |\ \ B(Av, Aw) = B(v, w) \text{ for all } v, w \in V\}. \tag{2}$$

The subgroup of orthogonal transformations of determinant 1 is denoted $\mathrm{SO}(V)$, and is called the *special orthogonal group*.

For any subspace $F \subset V$, the *orthogonal* or *perpendicular* subspace is defined as

$$F^\perp = \{v \in V |\ B(v, v_1) = 0 \text{ for all } v_1 \in F\}.$$

The image of $B^\flat(F^\perp) \subset V^*$ is the annihilator of F. From this one deduces the dimension formula

$$\dim F + \dim F^\perp = \dim V \tag{3}$$

and the identities

$$(F^\perp)^\perp = F, \quad (F_1 \cap F_2)^\perp = F_1^\perp + F_2^\perp, \quad (F_1 + F_2)^\perp = F_1^\perp \cap F_2^\perp$$

for all $F, F_1, F_2 \subset V$. For any subspace $F \subset V$ the restriction of B to F has kernel $\ker(B|_{F\times F}) = F \cap F^\perp$.

Definition 1.2. A subspace $F \subset V$ is called a *quadratic subspace* if the restriction of B to F is non-degenerate, that is $F \cap F^\perp = 0$.

Using $(F^\perp)^\perp = F$ we see that F is quadratic $\Leftrightarrow$ $F^\perp$ is quadratic $\Leftrightarrow$ $F \oplus F^\perp = V$.

As a simple application, one finds that any non-degenerate symmetric bilinear form B on V can be 'diagonalized'. Let us call a basis $\epsilon_1, \ldots, \epsilon_n$ of V an *orthogonal basis* if $B(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$.

Proposition 1.3. *Any quadratic vector space (V, B) admits an orthogonal basis $\epsilon_1, \ldots, \epsilon_n$. If $\mathbb{K} = \mathbb{C}$ one can arrange that $B(\epsilon_i, \epsilon_i) = 1$ for all i. If $\mathbb{K} = \mathbb{R}$ or $K = \mathbb{Q}$, one can arrange that $B(\epsilon_i, \epsilon_i) = \pm 1$ for all i.*

Proof. The proof is by induction on $n = \dim V$, the case $\dim V = 1$ being obvious. If $n > 1$ choose any non-isotropic vector $\epsilon_1 \in V$. The span of ϵ_1 is a quadratic subspace, hence so is $\text{span}(\epsilon_1)^\perp$. By induction, there is an orthogonal basis $\epsilon_2, \ldots, \epsilon_n$ of $\text{span}(\epsilon_1)^\perp$. If $\mathbb{K} = \mathbb{C}$ (resp. $\mathbb{K} = \mathbb{R}, \mathbb{Q}$), one can rescale the ϵ_i such that $B(\epsilon_i, \epsilon_i) = 1$ (resp. $B(\epsilon_i, \epsilon_i) = \pm 1$). □

We will denote by $\mathbb{K}^{n,m}$ the vector space $\mathbb{K}^{n+m}$ with bilinear form given by $B(\epsilon_i, \epsilon_j) = \pm\delta_{ij}$, with a $+$ sign for $i = 1, \ldots, n$ and a $-$ sign for $i = n + 1, \ldots, n + m$. If $m = 0$ we simple write $\mathbb{K}^n = \mathbb{K}^{n,0}$, and refer to the bilinear form as *standard*. The Proposition above shows that for $\mathbb{K} = \mathbb{C}$, and quadratic vector space (V, B) is isomorphic to $\mathbb{C}^n$ with the standard bilinear form, while for $\mathbb{K} = \mathbb{R}$ it is isomorphic to some $\mathbb{R}^{n,m}$. (Here n, m are uniquely determined, although it is not entirely obvious at this point.)

2. Isotropic subspaces

Let (V, B) be a quadratic vector space.

Definition 2.1. A subspace $F \subset V$ is called *isotropic*[1] if $B|_{F\times F} = 0$, that is $F \subset F^\perp$.

[1] In some of the literature (e.g. C. Chevalley [?] or L. Grove [?]), a subspace is called isotropic if it contains at least one non-zero isotropic vector, and totally isotropic if all of its vectors are isotropic.

The polarization identity (1) shows that a subspace $F \subset V$ is isotropic if and only if all of its vectors are isotropic. If $F \subset V$ is isotropic, then

$$\dim F \leq \dim V/2 \tag{4}$$

since $\dim V = \dim F + \dim F^\perp \geq 2 \dim F$.

PROPOSITION 2.2. *For isotropic subspaces F, F' the following three conditions*

(a) $F + F'$ *is quadratic,*
(b) $V = F \oplus (F')^\perp$,
(c) $V = F' \oplus F^\perp$

are equivalent, and imply that $\dim F = \dim F'$. *Given an isotropic subspace $F \subset V$ one can always find an isotropic subspace F' satisfying these conditions.*

PROOF. We have

$$\begin{aligned}(F + F') \cap (F + F')^\perp &= (F + F') \cap F^\perp \cap (F')^\perp \\ &= (F + (F' \cap F^\perp)) \cap (F')^\perp \\ &= (F \cap (F')^\perp) + (F' \cap F^\perp).\end{aligned}$$

Thus

$$\begin{aligned}(F + F') \cap (F + F')^\perp = 0 &\Leftrightarrow F \cap (F')^\perp = 0 \text{ and } \ F' \cap F^\perp = 0 \\ &\Leftrightarrow F \cap (F')^\perp = 0, \text{ and } \ F + (F')^\perp = V.\end{aligned} \tag{5}$$

This shows (a)$\Leftrightarrow$(b), and similarly (a)$\Leftrightarrow$(c). Property (b) shows $\dim V = \dim F + (\dim F')^\perp = \dim F + \dim V - \dim F'$, hence $\dim F = \dim F'$. Given an isotropic subspace F, we find an isotropic subspace F' satisfying (c) as follows. Choose any complement W to $F^\perp$, so that

$$V = F^\perp \oplus W.$$

Thus $V = F^\perp + W$ and $0 = F^\perp \cap W$. Taking orthogonals, this is equivalent to $0 = F \cap W^\perp$ and $V = F + W^\perp$, that is

$$V = F \oplus W^\perp.$$

Let $S \colon W \to F \subset F^\perp$ be the projection along $W^\perp$. Then $w - S(w) \in W^\perp$ for all $w \in W$. The subspace

$$F' = \{w - \tfrac{1}{2}S(w) | \ w \in W\}.$$

(being the graph of a map $W \to F^\perp$) is again a complement to $F^\perp$, and since for all $w \in$

$$B(w - \tfrac{1}{2}S(w), w - \tfrac{1}{2}S(w)) = B(w, w - S(w)) + \tfrac{1}{4}B(S(w), S(w)) = 0$$

(the first term vanishes since $w - S(w) \in W^\perp$, the second term vanishes since $S(w) \in F$ is isotropic) it follows that F' is isotropic. $\square$

An isotropic subspace is called *maximal isotropic* if it is not properly contained in another isotropic subspace. Put differently, an isotropic subspace F is maximal isotropic if and only if it contains all $v \in F^\perp$ with $B(v, v) = 0$.

PROPOSITION 2.3. *Suppose F, F' are maximal isotropic. Then*

(a) *the kernel of the restriction of B to $F + F'$ equals $F \cap F'$. (In particular, $F + F'$ is quadratic if and only if $F \cap F' = 0$.)*
(b) *The images of F, F' in the quadratic vector space $(F + F')/(F \cap F')$ are maximal isotropic.*
(c) $\dim F = \dim F'$.

PROOF. Since F is maximal isotropic, it contains all isotropic vectors of $F^\perp$, and in particular it contains $F^\perp \cap F'$. Thus

$$F^\perp \cap F' = F \cap F'$$

Similarly $F \cap (F')^\perp = F \cap F'$ since F' is maximal isotropic. The calculation (5) hence shows

$$(F + F') \cap (F + F')^\perp = F \cap F',$$

proving (a). Let $W = (F + F')/(F \cap F')$ with the bilinear form B_W induced from B, and $\pi\colon F + F' \to W$ the quotient map. Clearly, B_W is non-degenerate, and $\pi(F), \pi(F')$ are isotropic. Hence the sum $W = \pi(F) + \pi(F')$ is a direct sum, and the two subspaces are maximal isotropic of dimension $\frac{1}{2}\dim W$. It follows that $\dim F = \dim \pi(F) + \dim(F \cap F') = \dim \pi(F') + \dim(F \cap F') = \dim F'$. □

DEFINITION 2.4. The *Witt index* of a non-degenerate symmetric bilinear form B is the dimension of a maximal isotropic subspace.

By (4), the maximal Witt index is $\frac{1}{2}\dim V$ if $\dim V$ is even, and $\frac{1}{2}(\dim V - 1)$ if $\dim V$ is odd.

3. Split bilinear forms

DEFINITION 3.1. The non-degenerate symmetric bilinear form B on an even-dimensional vector space V is called *split* if its Witt index is $\frac{1}{2}\dim V$. In this case, maximal isotropic subspaces are also called *Lagrangian subspaces*.

Equivalently, the Lagrangian subspaces are characterized by the property

$$F = F^\perp.$$

Split bilinear forms are easily classified:

PROPOSITION 3.2. *Let (V, B) be a quadratic vector space with a split bilinear form. Then there exists a basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ of V in which the bilinear form is given as follows:*

$$B(e_i, e_j) = 0,\ B(e_i, f_j) = \delta_{ij},\ B(f_i, f_j) = 0. \tag{6}$$

PROOF. Choose a pair of complementary Lagrangian subspaces, F, F'. Since B defines a non-degenerate pairing between F and F', it defines an isomorphism, $F' \cong F^*$. Choose a basis $e_1, \ldots, e_k$, and let $f_1, \ldots, f_k$ be the dual basis of F' under this identification. Then $B(e_i, f_j) = \delta_{ij}$ by definition of dual basis, and $B(e_i, e_j) = B(f_i, f_j) = 0$ since F, F' are Lagrangian. □

Our basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ for a quadratic vector space (V, B) with split bilinear form is not orthogonal. However, it may be replaced by an orthogonal basis

$$\epsilon_i = e_i + \tfrac{1}{2} f_i, \quad \tilde{\epsilon}_i = e_i - \tfrac{1}{2} f_i.$$

In the new basis, the bilinear form reads,

$$B(\epsilon_i, \epsilon_j) = \delta_{ij}, \; B(\epsilon_i, \tilde{\epsilon}_j) = 0, \; B(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = -\delta_{ij}. \tag{7}$$

Put differently, Proposition 3.2 (and its proof) say that any quadratic vector space with split bilinear form is isometric to a vector space

$$V = F^* \oplus F,$$

where the bilinear form is given by the pairing:

$$B((\mu, v), (\mu', v')) = \langle \mu', v \rangle + \langle \mu, v' \rangle.$$

The corresponding orthogonal group will be discussed in Section **??** below. At this point we will only need the following fact:

LEMMA 3.3. *Let $V = F^* \oplus F$, with the split bilinear form B given by the pairing. Then the subgroup $\mathrm{O}(V)_F \subset \mathrm{O}(V)$ fixing F pointwise consists of transformations of the form*

$$A_D \colon (\mu, v) \mapsto (\mu, v + D\mu)$$

where $D \colon F^* \to F$ is skew-adjoint: $D^* = -D$. In particular, $\mathrm{O}(V)_F \subset \mathrm{SO}(V)$.

PROOF. A linear transformation $A \in \mathrm{GL}(V)$ fixes F pointwise if and only if it is of the form

$$A(\mu, v) = (S\mu, v + D\mu)$$

for some linear maps $D \colon F^* \to F$ and $S \colon F^* \to F^*$. Suppose A is orthogonal. Then

$$0 = B(A(\mu, 0), A(0, v)) - B((\mu, 0), (0, v)) = \langle S\mu - \mu, v \rangle$$

for all $v \in F, \; \mu \in F^*$; hence $S = I$. Furthermore

$$0 = B(A(\mu, 0), A(\mu', 0)) - B((\mu, 0), (\mu', 0)) = \langle \mu, D\mu' \rangle + \langle \mu', D\mu \rangle,$$

so that $D = -D^*$. Conversely, it is straightforward to check that transformations of the form $A = A_D$ are orthogonal. □

4. E.Cartan-Dieudonné's Theorem

Throughout this Section, we assume that (V, B) is a quadratic vector space. The following simple result will be frequently used.

LEMMA 4.1. *For any $A \in \mathrm{O}(V)$, the orthogonal of the space of A-fixed vectors equals the range of $A - I$:*

$$\mathrm{ran}(A - I) = \ker(A - I)^{\perp}.$$

PROOF. For any $L \in \mathrm{End}(V)$, the transpose $L^{\top}$ relative to B satisfies $\mathrm{ran}(L) = \ker(L^{\top})^{\perp}$. We apply this to $L = A - I$, and observe that $\ker(A^{\top} - I) = \ker(A - I)$ since a vector is fixed under A if and only if it is fixed under $A^{\top} = A^{-1}$. □

DEFINITION 4.2. An orthogonal transformation $R \in \mathrm{O}(V)$ is called a *reflection* if its fixed point set $\ker(R - I)$ has codimension 1.

Equivalently, $\mathrm{ran}(R - I) = \ker(R - I)^{\perp}$ is 1-dimensional. If $v \in V$ is a non-isotropic vector, then the formula

$$R_v(w) = w - 2\frac{B(v, w)}{B(v, v)}v,$$

defines a reflection, since $\mathrm{ran}(R_v - I) = \mathrm{span}(v)$ is 1-dimensional.

PROPOSITION 4.3. *Any reflection R is of the form R_v, where the non-isotropic vector v is unique up to a non-zero scalar.*

PROOF. Suppose R is a reflection, and consider the 1-dimensional subspace $F = \mathrm{ran}(R - I)$. We claim that F is a quadratic subspace of V. Once this is established, we obtain $R = R_v$ for any non-zero $v \in F$, since R_v then acts as -1 on F and as $+1$ on $F^{\perp}$. To prove the claim, suppose on the contrary that F is not quadratic. Since $\dim F = 1$ it is then isotropic. Let F' be an isotropic subspace such that $F + F'$ is quadratic. Since R fixes $(F + F')^{\perp} \subset F^{\perp} = \ker(R - I)$, it may be regarded as a reflection of $F + F'$. This reduces the problem to the case $\dim V = 2$, with $F \subset V$ maximal isotropic and $R \in \mathrm{O}(V)_F$. As we had seen, $\mathrm{O}(V)_F$ is identified with the group of skew-symmetric maps $F^* \to F$, but for $\dim F = 1$ this group is trivial. Hence R is the identity, contradicting $\dim \mathrm{ran}(R - I) = 1$. □

Some easy properties of reflections are,

(1) $\det(R) = -1$,
(2) $R^2 = I$,
(3) if v is non-isotropic, $AR_vA^{-1} = R_{Av}$ for all $A \in \mathrm{O}(V)$,
(4) distinct reflections $R_1 \neq R_2$ commute if and only if the lines $\mathrm{ran}(R_1 - I)$ and $\mathrm{ran}(R_2 - I)$ are orthogonal.

The last Property may be seen as follows: suppose $R_1R_2 = R_2R_1$ and apply to $v_1 \in \mathrm{ran}(R_1 - I)$. Then $R_1(R_2v_1) = -R_2v_1$, which implies that R_2v_1 is a multiple of v_1; in fact $R_2v_1 = \pm v_1$ since R_2 is orthogonal. Since $R_2v_1 = -v_1$ would imply that $R_1 = R_2$, we must have $R_2v_1 = v_1$, or $v_1 \in \ker(R_2 - I)$.

For any $A \in \mathrm{O}(V)$, let $l(A)$ denote the smallest number l such that $A = R_1 \cdots R_l$ where $R_i \in \mathrm{O}(V)$ are reflections. We put $l(I) = 0$, and for the time being we put $l(A) = \infty$ if A cannot be written as such a product. (The Cartan-Dieudonne theorem below states that $l(A) < \infty$ always.) The following properties are easily obtained from the definition, for all $A, g, A_1, A_2 \in \mathrm{O}(V)$,

$$
\begin{aligned}
l(A^{-1}) &= l(A), \\
l(gAg^{-1}) &= l(A), \\
|l(A_1) - l(A_2)| &\leq l(A_1A_2) \leq l(A_1) + l(A_2), \\
\det(A) &= (-1)^{l(A)}
\end{aligned}
$$

A little less obvious is the following estimate.

Proposition 4.4. *There is a lower bound*

$$\dim(\mathrm{ran}(A - I)) \leq l(A)$$

for any $A \in \mathrm{O}(V)$.

Proof. Let $n(A) = \dim(\mathrm{ran}(A - I))$. If $A_1, A_2 \in \mathrm{O}(V)$, we have

$$\ker(A_1A_2 - I) \supseteq \ker(A_1A_2 - I) \cap \ker(A_1 - I) = \ker(A_2 - I) \cap \ker(A_1 - I)$$

Taking orthogonals,

$$\mathrm{ran}(A_1A_2 - I) \subseteq \mathrm{ran}(A_2 - I) + \mathrm{ran}(A_1 - I)$$

which shows

$$n(A_1A_2) \leq n(A_1) + n(A_2).$$

Thus, if $A = R_1 \cdots R_l$ is a product of $l = l(A)$ reflections, we have

$$n(A) \leq n(R_1) + \ldots + n(R_l) = l(A).$$ □

The following upper bound for $l(A)$ is much more tricky:

Theorem 4.5 (E.Cartan-Dieudonné). *Any orthogonal transformation* $A \in \mathrm{O}(V)$ *can be written as a product of* $l(A) \leq \dim V$ *reflections.*

Proof. By induction, we may assume that the Theorem is true for quadratic vector spaces of dimension $\leq \dim V - 1$. We will consider three cases.

Case 1: $\ker(A - I)$ is non-isotropic. Choose any non-isotropic vector $v \in \ker(A - I)$. Then A fixes the span of v and restricts to an orthogonal transformation A_1 of $V_1 = \mathrm{span}(v)^\perp$. Using the induction hypothesis, we obtain

$$l(A) = l(A_1) \leq \dim V - 1. \tag{8}$$

Case 2: $\mathrm{ran}(A - I)$ is non-isotropic. We claim:

(C) There exists a non-isotropic element $w \in V$ such that $v = (A - I)w$ is non-isotropic.

Given v, w as in (C), we may argue as follows. Since $v = (A - I)w$, and hence $(A + I)w \in \mathrm{span}(v)^\perp$, we have

$$R_v(A - I)w = -(A - I)w, \quad R_v(A + I)w = (A + I)w.$$

Adding and dividing by 2 we find $R_v A w = w$. Since w is non-isotropic, this shows that the kernel of $R_v A - I$ is non-isotropic. Equation (8) applied to the orthogonal transformation $R_v A$ shows $l(R_v A) \le \dim V - 1$. Hence $l(A) \le \dim V$. It remains to prove the claim (C). Suppose it is false, so that we have:

($\neg C$) The transformation $A - I$ takes the set of non-isotropic elements into the set of isotropic elements.

Let $v = (A - I)w$ be a non-isotropic element in $\mathrm{ran}(A - I)$. By ($\neg C$) the element w is isotropic. The orthogonal space $\mathrm{span}(w)^\perp$ is non-isotropic for dimensional reasons, hence there exists a non-isotropic element w_1 with $B(w, w_1) = 0$. Then $w_1,\ w + w_1,\ w - w_1$ are all non-isotropic, and by ($\neg C$) their images

$$v_1 = (A - I)w_1,\ v + v_1 = (A - I)(w + w_1),\ v - v_1 = (A - I)(w - w_1)$$

are isotropic. But then the polarization identity

$$Q_B(v) = \tfrac{1}{2}(Q_B(v + v_1) + Q_B(v - v_1)) - Q_B(v_1) = 0$$

shows that v is isotropic, a contradiction. This proves (C).

Case 3: Both $\ker(A - I)$ and $\mathrm{ran}(A - I)$ are isotropic. Since these two subspaces are orthogonal, it follows that they are equal, and are both Lagrangian. This reduces the problem to the case $V = F^* \oplus F$, where $F = \ker(A - I)$, that is $A \in \mathrm{O}(V)_F$. In particular $\det(A) = 1$. Let R_v be *any* reflection, then $A_1 = R_v A \in \mathrm{O}(V)$ has $\det(A_1) = -1$. Hence $\ker(A_1 - I)$ and $\mathrm{ran}(A_1 - I)$ cannot be both isotropic, and by the first two cases $l(A_1) \le \dim V = 2 \dim F$. But since $\det(A_1) = -1$, $l(A_1)$ must be *odd*, hence $l(A_1) < \dim V$ and therefore $l(A) \le \dim V$. □

Remark 4.6. Our proof of Cartan-Dieudonne's theorem is a small modification of Artin's proof in [**?**]. If $\mathrm{char}(\mathbb{K}) = 2$, the statement of the Cartan-Dieudonne theorem is still true, except in some very special cases. See Chevalley [**?**, page 83].

Example 4.7. Let $\dim F = 2$, and $V = F^* \oplus F$ with bilinear form given by the pairing. Suppose $A \in \mathrm{O}(V)_F$, so that $A(\mu, v) = (\mu, v + D\mu)$ where $D\colon F^* \to F$ is skew-adjoint: $D^* = -D$. Assuming $D \neq 0$ we will show how to write A as a product of four reflections. Choose a basis e_1, e_2 of F, with dual basis f_1, f_2 of F^*, such that D has the normal form $Df_1 = e_2,\ Df_2 = -e_1$. Let $Q \in \mathrm{GL}(F)$ be the diagonal transformation,

$$Q(e_1) = 2e_1, Q(e_2) = e_2$$

and put

$$g = \begin{pmatrix} (Q^*)^{-1} & 0 \\ 0 & Q \end{pmatrix}, \quad .$$

Then g is a product of two reflections, for example $g = R'R$ where

$$R = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \; R' = gR.$$

On the other hand, using $QD(Q^*)^{-1} = 2D$ we see

$$gAg^{-1} = \begin{pmatrix} I & 0 \\ QD(Q^*)^{-1} & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 2D & I \end{pmatrix} = A^2,$$

or $A = gAg^{-1}A^{-1}$. Since $g = R'R$ we obtain the desired presentation of A as a product of 4 reflections:

$$A = R'R(ARA^{-1})(AR'A^{-1}).$$

5. Witt's Theorem

The following result is of fundamental importance in the theory of quadratic forms.

THEOREM 5.1 (Witt's Theorem). *Suppose $F, \tilde{F}$ are subspaces of a quadratic vector space (V, B), such that there exists an isometric isomorphism $\phi\colon F \to \tilde{F}$, i.e. $B(\phi(v), \phi(w)) = B(v, w)$ for all $v, w \in F$. Then ϕ extends to an orthogonal transformation $A \in \mathrm{O}(V)$.*

PROOF. By induction, we may assume that the Theorem is true for quadratic vector spaces of dimension $\leq \dim V - 1$. We will consider two cases.

Case 1: F is non-isotropic. Let $v \in F$ be a non-isotropic vector, and let $\tilde{v} = \phi(v)$. Then $Q_B(v) = Q_B(\tilde{v}) \neq 0$, and $v + \tilde{v}$ and $v - \tilde{v}$ are orthogonal. The polarization identity $Q_B(v) + Q_B(\tilde{V}) = \frac{1}{2}(Q_B(v + \tilde{v}) + Q_B(v - \tilde{v}))$ show that are not both isotropic; say $w = v + \tilde{v}$ is non-isotropic. The reflection R_w satisfies

$$R_w(v + \tilde{v}) = -(v + \tilde{v}), \quad R_w(v - \tilde{v}) = v - \tilde{v}.$$

Adding, and dividing by 2 we find that $R_w(v) = -\tilde{v}$. Let $Q = R_w R_v$. Then Q is an orthogonal transformation with $Q(v) = \tilde{v} = \phi(v)$.

Replacing F with $F' = Q(F)$, v with $v' = Q(v)$ and ϕ with $\phi' = \phi \circ Q^{-1}$, we may thus assume that $F \cap \tilde{F}$ contains a non-isotropic vector v such that $\phi(v) = v$. Let

$$V_1 = \mathrm{span}(v)^\perp, \quad F_1 = F \cap V_1, \quad \tilde{F}_1 = \tilde{F} \cap V_1$$

and $\phi_1 \colon F_1 \to \tilde{F}_1$ the restriction of ϕ. By induction, there exists an orthogonal transformation $A_1 \in \mathrm{O}(V_1)$ extending ϕ_1. Let $A \in \mathrm{O}(V)$ with $A(v) = v$ and $A|_{V_1} = A_1$; then A extends ϕ.

Case 2: F is isotropic. Let F' be an isotropic complement to $F^\perp$, and let $\tilde{F}'$ be an isotropic complement to $\tilde{F}^\perp$. The pairing given by B identifies

$F' \cong F^*$ and $\tilde{F}' \cong \tilde{F}^*$. The isomorphism $\phi\colon F \to \tilde{F}$ extends to an isometry $\psi\colon F \oplus F' \to \tilde{F} \oplus \tilde{F}'$, given by $(\phi^{-1})^*$ on $F' \cong F^*$. By Case 1 above, ψ extends further to an orthogonal transformation of V. □

Some direct consequences are:

(1) $\mathrm{O}(V)$ acts transitively on the set of isotropic subspaces of any given dimension.
(2) If $F, \tilde{F}$ are isometric, then so are $F^\perp, \tilde{F}^\perp$. Indeed, any orthogonal extension of an isometry $\phi\colon F \to \tilde{F}$ restricts to an isometry of their orthogonals.
(3) Suppose $F \subset V$ is a subspace isometric to $\mathbb{K}^n$, with standard bilinear form $B(\epsilon_i, \epsilon_j) = \delta_{ij}$, and F is maximal relative to this property. If $F' \subset V$ is isometric to $\mathbb{K}^{n'}$, then there exists an orthogonal transformation $A \in \mathrm{O}(V)$ with $F' \subset A(F)$. In particular, the dimension of such a subspace F is an invariant of (V, B).

A subspace $W \subset V$ of a quadratic vector space is called *anisotropic* if it does not contain isotropic vectors other than 0. In particular, W is a quadratic subspace.

Proposition 5.2 (Witt decomposition). *Any quadratic vector space (V, B) admits a decomposition $V = F \oplus F' \oplus W$ where F, F' are maximal isotropic, W is anisotropic, and $W^\perp = F \oplus F'$. If $V = F_1 \oplus F_1' \oplus W_1$ is another such decomposition, then there exists $A \in \mathrm{O}(V)$ with $A(F) = F_1,\ A(F') = F_1',\ A(W) = W_1$.*

Proof. To construct such a decomposition, let F be a maximal isotropic subspace, and F' an isotropic complement to $F^\perp$. Then $F \oplus F'$ is quadratic, hence so is $W = (F \oplus F')^\perp$. Since F is maximal isotropic, the subspace W cannot contain isotropic vectors other than 0. Hence W is anisotropic. Given another such decomposition $V = F_1 \oplus F_1' \oplus W_1$, choose an isomorphism $F \cong F_1$. As we had seen (e.g. in the proof of Witt's Theorem), this extends canonically to an isometry $\phi\colon F \oplus F' \to F_1 \oplus F_1'$. Witt's Theorem gives an extension of ϕ to an orthogonal transformation $A \in \mathrm{O}(V)$. It is automatic that A takes $W = (F \oplus F')^\perp$ to $W = (F_1 \oplus F_1')^\perp$. □

Example 5.3. If $\mathbb{K} = \mathbb{R}$, the bilinear form on the anisotropic part of the Witt decomposition is either positive definite (i.e. $Q_B(v) > 0$ for non-zero $v \in W$) or negative definite (i.e. $Q_B(v) < 0$ for non-zero $v \in W$). By Proposition 1.3, any quadratic vector space (V, B) over $\mathbb{R}$ is isometric to $\mathbb{R}^{n,m}$ for some n, m. The Witt decomposition shows that n, m are uniquely determined by B. Indeed $\min(n, m)$ is the Witt index of B, while the sign of $n - m$ is given by the sign of Q_B on the anisotropic part.

6. Orthogonal groups for $\mathbb{K} = \mathbb{R}, \mathbb{C}$

In this Section we discuss the structure of the orthogonal group $\mathrm{O}(V)$ for quadratic vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Being a closed subgroup of $\mathrm{GL}(V)$, the orthogonal group $\mathrm{O}(V)$ is a Lie group. (If $\mathbb{K}=\mathbb{C}$ it is an algebraic Lie group since the defining equations are polynomial.) Recall that for a Lie subgroup $G\subset \mathrm{GL}(V)$, the corresponding Lie algebra $\mathfrak{g}$ is the subspace of all $\xi\in\mathrm{End}(V)$ with the property $\exp(t\xi)\in G$ for all $t\in\mathbb{K}$ (using the exponential map of matrices). We have:

Proposition 6.1. *The Lie algebra of* $\mathrm{O}(V)$ *is given by*

$$\mathfrak{o}(V)=\{A\in\mathrm{End}(V)|\ \ B(Av,w)+B(v,Aw)=0 \text{ for all } v,w\in V\},$$

with bracket given by commutator.

Proof. Suppose $A\in\mathfrak{o}(V)$, so that $\exp(tA)\in\mathrm{O}(V)$ for all t. Taking the t-derivative of $B(\exp(tA)v,\exp(tA)w)=B(v,w)$ we obtain $B(Av,w)+B(v,Aw)=0$ for all $v,w\in V$. Conversely, given $A\in\mathfrak{gl}(V)$ with $B(Av,w)+B(v,Aw)=0$ for all $v,w\in V$ we have

$$\begin{aligned}
B(\exp(tA)v,\exp(tA)w) &= \sum_{k,l=0}^{\infty}\frac{t^{k+l}}{k!l!}B(A^kv,A^lw)\\
&=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{t^k}{i!(k-i)!}B(A^iv,A^{k-i}w)\\
&=\sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{i=0}^{k}\binom{k}{i}B(A^iv,A^{k-i}w)\\
&=\sum_{k=0}^{\infty}\frac{t^k}{k!}B(v,A^kw)\sum_{i=0}^{k}(-1)^i\binom{k}{i}\\
&=B(v,w)
\end{aligned}$$

since $\sum_{i=0}^{k}(-1)^i\binom{k}{i}=\delta_{k,0}$. ∎

Thus $A\in\mathfrak{o}(V)$ if and only if $B^\flat\circ A\colon V\to V^*$ is a skew-adjoint map. In particular

$$\dim_{\mathbb{K}}\mathfrak{o}(V)=N(N-1)/2$$

where $N=\dim V$.

Let us now first discuss the case $\mathbb{K}=\mathbb{R}$. We have shown that any quadratic vector space (V,B) over $\mathbb{R}$ is isometric to $\mathbb{R}^{n,m}$, for unique n,m. The corresponding orthogonal group will be denoted $\mathrm{O}(n,m)$, the special orthogonal group $\mathrm{SO}(n,m)$, and its identity component $\mathrm{SO}_0(n,m)$. The dimension of $\mathrm{O}(n,m)$ coincides with the dimension of its Lie algebra $\mathfrak{o}(n,m)$, $N(N-1)/2$ where $N=n+m$. If $m=0$ we will write $\mathrm{O}(n)=\mathrm{O}(n,0)$ and $\mathrm{SO}(n)=\mathrm{SO}(n,0)$. These groups are compact, since they are closed subsets of the unit ball in $\mathrm{Mat}(n,\mathbb{R})$.

Lemma 6.2. *The groups* $\mathrm{SO}(n)$ *are connected for all* $n\geq 1$, *and have fundamental group* $\pi_1(\mathrm{SO}(n))=\mathbb{Z}_2$ *for* $n\geq 3$.

PROOF. The defining action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ restricts to a transitive action on the unit sphere S^{n-1}, with stabilizer at $(0, \ldots, 0, 1)$ equal to $\mathrm{SO}(n-1)$. Hence, for $n \geq 2$ the Lie group $\mathrm{SO}(n)$ is the total space of a principal fiber bundle over S^{n-1}, with fiber $\mathrm{SO}(n-1)$. This shows by induction that $\mathrm{SO}(n)$ is connected. The long exact sequence of homotopy groups

$$\cdots \to \pi_2(S^{n-1}) \to \pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n)) \to \pi_1(S^{n-1})$$

shows furthermore that the map $\pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n))$ is an isomorphism for $n > 3$ (since $\pi_2(S^{n-1}) = 0$ in that case). But $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, since $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}P(3) = S^3/\mathbb{Z}_2$ (see below). □

The groups $\mathrm{SO}(3)$ and $\mathrm{SO}(4)$ have a well-known relation with the group $\mathrm{SU}(2)$ of complex 2×2-matrices X satisfying $X^\dagger = X^{-1}$ and $\det X = 1$. Recall that the center of $\mathrm{SU}(2)$ is $\mathbb{Z}_2 = \{+I, -I\}$.

PROPOSITION 6.3. *There are isomorphisms of Lie groups,*

$$\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2, \quad \mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$$

where in the second equality the quotient is by the diagonal subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.

PROOF. Consider the algebra of quaternions $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$,

$$\mathbb{H} = \left\{ X = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}, \ z, w \in \mathbb{C} \right\}.$$

For any $X \in \mathbb{H}$ let $||X|| = (|z|^2 + |w|^2)^{\frac{1}{2}}$. Note that $X^\dagger X = X X^\dagger = ||X||^2 \, I$ for all $X \in \mathbb{H}$. Define a symmetric $\mathbb{R}$-bilinear form on $\mathbb{H}$ by

$$B(X_1, X_2) = \tfrac{1}{2} \operatorname{tr}(X_1^\dagger X_2).$$

The identification $\mathbb{H} \cong \mathbb{R}^4$ takes this to the standard bilinear form on $\mathbb{R}^4$ since $B(X, X) = \frac{1}{2}||X||^2 \operatorname{tr}(I) = ||X||^2$. The unit sphere $S^3 \subset \mathbb{H}$, characterized by $||X||^2 = 1$ is the group $\mathrm{SU}(2) = \{X | \ X^\dagger = X^{-1}, \ \det(X) = 1\}$. Define an action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{H}$ by

$$(X_1, X_2) \cdot X = X_1 X X_2^{-1}.$$

This action preserves the bilinear form on $\mathbb{H} \cong \mathbb{R}^4$, and hence defines a homomorphism $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$. The kernel of this homomorphism is the finite subgroup $\{\pm(I, I)\} \cong \mathbb{Z}_2$. (Indeed, $X_1 X X_2^{-1} = X$ for all X implies in particular $X_1 = X X_2 X^{-1}$ for all invertible X. But this is only possible if $X_1 = X_2 = \pm I$.) Since $\dim \mathrm{SO}(4) = 6 = 2 \dim \mathrm{SU}(2)$, and since $\mathrm{SO}(4)$ is connected, this homomorphism must be onto. Thus $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(I, I)\}$.

Similarly, identify $\mathbb{R}^3 \cong \{X \in \mathbb{H} | \operatorname{tr}(X) = 0\} = \operatorname{span}(I)^\perp$. The conjugation action of $\mathrm{SU}(2)$ on $\mathbb{H}$ preserves this subspace; hence we obtain a group homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. The kernel of this homomorphism is $\mathbb{Z}_2 \cong \{\pm I\} \subset \mathrm{SU}(2)$. Since $\mathrm{SO}(3)$ is connected and $\dim \mathrm{SO}(3) = 3 = \dim \mathrm{SU}(2)$, it follows that $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm I\}$. □

To study the more general groups $\mathrm{SO}(n,m)$ and $\mathrm{O}(n,m)$, we recall the polar decomposition of matrices. Let

$$\mathrm{Sym}(k) = \{A \mid A^\top = A\} \subset \mathfrak{gl}(k,\mathbb{R})$$

be the space of real symmetric $k \times k$-matrices, and $\mathrm{Sym}^+(k)$ its subspace of positive definite matrices. As is well-known, the exponential map for matrices restricts to a diffeomorphism,

$$\exp\colon \mathrm{Sym}(k) \to \mathrm{Sym}^+(k),$$

with inverse $\log\colon \mathrm{Sym}^+(k) \to \mathrm{Sym}(k)$. Furthermore, the map

$$\mathrm{O}(k) \times \mathrm{Sym}(k) \to \mathrm{GL}(k,\mathbb{R}),\ (O,X) \mapsto Oe^X$$

is a diffeomorphism. The inverse map

$$\mathrm{GL}(k,\mathbb{R}) \to \mathrm{O}(k) \times \mathrm{Sym}(k),\ \mapsto (A|A|^{-1}, \log|A|),$$

where $|A| = (A^\top A)^{1/2}$, is called the *polar decomposition* for $\mathrm{GL}(k,\mathbb{R})$. We will need the following simple observation:

Lemma 6.4. *Suppose $X \in \mathrm{Sym}(k)$ is non-zero. Then the closed subgroup of $\mathrm{GL}(k,\mathbb{R})$ generated by e^X is non-compact.*

Proof. Replacing X with $-X$ if necessary, we may assume $||e^X|| > 1$. But then $||e^{nX}|| = ||e^X||^n$ goes to ∞ for $n \to \infty$. □

This shows that $\mathrm{O}(k)$ is a maximal compact subgroup of $\mathrm{GL}(k,\mathbb{R})$. The polar decomposition for $\mathrm{GL}(k,\mathbb{R})$ restricts to a polar decomposition for any closed subgroup G that is invariant under the involution $A \mapsto A^\top$. Let

$$K = G \cap \mathrm{O}(k,\mathbb{R}),\ P = G \cap \mathrm{Sym}^+(k),\ \mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(k).$$

The diffeomorphism $\exp\colon \mathrm{Sym}(k) \to \mathrm{Sym}^+(k)$ restricts to a diffeomorphism $\exp\colon \mathfrak{p} \to P$, with inverse the restriction of log. Hence the polar decomposition for $\mathrm{GL}(k,\mathbb{R})$ restricts to a diffeomorphism

$$K \times \mathfrak{p} \to G$$

whose inverse is called the polar decomposition of G. (It is a special case of a *Cartan decomposition.*) Using Lemma 6.4, we see that K is a maximal compact subgroup of G. Since $\mathfrak{p}$ is just a vector space, K is a deformation retract of G.

We will now apply these considerations to $G = \mathrm{O}(n,m)$. Let B_0 be the standard bilinear form on $\mathbb{R}^{n+m}$, and define the endomorphism τ by

$$B(v,w) = B_0(\tau v, w).$$

Thus τ acts as the identity on $\mathbb{R}^n \oplus 0$ and as minus the identity $0 \oplus \mathbb{R}^m$, and an endomorphism of $\mathbb{R}^{n+m}$ commutes with τ if and only if it preserves the direct sum decomposition $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$. A matrix $A \in \mathrm{Mat}(n+m,\mathbb{R})$ lies in $\mathrm{O}(n,m)$ if and only if $A^\top \tau A = \tau$, where $\top$ denotes as before the usual transpose of matrices, i.e. the transpose relative to B_0 (not relative to B). Similarly $X \in \mathfrak{o}(n,m)$ if and only if $X^\top \tau + \tau X = 0$.

Remark 6.5. In block form we have

$$\tau = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$$

For $A \in \mathrm{Mat}(n+m, \mathbb{R})$ in block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{9}$$

we have $A \in \mathrm{O}(n, m)$ if and only if

$$a^\top a = I + c^\top c, \quad d^\top d = I + b^\top b, \quad a^\top b = c^\top d. \tag{10}$$

Similarly, for $X \in \mathrm{Mat}(n+m, \mathbb{R})$, written in block form

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{11}$$

we have $X \in \mathfrak{o}(n, m)$ if and only if

$$\alpha^\top = -\alpha, \; \beta^\top = \gamma, \; \delta^\top = -\delta. \tag{12}$$

Since $\mathrm{O}(n, m)$ is invariant under $A \mapsto A^\top$, (and likewise for the special orthogonal group and its identity component) the polar decomposition applies. We find:

Proposition 6.6. *Relative to the polar decomposition of* $\mathrm{GL}(n+m, \mathbb{R})$*, the maximal subgroups of*

$$G = \mathrm{O}(n, m), \; \mathrm{SO}(n, m), \; \mathrm{SO}_0(n, m),$$

are, respectively,

$$K = \mathrm{O}(n) \times \mathrm{O}(m), \; \mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m)), \; \mathrm{SO}(n) \times \mathrm{SO}(m).$$

(Here $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m))$ *are elements of* $(\mathrm{O}(n) \times \mathrm{O}(m))$ *of determinant* 1*.) In all of these cases, the space* $\mathfrak{p}$ *in the Cartan decomposition is given by matrices of the form*

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ x^\top & 0 \end{pmatrix} \right\}$$

where x is an arbitrary $n \times m$-matrix.

Proof. We start with $G = \mathrm{O}(n, m)$. Elements in $K = G \cap \mathrm{O}(n+m)$ are characterized by $A^\top \tau A = \tau$ and $A^\top A = I$. The two conditions give $A\tau = \tau A$, so that A is a block diagonal element of $\mathrm{O}(n+m)$. Hence $A \in \mathrm{O}(n) \times \mathrm{O}(m) \subset \mathrm{O}(n, m)$. This shows $K = \mathrm{O}(n) \times \mathrm{O}(m)$. Elements $X \in \mathfrak{p} = \mathfrak{o}(n, m) \cap \mathrm{Sym}(n+m)$ satisfy $X^\top \tau + \tau X = 0$ and $X^\top = X$, hence they are symmetric block off-diagonal matrices. This proves our characterization of $\mathfrak{p}$, and proves the polar decomposition for $\mathrm{O}(n, m)$. The polar decompositions for $\mathrm{SO}(n, m)$ is an immediate consequence, and the polar decomposition for $\mathrm{SO}_0(n, m)$ follows since $\mathrm{SO}(n) \times \mathrm{SO}(m)$ is the identity component of $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m))$. $\square$

COROLLARY 6.7. *Unless $n = 0$ or $m = 0$ the group $\mathrm{O}(n,m)$ has four connected components and $\mathrm{SO}(n,m)$ has two connected components.*

We next describe the space $P = \exp(\mathfrak{p})$.

PROPOSITION 6.8. *The space $P = \exp(\mathfrak{p}) \subset G$ consists of matrices*

$$P = \left\{ \begin{pmatrix} (I + bb^\top)^{1/2} & b \\ b^\top & (I + b^\top b)^{1/2} \end{pmatrix} \right\}$$

where b ranges over all $n \times m$-matrices. In fact,

$$\log \begin{pmatrix} (I + bb^\top)^{1/2} & b \\ b^\top & (I + b^\top b)^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & x \\ x^\top & 0 \end{pmatrix}$$

where x and b are related as follows,

$$b = \frac{\sinh(xx^\top)}{xx^\top} x, \quad x = \frac{\operatorname{arsinh}((bb^\top)^{1/2})}{(bb^\top)^{1/2}} b. \tag{13}$$

Note that $xx^\top$ (resp. $bb^\top$) need not be invertible. The quotient $\frac{\sinh(xx^\top)}{xx^\top}$ is to be interpreted as $f(xx^\top)$ where $f(z)$ is the entire holomorphic function $\frac{\sinh z}{z}$, and $f(xx^\top)$ is given in terms of the spectral theorem or equivalently in terms of the power series expansion of f.

PROOF. Let $X = \begin{pmatrix} 0 & x \\ x^\top & 0 \end{pmatrix}$. By induction on k,

$$X^{2k} = \begin{pmatrix} (xx^\top)^k & 0 \\ 0 & (x^\top x)^k \end{pmatrix}, \ X^{2k+1} = \begin{pmatrix} 0 & (xx^\top)^k x \\ x(x^\top x)^k & 0 \end{pmatrix}.$$

This gives

$$\exp(X) = \begin{pmatrix} \cosh(xx^\top) & \frac{\sinh(xx^\top)}{xx^\top} x \\ x\frac{\sinh(x^\top x)}{x^\top x} & \cosh(x^\top x) \end{pmatrix},$$

which is exactly the form of elements in P with $b = \frac{\sinh(xx^\top)}{xx^\top} x$. The equation $\cosh(xx^\top) = (1 + bb^\top)^{1/2}$ gives $\sinh(xx^\top) = (bb^\top)^{1/2}$. Plugging this into the formula for b, we obtain the second equation in (13). □

For later reference, we mention one more simple fact about the orthogonal and special orthogonal groups. Let $\mathbb{Z}_2$ be the center of $\mathrm{GL}(n + m, \mathbb{R})$ consisting of $\pm I$.

PROPOSITION 6.9. *For all n, m, the center of the group $\mathrm{O}(n,m)$ is $\mathbb{Z}_2$. Except in the cases $(n,m) = (0,2), (2,0)$, the center of $\mathrm{SO}(n,m)$ is $\mathbb{Z}_2$ if $-I$ lies in $\mathrm{SO}(n,m)$, and is trivial otherwise. The statement for the identity component is similar.*

The proof is left as an exercise. (Note that the elements of the center of G commute in particular with the diagonal elements of G. In the case of hand, one uses this fact to argue that the central elements are themselves diagonal, and finally that they are multiples of the identity.)

The discussion above carries over to $\mathbb{K} = \mathbb{C}$, with only minor modifications. It is enough to consider the case $V = \mathbb{C}^n$, with the standard symmetric bilinear form. Again, our starting point is the polar decomposition, but now for complex matrices. Let $\mathrm{Herm}(n) = \{A|\ A^\dagger = A\}$ be the space of Hermitian $n \times n$ matrices, and $\mathrm{Herm}^+(n)$ the subset of positive definite matrices. The exponential map gives a diffeomorphism

$$\mathrm{Herm}(n) \to \mathrm{Herm}^+(n),\ \ X \mapsto e^X.$$

This is used to show that the map

$$\mathrm{U}(n) \times \mathrm{Herm}(n) \to \mathrm{GL}(n, \mathbb{C}),\quad (U, X) \mapsto Ue^X$$

is a diffeomorphism; the inverse map takes A to (Ae^{-X}, X) with $X = \frac{1}{2}\log(A^\dagger A)$. The polar decomposition of $\mathrm{GL}(n, \mathbb{C})$ gives rise to polar decompositions of any closed subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ that is invariant under the involution $\dagger$. In particular, this applies to $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$. Indeed, if $A \in \mathrm{O}(n, \mathbb{C})$, the matrix $A^\dagger A$ lies in $\mathrm{O}(n, \mathbb{C}) \cap \mathrm{Herm}(n)$, and hence its logarithm $X = \frac{1}{2}\log(A^\dagger A)$ lies in $\mathfrak{o}(n, \mathbb{C}) \cap \mathrm{Herm}(n)$. But clearly,

$$\mathrm{O}(n, \mathbb{C}) \cap \mathrm{U}(n) = \mathrm{O}(n, \mathbb{R}),$$
$$\mathrm{SO}(n, \mathbb{C}) \cap \mathrm{U}(n) = \mathrm{SO}(n, \mathbb{R})$$

while

$$\mathfrak{o}(n, \mathbb{C}) \cap \mathrm{Herm}(n) = \sqrt{-1}\mathfrak{o}(n, \mathbb{R}).$$

Hence, the maps $(U, X) \mapsto Ue^X$ restrict to polar decompositions

$$\mathrm{O}(n, \mathbb{R}) \times \sqrt{-1}\mathfrak{o}(n, \mathbb{R}) \to \mathrm{O}(n, \mathbb{C}),$$
$$\mathrm{SO}(n, \mathbb{R}) \times \sqrt{-1}\mathfrak{o}(n, \mathbb{R}) \to \mathrm{SO}(n, \mathbb{C}),$$

which shows that the algebraic topology of the complex orthogonal and special orthogonal group coincides with that of its real counterparts. Arguing as in the real case, the center of $\mathrm{O}(n, \mathbb{C})$ is given by $\{+I, -I\}$ while the center of $\mathrm{SO}(n, \mathbb{C})$ is trivial for n odd and $\{+I, -I\}$ for n even, provided $n \geq 3$.

7. Lagrangian Grassmannians

If (V, B) is a quadratic vector space with split bilinear form, denote by $\mathrm{Lag}(V)$ the set of Lagrangian subspaces. Recall that any such V is isomorphic to $\mathbb{K}^{n,n}$ where $\dim V = 2n$. For $\mathbb{K} = \mathbb{R}$ we have the following result.

THEOREM 7.1. *Let $V = \mathbb{R}^{n,n}$ with the standard basis satisfying* (7)*. Then the maximal compact subgroup $\mathrm{O}(n) \times \mathrm{O}(n)$ of $\mathrm{O}(n, n)$ acts transitively on the space $\mathrm{Lag}(\mathbb{R}^{n,n})$ of Lagrangian subspaces, with stabilizer at*

$$L_0 = \mathrm{span}\{\epsilon_1 + \tilde{\epsilon}_1, \ldots, \epsilon_n + \tilde{\epsilon}_n\} \tag{14}$$

the diagonal subgroup $\mathrm{O}(n)_\Delta$. Thus

$$\mathrm{Lag}(\mathbb{R}^{n,n}) \cong \mathrm{O}(n) \times \mathrm{O}(n)/\,\mathrm{O}(n)_\Delta \cong \mathrm{O}(n).$$

In particular, it is a compact space with two connected components.

PROOF. Let B_0 be the standard positive definite bilinear form on the vector space $\mathbb{R}^{n,n} = \mathbb{R}^{2n}$, with corresponding orthogonal group $\mathrm{O}(2n)$. Introduce an involution $\tau \in \mathrm{O}(2n)$, by

$$B(v,w) = B_0(\tau v, w).$$

That is $\tau\epsilon_i = \epsilon_i$, $\tau\tilde{\epsilon}_i = -\tilde{\epsilon}_i$. Then the maximal compact subgroup $\mathrm{O}(n) \times \mathrm{O}(n)$ consists of all those transformations $A \in \mathrm{O}(n,n)$ which commute with τ. At the same time, $\mathrm{O}(n) \times \mathrm{O}(n)$ is characterized as the orthogonal transformations in $\mathrm{O}(2n)$ commuting with τ.

The ± 1 eigenspaces $V_\pm$ of τ are both anisotropic, i.e. they do not contain any isotropic vectors. Hence, for any $L \subset \mathbb{R}^{n,n}$ is Lagrangian, then $\tau(L)$ is transverse to L:

$$L \cap \tau(L) = (L \cap V_+) \oplus (L \cap V_-) = 0.$$

For any L, we may choose a basis $v_1, \ldots, v_n$ that is orthonormal relative to B_0. Then $v_1, \ldots, v_n, \tau(v_1), \ldots, \tau(v_n)$ is a B_0-orthonormal basis of $\mathbb{R}^{n,n}$. If L' is another Lagrangian subspace, with B_0-orthonormal basis $v_1', \ldots, v_n'$, then the orthogonal transformation $A \in \mathrm{O}(2n)$ given by

$$Av_i = v_i', \quad A\tau(v_i) = \tau(v_i'), \; i = 1, \ldots, n$$

commutes with τ, hence $A \in \mathrm{O}(n) \times \mathrm{O}(n)$. This shows that $\mathrm{O}(n) \times \mathrm{O}(n)$ acts transitively on $\mathrm{Lag}(\mathbb{R}^{n,n})$. For the Lagrangian subspace (14), with $v_i = \frac{1}{\sqrt{2}}(\epsilon_i + \tilde{\epsilon}_i)$, the stabilizer of L_0 under the action of $\mathrm{O}(n) \times \mathrm{O}(n)$ consists of those transformations $A \in \mathrm{O}(n) \times \mathrm{O}(n)$ for which $v_1', \ldots, v_n'$ is again a B_0-orthonormal basis of L_0. But this is just the diagonal subgroup $\mathrm{O}(n)_\Delta \subset \mathrm{O}(n) \times \mathrm{O}(n)$. Finally, since the multiplication map

$$(\mathrm{O}(n) \times \{1\}) \times \mathrm{O}(n)_\Delta \to \mathrm{O}(n) \times \mathrm{O}(n)$$

is a bijection, the quotient is just $\mathrm{O}(n)$. □

Theorem 7.1 does not, as it stands, hold for other fields $\mathbb{K}$. Indeed, for $V = \mathbb{K}^{n,n}$ the group $\mathrm{O}(n,\mathbb{K}) \times \mathrm{O}(n,\mathbb{K})$ takes (14) to a Lagrangian subspace transverse to $V_+ = \mathbb{K}^n \oplus 0$, $V_- = 0 \oplus \mathbb{K}^n$, and any Lagrangian subspace transverse to V_+, V_- is of this form. However, there may be other Lagrangian subspaces: E.g. if $\mathbb{K} = \mathbb{C}$ and $n = 2$, the span of $\epsilon_1 + \sqrt{-1}\epsilon_2$ and $\tilde{\epsilon}_1 + \sqrt{-1}\tilde{\epsilon}_2$ is a Lagrangian subspace that is not transverse to $V_\pm$. Nonetheless, there is a good description of the space Lag in the complex case $\mathbb{K} = \mathbb{C}$.

THEOREM 7.2. *Let* $V = \mathbb{C}^{2m}$ *with the standard bilinear form. The action of the maximal compact subgroup* $\mathrm{O}(2m) \subset \mathrm{O}(2m,\mathbb{C})$ *on* $\mathrm{Lag}(V)$ *is transitive, with stabilizer at the Lagrangian subspace*

$$L_0 = \mathrm{span}\{\epsilon_1 - \sqrt{-1}\epsilon_{m+1}, \ldots, \epsilon_m - \sqrt{-1}\epsilon_{2m}\}$$

the unitary group $\mathrm{U}(m)$. *That is,* $\mathrm{Lag}(\mathbb{C}^{2m})$ *is a homogeneous space*

$$\mathrm{Lag}(\mathbb{C}^{2m}) \cong \mathrm{O}(2m)/\,\mathrm{U}(m);$$

In particular, it is a compact and has two connected components.

PROOF. Let $\tau\colon v \mapsto \overline{v}$ be complex conjugation in $V = \mathbb{C}^{2m}$, so that $\langle v, w\rangle = B(\overline{v}, w)$ is the standard Hermitian inner product on $\mathbb{C}^{2m}$. Then $\tau \in \mathrm{O}(4m, \mathbb{R})$ is a real orthogonal transformation of $\mathbb{C}^{2m} \cong \mathbb{R}^{4m}$. Let $V_\pm \subset \mathbb{C}^{2m}$ be the ± 1 eigenspaces of τ; thus $V_+ = \mathbb{R}^{2m}$ (viewed as a real subspace) and $V_- = \sqrt{-1}\mathbb{R}^{2m}$. Note that $V_\pm$ do not contain non-zero isotropic vectors. Hence, for $L \in \mathrm{Lag}(V)$ we have $L \cap \tau(L) \neq 0$, and hence $V = L \oplus \tau(L)$ is a direct sum. Let $v_1, \ldots, v_n$ be a basis of L that is orthonormal for the Hermitian inner product. Then $v_1, \ldots, v_n, \overline{v}_1, \ldots, \overline{v}_n$ is an orthonormal basis of V. Given another Lagrangian subspace L' with orthonormal basis $v_1,', \ldots, v_n'$, the unitary transformation $A \in \mathrm{U}(2m)$ with $Av_i = v_i'$ and $A\overline{v}_i = \overline{v}_i'$ commutes with τ, hence it is contained in $\mathrm{O}(2m) \subset \mathrm{U}(2m)$. This shows that $\mathrm{O}(2m)$ acts transitively. Note that any unitary transformation $U\colon L \to L'$ between Lagrangian subspaces extends uniquely to an element A of the maximal compact subgroup $\mathrm{O}(2m) \subset \mathrm{O}(2m, \mathbb{C})$, where $Av = Uv$ for $v \in L$ and $A\tau v = \tau U v$ for $\tau v \in \tau L$. In particular, the stabilizer in $\mathrm{O}(2m)$ of L_0 is the unitary group $U(L_0) \cong U(n)$. □

REMARK 7.3. The orbit of L_0 under $\mathrm{O}(m, \mathbb{C}) \times \mathrm{O}(m, \mathbb{C})$ is open and dense in $\mathrm{Lag}(\mathbb{C}^{2m})$, and as in the real case is identified with $\mathrm{O}(m, \mathbb{C})$. Thus, $\mathrm{Lag}(\mathbb{C}^{2m})$ is a smooth compactification of the complex Lie group $\mathrm{O}(m, \mathbb{C})$.

Theorem 7.2 has a well-known geometric interpretation. View $\mathbb{C}^{2m}$ as the complexification of $\mathbb{R}^{2m}$. Recall that an *orthogonal complex structure* on $\mathbb{R}^{2m}$ is an automorphism $J \in \mathrm{O}(2m)$ with $J^2 = -I$. We denote by J_0 the standard complex structure.

Let $\mathcal{J}(2m)$ denote the space of all orthogonal complex structures. It carries a transitive action of $\mathrm{O}(2m)$, with stabilizer at J_0 equal to $\mathrm{U}(m)$. Hence the space of orthogonal complex structures is identified with the complex Lagrangian Grassmannian:

$$\mathcal{J}(2m) = \mathrm{O}(2m)/\,\mathrm{U}(m) = \mathrm{Lag}(\mathbb{C}^{2m}).$$

Explicitly, this correspondence takes $J \in \mathcal{J}(2m)$ to its $+\sqrt{-1}$ eigenspace

$$L = \ker(J - \sqrt{-1}I).$$

This has complex dimension m since $\mathbb{C}^{2m} = L \oplus \overline{L}$, and it is isotropic since $v \in L$ implies

$$B(v, v) = B(Jv, Jv) = B(\sqrt{-1}v, \sqrt{-1}v) = -B(v, v).$$

Any Lagrangian subspace L determines J, as follows: Given $w \in \mathbb{R}^{2n}$, we may uniquely write $w = v + \overline{v}$ where $v \in L$. Define a linear map J by $Jw := -2\,\mathrm{Im}(v)$. Then $v = w - \sqrt{-1}Jw$. Since L is Lagrangian, we have

$$\begin{aligned} 0 = B(v, v) &= B(w - \sqrt{-1}Jw, w - \sqrt{-1}Jw) \\ &= B(w, w) - B(Jw, Jw) - 2\sqrt{-1}B(w, Jw), \end{aligned}$$

which shows that $J \in \mathrm{O}(2m)$ and that $B(w, Jw) = 0$ for all w. Multiplying the definition of J by $\sqrt{-1}$, we get

$$\sqrt{-1}v = \sqrt{-1}w + Jw$$

which shows that $J(Jw) = -w$. Hence J is an orthogonal complex structure.

REMARK 7.4. There are parallel results in symplectic geometry, for vector spaces V with a non-degenerate *skew*-symmetric linear form ω. If $\mathbb{K} = \mathbb{R}$, any such V is identified with $\mathbb{R}^{2n} = \mathbb{C}^n$ with the standard symplectic form, $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$ is a Lagrangian subspace, and the action of $\mathrm{U}(n) \subset \mathrm{Sp}(V, \omega)$ on L_0 identifies

$$\mathrm{Lag}_\omega(\mathbb{R}^{2n}) \cong \mathrm{U}(n)/\,\mathrm{O}(n)$$

For the space $\mathrm{Lag}(V)$ of complex Lagrangian Grassmannian subspaces of the complex symplectic vector space $\mathbb{C}^{2n} \cong \mathbf{H}^n$ one has

$$\mathrm{Lag}_\omega(\mathbb{C}^{2n}) \cong \mathrm{Sp}(n)/U(n)$$

where $\mathrm{Sp}(n)$ is the *compact symplectic group* (i.e. the quaternionic unitary group). See e.g. [**?**, p.67].

CHAPTER 2

Clifford algebras

1. Exterior algebras

1.1. Definition. For any vector space V over a field $\mathbb{K}$, let $T(V) = \bigoplus_{k\in\mathbb{Z}} T^k(V)$ be the tensor algebra, with $T^k(V) = V \otimes \cdots \otimes V$ the k-fold tensor product. The quotient of $T(V)$ by the two-sided ideal $\mathcal{I}(V)$ generated by all $v \otimes w + w \otimes v$ is the exterior algebra, denoted $\wedge(V)$. The product in $\wedge(V)$ is usually denoted $\alpha_1 \wedge \alpha_2$, although we will frequently omit the wedge sign and just write $\alpha_1\alpha_2$. Since $\mathcal{I}(V)$ is a *graded* ideal, the exterior algebra inherits a grading

$$\wedge(V) = \bigoplus_{k\in\mathbb{Z}} \wedge^k(V)$$

where $\wedge^k(V)$ is the image of $T^k(V)$ under the quotient map. Clearly, $\wedge^0(V) = \mathbb{K}$ and $\wedge^1(V) = V$ so that we can think of V as a subspace of $\wedge(V)$. We may thus think of $\wedge(V)$ as the associative algebra linearly generated by V, subject to the relations $vw + wv = 0$.

We will write $|\phi| = k$ if $\phi \in \wedge^k(V)$. The exterior algebra is *commutative* (in the graded sense). That is, for $\phi_1 \in \wedge^{k_1}(V)$ and $\phi_2 \in \wedge^{k_2}(V)$,

$$[\phi_1, \phi_2] := \phi_1\phi_2 + (-1)^{k_1k_2}\phi_2\phi_1 = 0.$$

If V has finite dimension, with basis $e_1, \ldots, e_n$, the space $\wedge^k(V)$ has basis

$$e_I = e_{i_1} \cdots e_{i_k}$$

for all ordered subsets $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$. (If $k = 0$, we put $e_\emptyset = 1$.) In particular, we see that $\dim \wedge^k(V) = \binom{n}{k}$, and

$$\dim \wedge(V) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Letting $e^i \in V^*$ denote the dual basis to the basis e_i considered above, we define a dual basis to e_I to be the basis $e^I = e^{i_1} \cdots e^{i_k} \in \wedge(V^*)$.

1.2. Universal property, functoriality. The exterior algebra is characterized by its *universal property*: If $\mathcal{A}$ is an algebra, and $f\colon V \to \mathcal{A}$ a linear map with $f(v)f(w) + f(w)f(v) = 0$ for all $v, w \in V$, then f extends uniquely to an algebra homomorphism $f_\wedge\colon \wedge(V) \to \mathcal{A}$. Thus, is $\widetilde{\wedge}(V)$ is another algebra with a homomorphism $V \to \widetilde{\wedge}(V)$, satisfying this universal property, then there is a unique isomorphism $\wedge(V) \to \widetilde{\wedge}(V)$ intertwining the two inclusions of V.

Any linear map $L\colon V \to W$ extends uniquely (by the universal property, applied to L viewed as a map into $V \to \wedge(W)$) to an algebra homomorphism $\wedge(L)\colon \wedge(V) \to \wedge(W)$. One has $\wedge(L_1 \circ L_2) = \wedge(L_1) \circ \wedge(L_2)$. As a special case, taking L to be the zero map $0\colon V \to V$ the resulting algebra homomorphism $\wedge(L)$ is the *augmentation map* (taking $\phi \in \wedge(V)$ to its component in $\wedge^0(V) \cong \mathbb{K}$). Taking L to be the map $v \mapsto -v$, the map $\wedge(L)$ is the *parity homomorphism* $\Pi \in \mathrm{Aut}(\wedge(V))$, equal to $(-1)^k$ on $\wedge^k(V)$.

The functoriality gives in particular a group homomorphism [1]

$$\mathrm{GL}(V) \to \mathrm{Aut}(\wedge(V)),\ g \mapsto \wedge(g)$$

into the group of algebra automorphisms of V. We will often write g in place of $\wedge(g)$, but reserve this notation for invertible transformations since e.g. $\wedge(0) \neq 0$.

As another application of the universal property, suppose V_1, V_2 are two vector spaces, and define $\wedge(V_1) \otimes \wedge(V_2)$ as the tensor product of graded algebras. This tensor product contains $V_1 \oplus V_2$ as a subspace, and satisfies the universal property of the exterior algebra over $V_1 \oplus V_2$. Hence there is a unique algebra isomorphism

$$\wedge(V_1 \oplus V_2) \to \wedge(V_1) \otimes \wedge(V_2)$$

intertwining the inclusions of $V_1 \oplus V_2$. It is clear that this isomorphism preserves gradings.

For $\alpha \in V^*$, define the contraction operators $\iota(\alpha) \in \mathrm{End}(\wedge(V))$ by $\iota(\alpha)1 = 0$ and

$$\iota(\alpha)(v_1 \wedge \cdots v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i \rangle\ v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_k. \tag{15}$$

On the other hand, for $v \in V$ we have the operator $\epsilon(v) \in \mathrm{End}(\wedge V)$ of exterior multiplication by v. These satisfy the relations

$$\begin{aligned} \iota(v)\epsilon(w) + \epsilon(w)\epsilon(v) &= 0, \\ \iota(\alpha)\iota(\beta) + \iota(\beta)\iota(\alpha) &= 0, \\ \iota(\alpha)\epsilon(v) + \epsilon(v)\iota(\alpha) &= \langle \alpha, v \rangle. \end{aligned} \tag{16}$$

For later reference, let us also observe that the kernel of $\iota(\alpha)$ is the exterior algebra over $\ker(\alpha) \subset V$; hence $\bigcap_{\alpha \in V^*} \ker(\iota(\alpha)) = 0$.

2. Clifford algebras

2.1. Definition and first properties. Let V be a vector space over $\mathbb{K}$, with a symmetric bilinear form $B\colon V \times V \to \mathbb{K}$ (possibly degenerate).

[1] If $\mathcal{A}$ is any algebra, we denote by $\mathrm{End}(\mathcal{A})$ (resp. $\mathrm{Aut}(\mathcal{A})$) the vector space homomorphisms (res. automorphisms) $\mathcal{A} \to \mathcal{A}$, while $\mathrm{End}_{\mathrm{alg}}(\mathcal{A})$ (resp. $\mathrm{Aut}_{\mathrm{alg}}(V)$) denotes the set of algebra homomorphisms (resp. group of algebra automorphisms).

DEFINITION 2.1. The *Clifford algebra* $\mathrm{Cl}(V;B)$ is the quotient

$$\mathrm{Cl}(V;B)=T(V)/\mathcal{I}(V;B)$$

where $\mathcal{I}(V;B)\subset T(V)$ is the two-sided ideal generated by all

$$v\otimes w+w\otimes v-2B(v,w)1,\quad v,w\in V$$

Clearly, $\mathrm{Cl}(V;0)=\wedge(V)$. It is not obvious from the definition that $\mathrm{Cl}(V;B)$ is non-trivial, but this follows from the following Proposition.

PROPOSITION 2.2. *The inclusion* $\mathbb{K}\to T(V)$ *descends to an inclusion* $\mathbb{K}\to\mathrm{Cl}(V;B)$. *The inclusion* $V\to T(V)$ *descends to an inclusion* $V\to\mathrm{Cl}(V;B)$.

PROOF. Consider the linear map

$$f\colon V\to\mathrm{End}(\wedge(V)),\ v\mapsto\epsilon(v)+\iota(B^\flat(v)).$$

and its extension to an algebra homomorphism $f_T\colon T(V)\to\mathrm{End}(\wedge(V))$. The commutation relations (16) show that $f(v)f(w)+f(w)f(v)=2B(v,w)1$. Hence f_T vanishes on the ideal $\mathcal{I}(V;B)$, and therefore descends to an algebra homomorphism

$$f_{\mathrm{Cl}}\colon\mathrm{Cl}(V;B)\to\mathrm{End}(\wedge(V)),\tag{17}$$

i.e. $f_{\mathrm{Cl}}\circ\pi=f_T$ where $\pi\colon T(V)\to\mathrm{Cl}(V;B)$ is the projection. Since $f_T(1)=1$, we see that $\pi(1)\neq 0$, i.e. the inclusion $\mathbb{K}\hookrightarrow T(V)$ descends to an inclusion $\mathbb{K}\hookrightarrow\mathrm{Cl}(V;B)$. Similarly, from $f_T(v).1=v$ we see that the inclusion $V\hookrightarrow T(V)$ descends to an inclusion $V\hookrightarrow\mathrm{Cl}(V;B)$. ☐

The Proposition shows that V is a subspace of $\mathrm{Cl}(V;B)$. We may thus characterize $\mathrm{Cl}(V;B)$ as the unital associative algebra, with generators $v\in V$ and relations

$$vw+wv=2B(v,w)1,\quad v,w\in V.\tag{18}$$

Let $T(V)$ carry the $\mathbb{Z}_2$-grading

$$T^{\bar 0}(V)=\bigoplus_{k=0}^{\infty}T^{2k}(V),\ T^{\bar 1}(V)=\bigoplus_{k=0}^{\infty}T^{2k+1}(V).$$

(Here $\bar k$ denotes $k \mod 2$.) Since the elements $v\otimes w+w\otimes v-2B(v,w)1$ are even, the ideal $\mathcal{I}(V;B)$ is $\mathbb{Z}_2$ graded, i.e. it is a direct sum of the subspaces $\mathcal{I}^{\bar k}(V;B)=\mathcal{I}(V;B)\cap T^{\bar k}(V)$ for $k=0,1$. Hence the Clifford algebra inherits a $\mathbb{Z}_2$-grading,

$$\mathrm{Cl}(V;B)=\mathrm{Cl}^{\bar 0}(V;B)\oplus\mathrm{Cl}^{\bar 1}(V;B).$$

The two summands are spanned by products $v_1\cdots v_k$ with k even, respectively odd. From now on, commutators $[\cdot,\cdot]$ in the Clifford algebra $\mathrm{Cl}(V;B)$ will denote $\mathbb{Z}_2$-graded commutators. (We will write $[\cdot,\cdot]_{\mathrm{Cl}}$ if there is risk of confusion.) In this notation, the defining relations for the Clifford algebra become

$$[v,w]=2B(v,w),\quad v,w\in V.$$

If $\dim V = n$, and e_i are an orthogonal basis of V, then (using the same notation as for the exterior algebra), the products

$$e_I = e_{i_1} \cdots e_{i_k}, \quad I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\},$$

with the convention $e_\emptyset = 1$, span $\mathrm{Cl}(V; B)$. We will see in Section 2.5 that the e_I are a basis.

2.2. Universal property, functoriality. The Clifford algebra is characterized by the following by a universal property:

PROPOSITION 2.3. *Let $\mathcal{A}$ be an associative unital algebra, and $f\colon V \to \mathcal{A}$ a linear map satisfying*

$$f(v_1)f(v_2) + f(v_2)f(v_1) = 2B(v_1, v_2) \cdot 1, \quad v_1, v_2 \in V.$$

Then f extends uniquely to an algebra homomorphism $\mathrm{Cl}(V; B) \to \mathcal{A}$.

PROOF. By the universal property of the tensor algebra, f extends to an algebra homomorphism $f_{T(V)}\colon T(V) \to \mathcal{A}$. The property $f(v_1)f(v_2) + f(v_2)f(v_1) = 2B(v_1, v_2) \cdot 1$ shows that f vanishes on the ideal $\mathcal{I}(V; B)$, and hence descends to the Clifford algebra. Uniqueness is clear, since the Clifford algebra is generated by elements of V. □

Suppose B_1, B_2 are symmetric bilinear forms on V_1, V_2, and $f\colon V_1 \to V_2$ is a linear map such that

$$B_2(f(v), f(w)) = B_1(v, w), \quad v, w \in V_1.$$

Viewing f as a map into $\mathrm{Cl}(V_2; B_2)$, the universal property provides a unique extension

$$\mathrm{Cl}(f)\colon \mathrm{Cl}(V_1; B_1) \to \mathrm{Cl}(V_2; B_2).$$

For instance, if $F \subset V$ is an isotropic subspace of V, there is an algebra homomorphism $\wedge(F) = \mathrm{Cl}(F) \to \mathrm{Cl}(V; B)$. Clearly, $\mathrm{Cl}(f_1 \circ f_2) = \mathrm{Cl}(f_1) \circ \mathrm{Cl}(f_2)$. Taking $V_1 = V_2 = V$, and restricting attention to invertible linear maps, one obtains a group homomorphism

$$\mathrm{O}(V; B) \to \mathrm{Aut}(\mathrm{Cl}(V; B)), \; g \mapsto \mathrm{Cl}(g).$$

We will usually just write g in place of $\mathrm{Cl}(g)$. For example, the involution $v \mapsto -v$ lies in $O(V; B)$, hence it defines an involutive algebra automorphism Π of $\mathrm{Cl}(V; B)$ called the *parity automorphism*. The ± 1 eigenspaces are the even and odd part of the Clifford algebra, respectively.

Suppose again that (V, B_1) and (V_2, B_2) be two vector spaces with symmetric bilinear forms, and consider the direct sum $(V_1 \oplus V_2, \; B_1 \oplus B_2)$. Then

$$\mathrm{Cl}(V_1 \oplus V_2; B_1 \oplus B_2) = \mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2)$$

as $\mathbb{Z}_2$-graded algebras. This follows since $\mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2)$ satisfies the universal property of the Clifford algebra over $(V_1 \oplus V_2; B_1 \oplus B_2)$. In particular, if $\mathrm{Cl}(n, m)$ denotes the Clifford algebra for $\mathbb{K}^{n,m}$ we have

$$\mathrm{Cl}(n, m) = \mathrm{Cl}(1, 0) \otimes \cdots \otimes \mathrm{Cl}(1, 0) \otimes \mathrm{Cl}(0, 1) \otimes \cdots \otimes \mathrm{Cl}(0, 1),$$

with $\mathbb{Z}_2$-graded tensor products.

2.3. The Clifford algebras $\mathrm{Cl}(n,m)$**.** Consider the case $\mathbb{K}=\mathbb{R}$. For n,m small one can determine the algebras $\mathrm{Cl}(n,m)=\mathrm{Cl}(\mathbb{R}^{n,m})$ by hand.

PROPOSITION 2.4. *For $\mathbb{K}=\mathbb{R}$, one has the following isomorphisms of the Clifford algebras $\mathrm{Cl}(n,m)$ with $n+m\le 2$, as ungraded algebras:*

$$\begin{aligned}\mathrm{Cl}(0,1)&\cong\mathbb{C}\\ \mathrm{Cl}(1,0)&\cong\mathbb{R}\oplus\mathbb{R},\\ \mathrm{Cl}(0,2)&\cong\mathbb{H},\\ \mathrm{Cl}(1,1)&\cong\mathrm{Mat}_2(\mathbb{R}),\\ \mathrm{Cl}(2,0)&\cong\mathrm{Mat}_2(\mathbb{R}).\end{aligned}$$

Here $\mathbb{C}$ and $\mathbb{H}$ are viewed as algebras over $\mathbb{R}$, and $\mathrm{Mat}_2(\mathbb{R})=\mathrm{End}(\mathbb{R}^2)$ is the algebra of real 2×2-matrices.

PROOF. By the universal property, an algebra $\mathcal{A}$ of dimension 2^{n+m} is isomorphic to $\mathrm{Cl}(n,m)$ if there exists a linear map $f\colon\mathbb{R}^{n,m}\to\mathcal{A}$ satisfying $f(e_i)f(e_j)+f(e_j)f(e_i)=\pm 2\delta_{ij}$, with a plus sign for $i\le n$ and a minus sign for $i>n$. We will describe these maps for $n+m\le 2$. For $(n,m)=(0,1)$ we take $f\colon\mathbb{R}^{0,1}\to\mathbb{C},\ e_1\mapsto i=\sqrt{-1}$. For $(n,m)=(1,0)$, we use $f\colon\mathbb{R}^{1,0}\to\mathbb{R}\oplus\mathbb{R},\ e_1\mapsto(1,-1)$. For $(n,m)=(0,2)$ we use

$$f(e_1)=\begin{pmatrix}\sqrt{-1}&0\\0&\sqrt{-1}\end{pmatrix},\quad f(e_2)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

(The matrices represent the first two of the standard unit quaternions $i,j,k=ij\in\mathcal{H}$.) For $(n,m)=(1,1)$ the relevant map is

$$f(e_1)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad f(e_2)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

The case $(n,m)=(2,0)$ is left as an exercise. □

The full classification of the Clifford algebras $\mathrm{Cl}(n,m)$ may be found in the book by Lawson-Michelsohn [**?**] or in the monograph by Budinich-Trautman [**?**]. The Clifford algebras exhibit a remarkable mod 8 periodicity,

$$\mathrm{Cl}(n+8,m)\cong\mathrm{Mat}_{16}(\mathrm{Cl}(n,m))\cong\mathrm{Cl}(n,m+8)$$

which is related to the mod 8 periodicity in real K-theory [**?**].

2.4. The Clifford algebras $\mathbb{C}l(n)$**.** For $\mathbb{K}=\mathbb{C}$ the pattern is simpler. Denote by $\mathbb{C}l(n)$ the Clifford algebra of $\mathbb{C}^n$.

PROPOSITION 2.5. *One has the following isomorphisms of algebras over $\mathbb{C}$,*

$$\mathbb{C}l(2m)=\mathrm{Mat}_{2^m}(\mathbb{C}),\quad\mathbb{C}l(2m+1)=\mathrm{Mat}_{2^m}(\mathbb{C})\oplus\mathrm{Mat}_{2^m}(\mathbb{C}).$$

More precisely, $\mathrm{Cl}(2m)=\mathrm{End}(\wedge\mathbb{C}^m)$ as a $\mathbb{Z}_2$-graded algebra, where the exterior algebra $\wedge\mathbb{C}^m$ carries its usual $\mathbb{Z}_2$-grading.

PROOF. Consider first $\mathbb{C}^2$. The map $f\colon \mathbb{C}^2 \to \mathrm{End}(\mathbb{C}^2)$,

$$f(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$$

extends to an isomorphism $\mathbb{C}l(2) \to \mathrm{End}(\mathbb{C}^2)$. The resulting $\mathbb{Z}_2$-grading on $\mathrm{End}(\mathbb{C}^2)$ is induced by the $\mathbb{Z}_2$-grading on $\mathbb{C}^2$ where the first component is even and the second is odd. Equivalently, it corresponds to the identification $\mathbb{C}^2 \cong \wedge\mathbb{C}$. This shows $\mathbb{C}l(2) \cong \mathrm{End}(\wedge\mathbb{C})$ as $\mathbb{Z}_2$-graded vector algebras. For $\mathbb{C}^{2m} = \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2$ we hence obtain

$$\begin{aligned} \mathbb{C}l(2m) &= \mathrm{Cl}(2) \otimes \cdots \otimes \mathrm{Cl}(2) \\ &\cong \mathrm{End}(\wedge\mathbb{C}) \otimes \cdots \otimes \mathrm{End}(\wedge\mathbb{C}) \\ &= \mathrm{End}(\wedge\mathbb{C} \otimes \cdots \otimes \wedge\mathbb{C}) \\ &= \mathrm{End}(\wedge\mathbb{C}^m), \end{aligned}$$

as $\mathbb{Z}_2$-graded algebras. The even subalgebra $\mathbb{C}l^{\bar{0}}(2m)$ preserves the $\mathbb{Z}_2$-grading on $\wedge\mathbb{C}^m$, hence we have an isomorphism

$$\mathbb{C}l^{\bar{0}}(2m) \to \mathrm{End}((\wedge\mathbb{C}^m)^{\bar{0}}) \oplus \mathrm{End}((\wedge\mathbb{C}^m)^{\bar{1}}) = \mathrm{Mat}_{2^{m-1}}(\mathbb{C}) \oplus \mathrm{Mat}_{2^{m-1}}(\mathbb{C})$$

as ungraded algebras. On the other hand, there is an (ungraded) isomorphism of algebras $\mathbb{C}l(2m-1) \to \mathbb{C}l^{\bar{0}}(2m)$, given on generators by $e_i \mapsto \sqrt{-1}\, e_i e_{2m}$ for $i < 2m$. □

The mod 2 periodicity

$$\mathbb{C}l(n+2) \cong \mathrm{Mat}_2(\mathbb{C}l(n))$$

appearent in this classification result is related to the mod 2 periodicity in complex K-theory [**?**]. For later reference, let us highlight the isomorphism (of ungraded algebras)

$$\mathbb{C}l(n) \to \mathbb{C}l^{\bar{0}}(n+1), \quad e_i \mapsto \sqrt{-1} e_i e_{n+1} \tag{19}$$

used in this argument.

2.5. Symbol map and quantization map. Returning to the algebra homomorphism $f_{\mathrm{Cl}}\colon \mathrm{Cl}(V;B) \to \mathrm{End}(\wedge V)$ (see (17)), given on generators by $f_{\mathrm{Cl}}(v) = \epsilon(v) + \iota(B^\flat(v))$, one defines the *symbol map*,

$$\sigma\colon \mathrm{Cl}(V;B) \to \wedge(V),\ x \mapsto f_{\mathrm{Cl}}(x).1$$

where $1 \in \wedge^0(V) = \mathbb{K}$.

PROPOSITION 2.6. *The symbol map is an isomorphism of vector spaces. In low degrees,*

$$\begin{aligned} \sigma(1) &= 1 \\ \sigma(v) &= v \\ \sigma(v_1 v_2) &= v_1 \wedge v_2 + B(v_1, v_2), \\ \sigma(v_1 v_2 v_3) &= v_1 \wedge v_2 \wedge v_3 + B(v_2, v_3) v_1 - B(v_1, v_3) v_2 + B(v_1, v_2) v_3. \end{aligned}$$

PROOF. Let $e_i \in V$ be an orthogonal basis. Since the operators $f(e_i)$ commute (in the grade sense), we find

$$\sigma(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

for $i_1 < \cdots < i_k$. This directly shows that the symbol map is an isomorphism: It takes the element $e_I \in \mathrm{Cl}(V;B)$ to the corresponding element $e_I \in \wedge(V)$. The formulas in low degrees are obtained by straightforward calculation. □

The inverse map is called the *quantization map*

$$q\colon \wedge(V) \to \mathrm{Cl}(V;B).$$

In terms of the basis, $q(e_I) = e_I$. In low degrees,

$$\begin{aligned} q(1) &= 1, \\ q(v) &= v, \\ q(v_1 \wedge v_2) &= v_1 v_2 - B(v_1, v_2), \\ q(v_1 \wedge v_2 \wedge v_3) &= v_1 v_2 v_3 - B(v_2, v_3) v_1 + B(v_1, v_3) v_2 - B(v_1, v_2) v_3. \end{aligned}$$

If $\mathbb{K}$ has characteristic 0 (so that division by all non-zero integers is defined), the quantization map has the following alternative description.

PROPOSITION 2.7. *Suppose $\mathbb{K}$ has characteristic 0. Then the quantization map is given by graded symmetrization. That is, for $v_1, \ldots, v_k \in V$,*

$$q(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \mathrm{sign}(s) v_{s(1)} \cdots v_{s(k)}.$$

Here $\mathfrak{S}_k$ is the group of permutations of $1, \ldots, k$ and $\mathrm{sign}(s) = \pm 1$ is the parity of a permutation s.

PROOF. By linearity, it suffices to check for the case that the v_j are elements of an orthonormal basis $e_1, \ldots, e_n$ of V, that is $v_j = e_{i_j}$ (the indices i_j need not be ordered or distinct). If the i_j are all distinct, then the e_{i_j} Clifford commute in the graded sense, and the right hand side equals $e_{i_1} \cdots e_{i_k} \in \mathrm{Cl}(V;B)$, which coincides with the left hand side. If any two e_{i_j} coincide, then both sides are zero. □

2.6. $\mathbb{Z}$-filtration. The increasing filtration

$$T_{(0)}(V) \subset T_{(1)}(V) \subset \cdots$$

with $T_{(k)}(V) = \bigoplus_{j \le k} T^j(V)$ descends to a filtration

$$\mathrm{Cl}_{(0)}(V;B) \subset \mathrm{Cl}_{(1)}(V;B) \subset \cdots$$

of the Clifford algebra, with $\mathrm{Cl}_{(k)}(V;B)$ the image of $T_{(k)}(V)$ under the quotient map. Equivalently, $\mathrm{Cl}_{(k)}(V;B)$ consists of linear combinations of products $v_1 \cdots v_l$ with $l \le k$ (including scalars, viewed as products of length 0). The filtration is compatible with product map, that is,

$$\mathrm{Cl}_{(k_1)}(V;B)\mathrm{Cl}_{(k_2)}(V;B) \subset \mathrm{Cl}_{(k_1+k_2)}(V;B).$$

Thus, $\mathrm{Cl}(V;B)$ is a *filtered algebra*. Let $\mathrm{gr}(\mathrm{Cl}(V;B))$ be the associated graded algebra.

PROPOSITION 2.8. *The symbol map induces an isomorphism of associated graded algebras*

$$\mathrm{gr}(\sigma)\colon\ \mathrm{gr}(\mathrm{Cl}(V;B)) \to \wedge(V).$$

PROOF. The symbol map and the quantization map are filtration preserving, hence they descend to isomorphisms of the associated graded vector spaces. Let $\pi_{\mathrm{Cl}}\colon T(V) \to \mathrm{Cl}(V;B)$ and $\pi_\wedge\colon T(V) \to \wedge(V)$ be the quotient maps. By definition of the symbol map, the composition $\sigma \circ \pi_{\mathrm{Cl}}\colon T_{(k)}(V) \to \wedge(V)$ coincides with $\pi_\wedge\colon T_{(k)}(V) \to \wedge(V)$ up to lower order terms. Passing to the associated graded maps, this gives

$$\mathrm{gr}(\sigma) \circ \mathrm{gr}(\pi_{\mathrm{Cl}}) = \pi_\wedge.$$

Since π_{Cl} is a surjective algebra homomorphism, so is $\mathrm{gr}(\pi_{\mathrm{Cl}})$. It hence follows that $\mathrm{gr}(\sigma)$ is an algebra homomorphism as well. □

Note that the symbol map $\sigma\colon \mathrm{Cl}(V;B) \to \wedge(V)$ preserves the $\mathbb{Z}_2$-grading. The even (resp. odd) elements of $\mathrm{Cl}(V;B)$ are linear combinations of products $v_1 \cdots v_k$ with k even (resp. odd). The filtration is also compatible with the $\mathbb{Z}_2$-grading, that is, each $\mathrm{Cl}_{(k)}(V;B)$ is a $\mathbb{Z}_2$-graded subspace. In fact,

$$\begin{aligned} \mathrm{Cl}^{\bar 0}_{(2k)}(V;B) &= \mathrm{Cl}^{\bar 0}_{(2k+1)}(V;B), \\ \mathrm{Cl}^{\bar 1}_{(2k+1)}(V;B) &= \mathrm{Cl}^{\bar 1}_{(2k+2)}(V;B). \end{aligned} \tag{20}$$

2.7. Transposition. An anti-automorphism of an algebra $\mathcal{A}$ is an invertible linear map $f\colon \mathcal{A} \to \mathcal{A}$ with the property $f(ab) = f(b)f(a)$ for all $a, b \in \mathcal{A}$. Put differently, if $\mathcal{A}^{\mathrm{op}}$ is $\mathcal{A}$ with the opposite algebra structure $a \cdot_{\mathrm{op}} b := ba$, an anti-automorphism is an algebra isomorphism $\mathcal{A} \to \mathcal{A}^{\mathrm{op}}$.

The tensor algebra carries a unique involutive anti-automorphism that is equal to the identity on $V \subset T(V)$. It is called the *canonical anti-automorphism* or *transposition*, and is given by

$$(v_1 \otimes \cdots \otimes v_k)^\top = v_k \otimes \cdots \otimes v_1.$$

Since transposition preserves the ideal $\mathcal{I}(V)$ defining the exterior algebra, it descends to an anti-automorphism of the exterior algebra, $\phi \mapsto \phi^\top$. In fact, since transposition is given by a permutation of length $(k-1)+\cdots+2+1 = k(k-1)/2$, we have

$$\phi^\top = (-1)^{k(k-1)/2}\phi, \quad \phi \in \wedge^k(V).$$

Given a symmetric bilinear form B ob V the transposition anti-automorphism of the tensor algebra also preserves the ideal $\mathcal{I}(V;B)$, and hence descends to an anti-automorphism of $\mathrm{Cl}(V;B)$, still called *canonical anti-automorphism* or *transposition*, with

$$(v_1 \cdots v_k)^\top = v_k \cdots v_1.$$

The symbol map and its inverse, the quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V;B)$ intertwines the transposition maps for $\wedge(V)$ and $\mathrm{Cl}(V;B)$. This information is sometimes useful for computations.

EXAMPLE 2.9. Suppose $\phi \in \wedge^k(V)$, and consider the square of $q(\phi)$. The element $q(\phi)^2 \in \mathrm{Cl}(V)$ is even, and is hence contained in $\mathrm{Cl}^{\bar{0}}_{(2k)}(V)$. But $(q(\phi)^2)^\top = (q(\phi)^\top)^2 = q(\phi)^2$ since $q(\phi)^\top = q(\phi^\top) = \pm q(\phi)$. It follows that

$$q(\phi)^2 \in q\big(\wedge^0(V) \oplus \wedge^4(V) \oplus \cdots \oplus \wedge^{4r}(V)\big),$$

where r is the largest number with $2r \le k$.

2.8. Chirality element, trace. Let $\dim V = n$. Then any generator $\Gamma_\wedge \in \det(V) := \wedge^n(V)$ quantizes to given an element $\Gamma = q(\Gamma_\wedge)$. This element (or suitable normalizations of this element) is called the *chirality element* of the Clifford algebra. The square Γ^2 of the chirality element is always a scalar, as is immediate by choosing an orthogonal basis e_i, and letting $\Gamma = e_1 \cdots e_n$. In fact, since $\Gamma^\top = (-1)^{n(n-1)/2}\Gamma$ we have

$$\Gamma^2 = (-1)^{n(n-1)/2} \prod_{i=1}^n B(e_i, e_i).$$

In the case $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^n$ we can always normalize Γ to satisfy $\Gamma^2 = 1$; this normalization determines Γ up to sign. For any $v \in V$, we have $\Gamma v = (-1)^{n-1} v\Gamma$, as one checks e.g. using an orthogonal basis. (If $v = e_i$, then v anti-commutes with all e_j for $j \neq i$ in the product $\Gamma = e_1 \cdots e_n$, and commutes with e_i. Hence we obtain $n-1$ sign changes.)

$$\Gamma v = \begin{cases} v\Gamma & \text{if } n \text{ is odd} \\ -v\Gamma & \text{if } n \text{ is even} \end{cases}$$

Thus, if n is odd then Γ lies in the center of $\mathrm{Cl}(V;B)$, viewed as an ordinary algebra. In the case that n is even, we obtain

$$\Pi(x) = \Gamma x \Gamma^{-1},$$

for all $x \in \mathrm{Cl}(V;B)$, i.e. the chirality element *implements* the parity automorphism.

For any $\mathbb{Z}_2$-graded algebra $\mathcal{A}$ and vector space Y, a Y-valued *super-trace* on $\mathcal{A}$ is a linear map $\mathrm{tr}_s\colon \mathcal{A} \to Y$ vanishing on the subspace $[\mathcal{A},\mathcal{A}]$ spanned by super-commutators: That is, $\mathrm{tr}_s([x,y]) = 0$ for $x, y \in \mathcal{A}$.

PROPOSITION 2.10. *Suppose $n = \dim V < \infty$. The linear map*

$$\mathrm{tr}_s\colon \mathrm{Cl}(V;B) \to \det(V)$$

given as the quotient map to $\mathrm{Cl}_{(n)}(V;B)/\mathrm{Cl}_{(n-1)}(V;B) \cong \wedge^n(V) = \det(V)$, is a super-trace on $\mathrm{Cl}(V;B)$.

PROOF. Let e_i be an orthogonal basis, and e_I the assocated basis of $\mathrm{Cl}(V;B)$. Then $\mathrm{tr}_s(e_I) = 0$ unless $I = \{1,\ldots,n\}$. The product e_I, e_J is of the form $e_I e_J = c e_K$ where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$. Hence $\mathrm{tr}_s(e_I e_J) = 0 = \mathrm{tr}_s(e_J e_I)$ unless $I \cap J = \emptyset$ and $I \cup J = \{1,\ldots,n\}$. But in case $I \cap J = \emptyset$, e_I, e_J super-commute: $[e_I, e_J] = 0$. □

The Clifford algebra also carries an *ordinary trace*, vanishing on ordinary commutators.

PROPOSITION 2.11. *The formula*

$$\mathrm{tr}\colon \mathrm{Cl}(V;B) \to \mathbb{K}, \quad x \mapsto \sigma(x)_{[0]}$$

defines an (ordinary) trace on $\mathrm{Cl}(V;B)$*, that is* $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ *for all* $x, y \in \mathrm{Cl}(V;B)$*. The trace satisfies* $\mathrm{tr}(x^\top) = \mathrm{tr}(x)$ *and* $\mathrm{tr}(1) = 1$*. For* $\dim V < \infty$*, the trace and the super-trace are related by the formula,*

$$\mathrm{tr}_s(\Gamma x) = \mathrm{tr}(x)\,\Gamma_\wedge$$

where $\Gamma = q(\Gamma_\wedge)$ *is the chirality element in the Clifford algebra defined by a choice of generator of* $\det(V)$*.*

PROOF. Again, we use an orthogonal basis e_i of V. The definition gives $\mathrm{tr}(e_\emptyset) = 1$, while $\mathrm{tr}(e_I) = 0$ for $I \neq \emptyset$. Consider a product $e_I e_J = c e_K$ where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$. The set K is non-empty (i.e. $\mathrm{tr}(e_I e_J) = 0$) unless $I = J$, but in the latter case the trace property is trivial. To check the formula relating trace and super-trace we may assume $\Gamma_\wedge = e_I$ with $I = \{1,\ldots,n\}$. For $x = e_J$ we see that $\mathrm{tr}_s(\Gamma x)$ vanishes unless $J = \emptyset$, in which case it is $\Gamma_\wedge$. □

2.9. Extension of the bilinear form. The symmetric bilinear form on V extends to a symmetric bilinear form on the exterior algebra $\wedge(V)$, by setting $B(\phi,\psi) = 0$ for $|\phi| \neq |\psi|$ and

$$B(v_1 \wedge \cdots \wedge v_k,\ w_1 \wedge \cdots \wedge w_k) = \det\big(B(v_i, w_j)_{i,j}\big).$$

On the other hand, using the trace on $\mathrm{Cl}(V;B)$ we also have an extension to the Clifford algebra:

$$B(x,y) = \mathrm{tr}(x^\top y).$$

PROPOSITION 2.12. *The quantization map* q *intertwines the bilinear forms on* $\wedge(V)$, $\mathrm{Cl}(V;B)$*.*

PROOF. We check in an orthogonal basis e_i of V. Indeed, for $I \neq J$ $B(e_I, e_J)$ vanishes in $\wedge(V)$, but also in $\mathrm{Cl}(V;B)$ since $e_I^\top e_J = \pm e_I e_J$ has trace zero. On the other hand, taking $I = J = \{i_1,\ldots,i_k\}$ we get $B(e_I, e_I) = \prod_{j=1}^k B(e_{i_j}, e_{i_j})$ in both the Clifford and exterior algebras. □

2.10. Lie derivatives and contractions. Let V be a vector space, and $\alpha \in V^*$. Then the map $\iota(\alpha)\colon V \to \mathbb{K},\ v \mapsto \langle \alpha, v\rangle$ extends uniquely to a degree -1 derivation of the tensor algebra $T(V)$, called *contraction*, by

$$\iota(\alpha)(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i\rangle\ v_1 \otimes \cdots \widehat{v_i} \cdots \otimes v_k$$

The contraction operators preserve the ideal $\mathcal{I}(V)$ defining the exterior algebra, and descend to the contraction operators on $\wedge(V)$. Given a symmetric bilinear form B on V, the contraction operators also preserve the ideal $\mathcal{I}(V;B)$ since

$$\iota(\alpha)(v_1 \otimes v_2 + v_2 \otimes v_1 - 2B(v_1, v_2)) = 0.\ \ v_1, v_2 \in V.$$

It follows that $\iota(\alpha)$ descends to an odd derivation of $\mathrm{Cl}(V;B)$ of filtration degree -1, with

$$\iota(\alpha)(v_1 \cdots v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i\rangle v_1 \cdots \widehat{v_i} \cdots v_k. \tag{21}$$

Similarly, any $A \in \mathfrak{gl}(V) = \mathrm{End}(V)$ extends to a derivation L_A of degree 0 on $T(V)$, called *Lie derivative*:

$$L_A(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} v_1 \otimes \cdots \otimes L_A(v_i) \otimes \cdots \otimes v_k.$$

L_A preserves the ideal $\mathcal{I}(V)$, and hence descends to a derivation of $\wedge(V)$. If $A \in \mathfrak{o}(V;B)$, that is $B(Av_1, v_2) + B(v_1, Av_2) = 0$ for all v_1, v_2, then L_A also preserves the ideal $\mathcal{I}(V;B)$ and consequently descends to an even derivation of $\mathrm{Cl}(V;B)$, of filtration degree 0.

One has (on the tensor algebra, and hence also on the exterior and Clifford algebras)

$$[\iota(\alpha_1), \iota(\alpha_2)] = 0,\ [L_{A_1}, L_{A_2}] = L_{[A_1, A_2]},\ [L_A, \iota(\alpha)] = \iota(A.\alpha),$$

where $A.\alpha = -A^*\alpha$ with A^* the dual map. This proves the first part of:

Proposition 2.13. *The map $A \mapsto L_A,\ \alpha \mapsto \iota(\alpha)$ defines an action of the graded Lie algebra $\mathfrak{o}(V;B) \ltimes V^*$ (where elements of V^* have degree -1) on $\mathrm{Cl}(V;B)$ by derivations. The symbol map intertwines this with the corresponding action by derivations of $\wedge(V)$.*

Proof. It suffices to check on elements $\phi = v_1 \wedge \cdots \wedge v_k \in \wedge(V)$ where $v_1, \ldots, v_k$ are pairwise orthogonal. Then $q(\phi) = v_1 \cdots v_k$, and the quantization of $\iota(\alpha)\phi$ (given by (15)) coincides with $\iota(\alpha)(q(\phi))$ (given by (21)). The argument for the Lie derivatives is similar. □

Any element $v \in V$ defines a derivation of $\mathrm{Cl}(V;B)$ by graded commutator: $x \mapsto [v, x]$. For generators $w \in V$, we have $[v, w] = 2B(v, w) =$

$2\langle B^\flat(v),\ w\rangle$. This shows that this derivation agrees with the contraction by $2B^\flat(v)$:

$$[v,\cdot] = 2\iota(B^\flat(v)) \tag{22}$$

As a simple application, we find:

LEMMA 2.14. *The super-center of the $\mathbb{Z}_2$-graded algebra $\mathrm{Cl}(V;B)$ is the exterior algebra over $\mathrm{rad}(B) = \ker B^\flat$.*

PROOF. Indeed, suppose x lies in the super-center. Then $0 = [v,x] = 2\iota(B^\flat(v))x$ for all $v \in V$. Hence $\sigma(x)$ is annihilated by all contractions $B^\flat(v))$, and is therefore an element of the exterior algebra over $\mathrm{ann}(\mathrm{ran}(B^\flat)) = \ker(B^\flat)$. Consequently $x = q(\sigma(x))$ is in $\mathrm{Cl}(\ker(B^\flat)) = \wedge(\ker(B^\flat))$. ☐

2.11. The homomorphism $\wedge^2 V \to \mathfrak{o}(V;B)$. Consider next the derivations of $\mathrm{Cl}(V;B)$ defined by elements of $q(\wedge^2 V)$. Define a map

$$\wedge^2 V \to \mathfrak{o}(V;B),\ \lambda \mapsto A_\lambda \tag{23}$$

where $A_\lambda(v) = -2\iota(B^\flat(v))\lambda$. This does indeed lie in $\mathfrak{o}(V;B)$, since

$$B(A_\lambda(v), w) = -2\iota(B^\flat(w))A_\lambda(v) = -2\iota(B^\flat(w))\iota(B^\flat(v))\lambda$$

is anti-symmetric in v, w. We have $[q(\lambda), v] = -[v, q(\lambda)] = -2\iota(B^\flat(v))\lambda = A_\lambda(v)$ for all $v \in V$, hence

$$[q(\lambda),\cdot] = L_{A_\lambda} \tag{24}$$

since both sides are derivations which agree on generators. Define a bracket $\{\cdot,\cdot\}$ on $\wedge^2(V)$ by

$$\{\lambda, \lambda'\} = L_{A_\lambda}\lambda'. \tag{25}$$

The calculation

$$[q(\lambda), q(\lambda')] = L_{A_\lambda} q(\lambda') = q(L_{A_\lambda}\lambda') = q(\{\lambda,\lambda'\})$$

shows that q intertwines $\{\cdot,\cdot\}$ with the Clifford commutator; in particular $\{\cdot,\cdot\}$ is a Lie bracket. Furthermore, from

$$[q(\lambda), [q(\lambda'), v]] - [q(\lambda'), [q(\lambda), v]] = [[q(\lambda), q(\lambda')], v] = [q(\{\lambda,\lambda'\}), v]$$

we see that $[A_\lambda, A_{\lambda'}] = A_{\{\lambda,\lambda'\}}$, that is, the map $\lambda \mapsto A_\lambda$ is a Lie algebra homomorphism. To summarize:

PROPOSITION 2.15. *The formula* (25) *defines a Lie bracket on $\wedge^2(V)$. Relative to this bracket, the map*

$$\wedge^2(V) \rtimes V[1] \to \mathfrak{o}(V;B) \rtimes V^*[1],\quad (\lambda, v) \mapsto (A_\lambda, B^\flat(v))$$

is a homomorphism of graded Lie algebras. (The symbol [1] *indicates a degree shift: We assign degree -1 to the elements of V, V^* while $\wedge^2(V)$, $\mathfrak{o}(V;B)$*

are assigned degree 0*.) It intertwines the actions on* $\mathrm{Cl}(V;B)$ *by derivations, producing a commutative diagram of* $\mathbb{Z}_2$*-graded Lie algebras,*

$$\begin{array}{ccc} \wedge^2(V) \rtimes V[1] & \longrightarrow & \mathfrak{o}(V;B) \rtimes V^*[1] \\ \Big\downarrow q & & \Big\downarrow \\ \mathrm{Cl}(V;B) & \underset{\mathrm{ad}}{\longrightarrow} & \mathrm{Der}(\mathrm{Cl}(V;B)) \end{array}$$

Note that we can also think of $\wedge^2(V) \rtimes V[1]$ as a graded subspace of $\wedge(V)[2]$, using the standard grading on $\wedge(V)$ shifted down by 2. We will see in the following Section 3 that the graded Lie bracket on this subspace extends to a graded Lie bracket on all of $\wedge(V)[2]$.

2.12. A formula for the Clifford product. It is sometimes useful to express the Clifford multiplication

$$m_{\mathrm{Cl}}\colon \mathrm{Cl}(V \oplus V) = \mathrm{Cl}(V) \otimes \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

in terms of the exterior algebra multiplication,

$$m_\wedge\colon \wedge(V \oplus V) = \wedge(V) \otimes \wedge(V) \to \wedge(V).$$

Recall that by definition of the isomorphism $\wedge(V \oplus V) = \wedge(V) \otimes \wedge(V)$, if $\phi, \psi \in \wedge(V^*)$, the element $\phi \otimes \psi \in \wedge(V^*) \otimes \wedge(V^*)$ is identified with the element $(\phi \oplus 0) \wedge (0 \oplus \psi) \in \wedge(V^* \oplus V^*)$. Similarly for the Clifford algebra.

Let $e_i \in V$ be an orthogonal basis, $e^i \in V^*$ the dual basis, and $e_I \in \wedge(V)$, $e^I \in \wedge(V^*)$ the corresponding dual bases indexed by subsets $I \subset \{1, \ldots, n\}$. Then the element

$$\Psi = \sum_I e^I \otimes B^\flat((e_I)^\top) \in \wedge(V^*) \otimes \wedge(V^*)$$

is independent of the choice of bases.

Proposition 2.16. *Under the quantization map, the exterior algebra product and the Clifford product are related as follows:*

$$m_{\mathrm{Cl}} \circ q = q \circ m_\wedge \circ \iota(\Psi)$$

Proof. Let V_i be the 1-dimensional subspace spanned by e_i. Then $\wedge(V)$ is the graded tensor product over all $\wedge(V_i)$, and similarly $\mathrm{Cl}(V)$ is the graded tensor product over all $\mathrm{Cl}(V_i)$. The formula for Ψ factorizes as

$$\Psi = \prod_{i=1}^n \big(1 - e^i \otimes B^\flat(e_i)\big). \tag{26}$$

It hence suffices to prove the formula for the case $V = V_1$. We have,

$$\begin{aligned} q \circ m_\wedge \circ \iota\big(1 - e^1 \otimes B^\flat(e_1)\big)(e_1 \otimes e_1) &= q \circ m_\wedge\big(e_1 \otimes e_1 + B(e_1, e_1)\big) \\ &= q\big(B(e_1, e_1)\big) \\ &= e_1 e_1. \end{aligned}$$

∎

If $\mathrm{char}(\mathbb{K}) = 0$, we may also write the element Ψ as an exponential:

$$\Psi = \exp\big(-\sum_i e^i \otimes B^\flat(e_i)\big).$$

This follows by rewriting (26) as $\prod_i \exp\big(-e^i \otimes B^\flat(e_i)\big)$, and then writing the product of exponentials as an exponential of a sum.

3. The Clifford algebra as a quantization of the exterior algebra

Using the quantization map, the Clifford algebra $\mathrm{Cl}(V;B)$ may be thought of as $\wedge(V)$ with a new associative product. We can make more precise (following Kostant-Sternberg [**?**]) in which sense the Clifford algebra is a quantization of the exterior algebra.

3.1. Differential operators. We begin with a review of some background from classical mechanics. Let $\mathbb{R}^n$ be the configuation space of a particle, with coordinates $q^1, \ldots, q^n$, and $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ the phase space, with coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$. The Poisson bracket of two functions $f, g \in C^\infty(\mathbb{R}^{2n})$ on phase space is given by the formula

$$\{f, g\} = \sum_{i=1}^n \Big(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\Big).$$

It is a Lie bracket, with the additional property

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

(That is, $\{f, \cdot\}$ is a derivation of the algebra of functions.). Let us denote by $\mathcal{Q}^k$ the functions f on phase space that are homogeneous polynomials of degree k in the p coordinate, and put $\mathcal{Q} = \bigoplus_{k=0}^\infty \mathcal{Q}^k$. Then $\mathcal{Q}$ is a commutative (in the usual sense) graded algebra, and under the Poisson bracket, $\{\mathcal{Q}^k, \mathcal{Q}^l\} \subset \mathcal{Q}^{k+l-1}$.

Now let $\mathcal{D}_{(k)}$ denote the degree k differential operators on $C^\infty(\mathbb{R}^n)$, and $\mathcal{D} = \bigcup_{k=0}^\infty \mathcal{D}_{(k)}$. Then $\mathcal{D}_{(k)}$ is a filtered algebra, that is, $\mathcal{D}_{(k)} \subset \mathcal{D}_{(k+1)}$ for all k and $\mathcal{D}_{(k)}\mathcal{D}_{(l)} \subset \mathcal{D}_{(k+l)}$. Elements of $\mathcal{D}_{(k)}$ are of the form

$$D = \sum_{|I| \le k} a_I(q)\partial_I$$

where the sum is over multi-indices $I = (i_1, \ldots, i_n)$ with $i_j \ge 0$, and $|I| = \sum i_j$, and $\partial_I = (\frac{\partial}{\partial q_1})^{i_1} \cdots (\frac{\partial}{\partial q_n})^{i_n}$. The *symbol* of such an operator is the function $\sum_{|I| \le k} a_I(q)p^I$ where $p^I = p_1^{i_1} \cdots p^{i_k}$. It defines an isomorphism

$$\sigma\colon \mathcal{D} \to \mathcal{Q}$$

of vector spaces, preserving filtrations. The (degree k) principal symbol is the leading term,

$$\sigma_k(D) = \sum_{|I| \le k} a_I(q)p^I$$

Thus $D \in \mathcal{D}_{(k)}$, $\sigma_k(D) = 0$ implies that $D \in \mathcal{D}_{(k-1)}$. If D_1, D_2 have degree k_1, k_2, one has $\sigma_{k_1+k_2}(D_1 \circ D_2) = \sigma_{k_1}(D_1)\sigma_{k_2}(D_2)$. The principal symbol therefore defines an isomorphism of graded algebras,

$$\mathrm{gr}(\mathcal{D}) \to \mathcal{Q}.$$

The degree k_1+k_2 principal symbol of the commutator $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is zero. Hence $[D_1, D_2]$ has degree $k_1 + k_2 - 1$. A calculation of the leading terms shows

$$\sigma_{k_1+k_2-1}([D_1, D_2]) = \{\sigma_{k_1}(D_1),\ \sigma_{k_2}(D_2)\}.$$

In this sense, the algebra $\mathcal{Q}$ is the 'classical limit' of the algebra $\mathcal{D}$ of differential operators: Under the symbol map, commutators go to Poisson brackets 'modulo lower order terms'.

3.2. Graded Poisson algebras. The symbol map for Clifford algebras may be put into a similar framework, but in a super-context. Recall that is $V = \bigoplus_{k\in\mathbb{Z}} V^k$ is a graded vector space, then $V[n]$ for $l \in \mathbb{Z}$ denotes V with the shifted grading $V[n]^k = V^{k+n}$. Thus, if $f \in V$ has degree $|f|$ for the original grading, then it has degree $|f| - n$ as an element of $V[n]$.

For the following notion, see e.g. Cattaneo-Fiorenza-Longini, 'graded Poisson algebras' (Preprint, 2005).

Definition 3.1. A *graded Poisson algebra of degree n* is a commutative graded algebra $\mathcal{P} = \bigoplus_{k\in\mathbb{Z}} \mathcal{P}^k$, together with a bilinear map $\{\cdot,\cdot\}\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ (called Poisson bracket) such that

(1) The space $\mathcal{P}[n]$ is a graded Lie algebra, with bracket $\{\cdot,\cdot\}$.
(2) The map $f \mapsto \{f, \cdot\}$ defines a graded Lie algebra homomorphism $\mathcal{P}[n] \to \mathrm{Der}(\mathcal{P})$.

That is, for any $f \in \mathcal{P}^k$, the map $\{f, \cdot\}$ is a degree $k - n$ derivation of the algebra structure. Note that any Poisson bracket on a graded algebra is uniquely determined by its values on generators for the algebra. The defining conditions are

$$\begin{aligned}\{f_1, \{f_2, f_3\}\} &= \{\{f_1, f_2\}, f_3\} + (-1)^{(|f_1|-n)(|f_2|-n)} f_1\{f_2, f_3\}\\ \{f_1, f_2\} &= -(-1)^{(|f_1|-n)\,(|f_2|-n)}\{f_2, f_1\},\\ \{f_1, f_2 f_3\} &= \{f_1, f_2\}f_3 + (-1)^{(|f_1|-n)|f_2|} f_2\{f_1, f_3\}.\end{aligned}$$

Example 3.2. The algebra $\mathcal{Q}$ from the last Section may be viewed as a graded Poisson algebra of degree 4 after doubling the degrees: $\mathcal{P}^{2k} = \mathcal{Q}^k$ and $\mathcal{P}^{2k+1} = 0$.

Example 3.3. Suppose $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is any Lie algebra, and let $\mathcal{P} = S(\mathfrak{g})$ be the symmetric algebra, with grading

$$S(\mathfrak{g})^{2k} = S^k(\mathfrak{g}), \quad S(\mathfrak{g})^{2k+1} = 0.$$

Then $S(\mathfrak{g})$ carries a graded Poisson bracket of degee 2, given on generators by the Lie bracket $\{\xi_1, \xi_2\} = [\xi_1, \xi_2]_{\mathfrak{g}}$. Conversely, if V is any vector space, then the structure of a graded Poisson algebra of degree 2 on $S(V)$ is *equivalent* to a Lie algebra structure on V.

Suppose now [**?**] that $\mathcal{A}$ is an $\mathbb{Z}_2$-graded algebra, equipped with a filtration $\mathcal{A}_{(k)}$ that is compatible with the $\mathbb{Z}_2$-grading in the sense that each $\mathcal{A}_{(k)}$ is a $\mathbb{Z}_2$-graded subspace, and the induced $\mathbb{Z}_2$-grading on the associated graded algebra $\mathrm{gr}(\mathcal{A})$ is just the mod 2 reduction of the $\mathbb{Z}$-grading. Explicitly, this means

$$\mathcal{A}^{\bar{0}}_{(2k)} = \mathcal{A}^{\bar{0}}_{(2k+1)}, \quad \mathcal{A}^{\bar{1}}_{(2k+1)} = \mathcal{A}^{\bar{1}}_{(2k+2)}. \tag{27}$$

Suppose furthermore that $\mathrm{gr}(\mathcal{A})$ is graded commutative. In other words, the multiplication in $\mathcal{A}$ is commutative 'up to lower order terms'. Thus $[\mathcal{A}_{(k)}, \mathcal{A}_{(l)}] \subset \mathcal{A}_{(k+l-1)}$. Taking parity into account, we see that in fact

$$[\mathcal{A}_{(k)}, \mathcal{A}_{(l)}] \subset \mathcal{A}_{(k+l-2)}.$$

Hence we can define a degree -2 bracket on $\mathrm{gr}(A)$ as follows,

$$\{f,\ g\} := [x, y]_{\mathcal{A}} \mod \mathcal{A}_{(k+l-3)}$$

for $x \in \mathcal{A}_{(k)}$ and $y \in \mathcal{A}_{(l)}$, where $f \in \mathrm{gr}(\mathcal{A})^k,\ g \in \mathrm{gr}(\mathcal{A})^l$ are the images in the associated graded algebra. It is easy to see that $\{\cdot,\cdot\}$ is a graded Poisson bracket of degree -2. (More generally, if $[\mathcal{A}_{(k)}, \mathcal{A}_{(l)}] \subset \mathcal{A}_{(k+l-2r)}$ a similar prescription gives a graded Poisson bracket of degree $2r$.)

3.3. Poisson structures on $\wedge(V)$. Any symmetric bilinear form B on a vector space induces on $\mathcal{A} = \wedge(V)$ the structure of a graded Poisson algebra of degree 2. The Poisson bracket is given on generators $v, w \in V = \wedge^1(V)$ by

$$\{v, w\} = 2B(v, w).$$

In this way, one obtains a one-to-one correspondence between Poisson brackets (of degree -2) on $\wedge(V)$ and symmetric bilinear forms B. Clearly, this Poisson bracket is induceded from the commutator on the Clifford algebra under the identification $\wedge(V) = \mathrm{gr}(\mathrm{Cl}(V; B))$.

For any $\phi \in \wedge^k(V)$, we may consider the corresponding derivation $\{\phi, \cdot\} \in \mathrm{Der}^{k-2}(\wedge(V))$. Let us consider some basic examples:

First, the Poisson bracket with $v \in V$ is a contraction:

$$\tfrac{1}{2}\{v, \cdot\} = \iota(B^\flat(v)),$$

since both sides are derivations given by $B(v, w)$ on generators $w \in V$. Next, for $\lambda \in \wedge^2(V)$ we have

$$\{\lambda, \cdot\} = L_{A_\lambda},$$

since both sides are derivations given by $A_\lambda(w)$ on generators $w \in V$. In particular, we recover our definition $\{\lambda, \lambda'\} = L_{A_\lambda}\lambda'$ of the Lie bracket on $\wedge^2(V)$, and the graded Lie algebra $\wedge^2(V) \rtimes V$ becomes a graded Lie subalgebra,

$$\wedge^2(V) \rtimes V \subset \wedge(V)[2]$$

where $\wedge(V)[2]$ carries the graded Lie bracket $\{\cdot,\cdot\}$. Under the quantization map, this graded Lie subalgebra becomes a (super-)Lie subalgebra of $\mathrm{Cl}(V;B)$ under the commutator bracket. That is, for quadratic elements the quantization map takes Poisson brackets to commutators. This is no longer true, in general, for higher order elements.

EXAMPLE 3.4. Let $\phi \in \wedge^3(V)$, so that $\{\phi,\phi\} \in \wedge^4(V)$. As we saw in Example **??**, the quantization of ϕ satisfies $[q(\phi),q(\phi)] = 2q(\phi)^2 \in q(\wedge^0(V)\oplus \wedge^4(V))$. The leading term is the Poisson bracket, hence the difference with $q(\{\phi,\phi\})$ is a scalar:

$$[q(\phi),q(\phi)] - q(\{\phi,\phi\}) \in \mathbb{K}.$$

In general, this scalar is non-zero. For instance, if $V = \mathbb{R}^3$ with standard bilinear product, and $\phi = e_1\wedge e_2\wedge e_3$ (the volume element) then $[q(\phi),q(\phi)] = 2q(\phi)^2 = 2(e_1e_2e_3)^2 = -2$.

If the bilinear form B is non-dgenerate, then the map $\wedge(V)[2] \to \mathrm{Der}(\wedge(V))$ given by the Poisson bracket is injective. Indeed for $\phi \in \wedge^k(V)$ we have, on generators $v \in V$,

$$\{\phi,v\} = -(-1)^k\{v,\phi\} = -(-1)^k\tfrac{1}{2}\iota(B^\flat(v))\phi.$$

This vanishes for all $v \in V$ if and only if $\phi = 0$. Derivations of the form $\{\phi,\cdot\}$ are called *inner derivations* of the Poisson algebra $\wedge(V)$. Note that the inner derivations are derivations not only of the product but also of the bracket.

Disregarding the bracket, the full space of derivations of $\wedge(V)$ (viewed simply as a graded algebra) is much larger. Assume $\dim V < \infty$, and let the commutative Lie algebra $V^*[1]$ act on $\wedge(V)$ by contractions. Let $\wedge(V)\otimes V^*[1]$ carry the Lie bracket

$$[\phi\otimes\alpha,\psi\otimes\beta] = (\phi\wedge\iota(\alpha)\psi)\otimes\beta - (-1)^{|\phi|}(\iota(\beta)\phi\wedge\psi)\otimes\alpha.$$

PROPOSITION 3.5. *For* $\dim(V) < \infty$*, the linear map*

$$\wedge(V)\otimes V^*[1] \to \mathrm{Der}(\wedge(V)),\ \ \phi = \sum_a \phi_a\otimes e^a \mapsto D_\phi = \epsilon(\phi_a)\iota(e^a), \tag{28}$$

is an isomorphism of graded Lie algebras.

PROOF. The map (28) is 1-1, since ϕ_a is recovered from the derivation D_ϕ as $\phi_a = D_\phi(e_a)$. Conversely, if $D \in \mathrm{Der}(\wedge(V))$ is any derivation and ϕ is defined by $\phi_a = D(e_a)$, then $D = D_\phi$ since both derivations agree on generators e_a. The description of the Lie bracket as a semi-direct product is equivalent to

$$[\phi,\psi] = \sum_a (D_\phi(\psi_a) - (-1)^{|\phi||\psi|}D_\psi(\phi_a))\otimes e^a = [D_\phi,D_\psi](e_a)\otimes e^a.$$

Hence $[D_\phi,D_\psi] = D_{[\phi,\psi]}$. ∎

For instance, any endomorphism $A \in \mathrm{End}(V)$ extends uniquely to a derivation $D_A \sum_a \epsilon(A(e_a))\,\iota(e^a) \in \mathrm{Der}^0(\wedge(V))$, and $A \mapsto D_A$ is a Lie algebra homomorphism. Given a non-dgenerate symmetric bilinear form B, the derivation D_A is inner if and only if $A \in \mathfrak{o}(V;B)$. In this case we have $D_A = \{\lambda,\cdot\}$ where $\lambda \in \wedge^2(V)$ is characterized by $2\iota(B^\flat(v))\lambda = -A(v)$ for all $v \in V$.

4. Spin groups

For any quadratic vector space (V,B) over $\mathbb{K}$, one can define a *Clifford group* $\Gamma(V) \subset \mathrm{Cl}(V;B)$, which is an extension of the orthogonal group by non-zero scalars $\mathbb{K}^\times$. If $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, one may reduce the kernel of the extension of the orthogonal group to $\mathbb{Z}_2 \subset \mathbb{K}^\times$, thus arriving at the Pin and Spin groups.

Throughout, we will assume that the bilinear form B on V is non-degenerate. We will write $\mathrm{Cl}(V)$ in place of $\mathrm{Cl}(V;B)$.

4.1. The Clifford group and the spin group.

Recall that $\Pi\colon \mathrm{Cl}(V) \to \mathrm{Cl}(V),\ x \mapsto (-1)^{|x|}x$ denotes the parity automorphism of the Clifford algebra. Let $\mathrm{Cl}(V)^\times$ be the group of invertible elements in $\mathrm{Cl}(V)$.

Definition 4.1. The *Clifford group* $\Gamma(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$, consisting of all $x \in \mathrm{Cl}(V)^\times$ such that $A_x(v) := \Pi(x)vx^{-1} \in V$ for all $v \in V \subset \mathrm{Cl}(V)$.

Hence, by definition the Clifford group comes with a natural representation, $\Gamma(V) \to \mathrm{GL}(V),\ x \mapsto A_x$. Let $S\Gamma(V) = \Gamma(V) \cap \mathrm{Cl}^{\bar{0}}(V)^\times$ denote the *special Clifford group.*

theorem 4.2. *The canonical representation of the Clifford group takes values in* $\mathrm{O}(V)$*, and defines an exact sequence,*

$$1 \longrightarrow \mathbb{K}^\times \longrightarrow \Gamma(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1.$$

It restricts to a similar exact sequence for the special Clifford group,

$$1 \longrightarrow \mathbb{K}^\times \longrightarrow S\Gamma(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

The elements of $\Gamma(V)$ *are all products* $x = v_1 \cdots v_k$ *where* $v_1,\ldots,v_k \in V$ *are non-isotropic.* $S\Gamma(V)$ *consists of similar products, with* k *even. The corresponding element* A_x *is a product of reflections:*

$$A_{v_1\cdots v_k} = R_{v_1}\cdots R_{v_k}.$$

Proof. Let $x \in \mathrm{Cl}(V)$. The transformation A_x is trivial if and only if $\Pi(x)v = vx$ for all $v \in V$, i.e. if and only if $[v,x]=0$ for all $v \in V$. That is, it is the intersection of the center $\mathbb{K} \subset \mathrm{Cl}(V)$ with $\Gamma(V)$. This shows that the kernel of the homomorphism $\Gamma(V) \to \mathrm{GL}(V)$ is the group $\mathbb{K}^\times$ of invertible scalars.

Applying $-\Pi$ to the definition of A_x, we obtain $A_x(v) = xv\Pi(x)^{-1} = A_{\Pi(x)}(v)$. This shows $A_{\Pi(x)} = A_x$ for $x \in \Gamma(V)$. For $x \in \Gamma(V)$ and $v, w \in V$ we have,

$$\begin{aligned} 2B(A_x(v), A_x(w)) &= \big(A_x(v)A_x(w) + A_x(w)A_x(v) \\ &= A_x(v)A_{\Pi(x)}(w) + A_x(w)A_{\Pi(x)}(v) \\ &= \Pi(x)(vw + wv)\Pi(x^{-1}) \\ &= 2B(v, w)\Pi(x)\Pi(x^{-1}) \\ &= 2B(v, w). \end{aligned}$$

This proves that $A_x \in \mathrm{O}(V)$ for all $x \in \Gamma(V)$. Suppose now that $v \in V$ is non-isotropic. If $w \in V$ is orthogonal to v, then

$$A_v(w) = \Pi(v)wv^{-1} = -\Pi(v)v^{-1}w = vv^{-1}w = w.$$

Since

$$A_v(v) = \Pi(v)vv^{-1} = \Pi(v) = -v$$

this shows that $A_v = R_v$ is the reflection defined by v. More generally, it follows that for all non-isotropic vectors $v_1, \ldots, b_k$,

$$A_{v_1 \cdots v_k} = R_{v_1} \cdots R_{v_k}.$$

By the E. Cartan-Dieudonné theorem, any $A \in \mathrm{O}(V)$ is of this form. This shows the map $x \mapsto A_x$ is onto $\mathrm{O}(V)$, and that $\Gamma(V)$ is generated by the non-isotropic vectors in V. □

Since all $x \in \Gamma(V)$ can be written in the form $x = v_1 \cdots v_k$ with non-isotropic vectors v_i, it follows that the element $x^\top x$ lies in $\mathbb{K}^\times$. This defines the *norm homomorphism*

$$\mathsf{N}\colon \Gamma(V) \to \mathbb{K}^\times,\ x \mapsto x^\top x.$$

It has the obvious property

$$\mathsf{N}(\lambda x) = \lambda^2 \mathsf{N}(x)$$

for $\lambda \in \mathbb{K}^\times$. If $\mathbb{K} = \mathbb{R}$, any x can be rescaled to satisfy $\mathsf{N}(x) = \pm 1$. One defines, [2]

Definition 4.3. Suppose $\mathbb{K} = \mathbb{R}$. The *Pin group* $\mathrm{Pin}(V)$ is the pre-image of $\{1, -1\}$ under the norm homomorphism $\mathsf{N}\colon \Gamma(V) \to \mathbb{K}^\times$. Its intersection with $S\Gamma(V)$ is called the *Spin group*, and is denoted $\mathrm{Spin}(V)$.

Since $N(\lambda) = \lambda^2$ for $\lambda \in \mathbb{K}^\times$, the only scalars in $\mathrm{Pin}(V)$ are ± 1. Hence, the exact sequence for the Clifford group restricts to an exact sequence,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1,$$

so that $\mathrm{Pin}(V)$ is a double cover of $\mathrm{O}(V)$. Similarly,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1,$$

[2] The definition also makes sense for arbitrary fields. However, the natural representation need not be onto. Cf. Grove [**?**, p. 78].

defines a double cover of $\mathrm{SO}(V)$. Elements in $\mathrm{Pin}(V)$ are products $x = v_1 \cdots v_k$ with $B(v_i, v_i) = \pm 1$. The group $\mathrm{Spin}(V)$ consists of similar products, with k even.

For $V = \mathbb{R}^{n,m}$, with the scalar product of signature n, m, let $\mathrm{Spin}(V) = \mathrm{Spin}(n,m)$ and $\mathrm{Pin}(V) = \mathrm{Pin}(n,m)$. Also, let $\mathrm{Spin}_0(n,m)$ denote the preimage of the identity component, $\mathrm{SO}_0(n,m)$. As usual, we will write $\mathrm{Pin}(n) = \mathrm{Pin}(n,0)$ and $\mathrm{Spin}(n) = \mathrm{Spin}(n,0)$.

THEOREM 4.4. *Let $\mathbb{K} = \mathbb{R}$, and $V \cong \mathbb{R}^{n,m}$. If $n \geq 2$ or $m \geq 2$, the group $\mathrm{Spin}_0(V)$ is connected.*

PROOF. The pre-image of the group unit $e \in \mathrm{SO}_0(V)$ in $\mathrm{Spin}(V)$ are the elements $+1, -1 \in \mathrm{Cl}(V)$. To show that $\mathrm{Spin}_0(V)$ is connected, it suffices to show that ± 1 are in the same connected component. Let

$$v(\theta) \in V, \quad 0 \leq \theta \leq \pi$$

be a continuous family of non-isotropic vectors with the property

$$v(\pi) = -v(0).$$

Such a family exists, since V contains a 2-dimensional subspace isomorphic to $\mathbb{R}^{2,0}$ or $\mathbb{R}^{0,2}$. We may normalize the vectors $v(\theta)$ to satisfy

$$B(v(\theta), v(\theta)) = \pm 1.$$

Then $v(\theta)v(0) \in \mathrm{Spin}(V) \subset \mathrm{Cl}^{\bar{0}}(V)$ equals ± 1 for $\theta = 0$, and ∓ 1 for $\theta = \pi$. This shows that 1 and -1 are in the same component of $\mathrm{Spin}_0(V)$, as desired. □

Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$, the connected double cover $\mathrm{Spin}(n)$ is the universal cover in that case. In low dimensions, we had determined these universal covers to be

$$\mathrm{Spin}(3) = \mathrm{SU}(2), \quad \mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2).$$

It can also be shown that $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ (the group of norm-preserving automorphisms of the quaternionic vector space $\mathbb{H}^2$) and $\mathrm{Spin}(6) = \mathrm{SU}(4)$. [3] For $n \geq 7$, the groups $\mathrm{Spin}(n)$ are all simple and non-isomorphic to the other classical groups.

The groups $\mathrm{Spin}_0(n,m)$ are usually not simply connected. Indeed since since $\mathrm{SO}_0(n,m)$ has maximal compact subgroup $\mathrm{SO}(n) \times \mathrm{SO}(m)$, the fundamental group is

$$\pi_1(\mathrm{SO}_0(n,m)) = \pi_1(\mathrm{SO}(n)) \times \pi_1(\mathrm{SO}(m))$$

[3] The last two isomorphisms are obtained using the spin representations. (Cf. Adams, lectures on exceptional Lie groups, p.31.) E.g.: Spin(6) has the two half-spin representations, both of which are irreducible representations of dimension 4. This gives homomorphisms $\mathrm{Spin}(6) \to \mathrm{SU}(4)$, which must be isomorphisms by dimension count. Similarly, for Spin(5) wa have the spin representation on $\mathbb{C}^4$ (obtained by restriction of any of the two half-spin representations of Spin(6)). We will show later that this spin representation is of quaternioinc type, so that it is a $\mathbb{H}$-linear represenbation on $\mathbb{H}^2$. This defines a homomorphism $\mathrm{Spin}(5) \to \mathrm{Sp}(2)$, which is an isomorphism by dimension count.

Hence, only in the cases $n \geq 3, \quad m = 0, 1$ or $n = 0, 1, \quad m \geq 3$ we obtain $\pi_1(\mathrm{SO}_0(n, m)) = \mathbb{Z}_2$, and only in those cases $\mathrm{Spin}_0(n, m)$ is a universal cover.

Let us now turn to the case $\mathbb{K} = \mathbb{C}$, so that $V \cong \mathbb{C}^n$ with the standard bilinear form. In that case, we can rescale any $x \in \Gamma(V) = \Gamma(n, \mathbb{C})$ to satisfy $N(x) = +1$. Hence define[4]

$$\mathrm{Pin}(n, \mathbb{C}) = \{x \in \Gamma(n, \mathbb{C}))| \, N(x) = +1\}$$

and $\mathrm{Spin}(n, \mathbb{C}) = \mathrm{Pin}(n, \mathbb{C}) \cap S\Gamma(n, \mathbb{C})$.

Proposition 4.5. $\mathrm{Pin}(n, \mathbb{C})$ *and* $\mathrm{Spin}(n, \mathbb{C})$ *are double covers of* $\mathrm{O}(n, \mathbb{C})$ *and* $\mathrm{SO}(n, \mathbb{C})$. *Furthermore,* $\mathrm{Spin}(n, \mathbb{C})$ *is connected and simply connected, i.e. it is the universal cover of* $\mathrm{SO}(n, \mathbb{C})$.

Proof. The first part is clear, since the condition $N(x) = 1$ determines the scalar multiple of x up to a sign. The second part follows by the same argument as in the real case, or alternatively by observing that ± 1 are in the same component of $\mathrm{Spin}(n, \mathbb{R}) \subset \mathrm{Spin}(n, \mathbb{C})$. ☐

Assume $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Recall the isomorphism $\lambda \colon \mathfrak{o}(V) \to \wedge^2(V)$, and let

$$\gamma = q \circ \lambda \colon \mathfrak{o}(V) \to \mathrm{Cl}(V).$$

Then $A(v) = [\gamma(A), v]$ for $v \in V$, and accordingly

$$\exp(A)(v) = e^{[\gamma(A), \cdot]} v = e^{\gamma(A)} v e^{-\gamma(A)}.$$

Here

$$e^{[\gamma(A), \cdot]} v = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[\gamma(A), [\gamma(A), [\cdots [\gamma(A)}_{n \text{ times}}, v] \cdots]]]$$

and $e^{\gamma(A)} = \sum_{n=0}^{\infty} \frac{1}{n!} \gamma(A)^n$. By definition of the Clifford group, this shows that $e^{\gamma(A)} \in S\Gamma(V)$. The element $\gamma(A)$ satisfies $\gamma(A)^{\top} = -\gamma(A)$. Hence,

$$(e^{\gamma(A)})^{\top} = e^{\gamma(A)^{\top}} = e^{-\gamma(A)},$$

and therefore $N(e^{\gamma(A)}) = 1$. That is,

$$e^{\gamma(A)} \in \mathrm{Spin}(V)$$

Since $\theta \mapsto e^{\theta \gamma(A)}$ defines a curve in $\mathrm{Spin}(V)$, connecting 1 with $e^{\gamma(A)}$, it follows that $e^{\gamma(A)}$ is in the identity component $\mathrm{Spin}_0(V)$.

In other words, the group $\mathrm{Spin}(V) \subset \mathrm{Cl}(V)^{\times}$ constructed above has Lie algebra $\gamma(\mathfrak{o}(V)) \subset \mathrm{Cl}^{\bar{0}}(V)$. Indeed, if $\mathbb{K} = \mathbb{R}$ and the bilinear form B is positive definite, we can directly define $\mathrm{Spin}(V)$ as the set of elements $e^{\gamma(\mathfrak{o}(V))}$. This follows because $\mathrm{Spin}(V)$, as a double cover of the compact group $\mathrm{SO}(V)$, is compact, and for compact Lie groups the exponential map is onto.

[4]There seem to be no standard conventions for the definitions for the complex case.

EXAMPLE 4.6. Let $V = \mathbb{R}^2$ with the standard bilinear form, and consider the element $A \in \mathfrak{o}(V)$ defined by $\lambda(A) = e_1 \wedge e_2$. Then $\gamma(A) = e_1 e_2$. Since $(e_1e_2)^2 = -1$, the 1-parameter group of elements

$$x(\theta) = \exp(\theta/2 e_1 e_2) \in \mathrm{Spin}(V).$$

is given by the formula,

$$x(\theta) = \cos(\theta/2) + \sin(\theta/2) e_1 e_2,$$

in particular $x(\theta + 2\pi) = -x(\theta)$. To find its action on V, we compute

$$\begin{aligned} x(\theta)e_1x(-\theta) &= (\cos(\theta/2) + \sin(\theta/2)e_1e_2)e_1(\cos(\theta/2) - \sin(\theta/2)e_1e_2) \\ &= (\cos(\theta/2)e_1 - \sin(\theta/2)e_2)(\cos(\theta/2) - \sin(\theta/2)e_1e_2) \\ &= (\cos^2(\theta/2) - \sin^2(\theta/2))e_1 - 2\sin(\theta/2)\cos(\theta/2)e_2 \\ &= \cos(\theta)e_1 - \sin(\theta)e_2 \end{aligned}$$

This verifies that $A_{x(\theta)}$ is given as rotations by θ.

4.2. The groups $\mathrm{Pin}_c(V)$ and $\mathrm{Spin}_c(V)$. Let V be a vector space over $\mathbb{K} = \mathbb{R}$, with a *positive definite* symmetric bilinear form B. Denote by $V^{\mathbb{C}}$ the complexification of V. The complex conjugation mapping $v \mapsto \overline{v}$ extends to an anti-linear algebra automorphism $x \mapsto \overline{x}$ of the complexified Clifford algebra

$$\mathrm{Cl}(V^{\mathbb{C}}) = \mathrm{Cl}(V)^{\mathbb{C}}.$$

PROPOSITION 4.7. *The Clifford algebra $\mathrm{Cl}(V^{\mathbb{C}})$ with involution $x^* = \overline{x}^\top$ is a C^*-algebra. That is, it admits a norm $||\cdot||$ relative to which it is a Banach algebra, and such that the C^*-identity $||x^*x|| = ||x||^2$ is satisfied.*

PROOF. It suffices to find a Hilbert space $\mathcal{H}$ with a faithful $*$-homomorphism $\pi\colon \mathrm{Cl}(V^{\mathbb{C}}) \to \mathrm{End}(\mathcal{H})$. Indeed, given π one obtains a C^*-norm on $\mathrm{Cl}(V^{\mathbb{C}})$ (necessarily unique, by a standard fact on C^*-algebras) by $||x|| = ||\pi(x)||$. Denote by $\mathrm{tr}\colon \mathrm{Cl}(V^{\mathbb{C}}) \to \mathbb{C}$ the normalized trace (cf. Proposition **??**) with $\mathrm{tr}(I) = 1$. We can take $\mathcal{H}$ simply to be $\mathrm{Cl}(V^{\mathbb{C}})$ itself, with Hermitian inner product $\langle x, y\rangle = \mathrm{tr}(x^*y)$, and with π the action by multiplication. □

REMARK 4.8. The C^*-norm on the Clifford algebra is explicitly given by the formula,

$$||a|| = \lim_{n\to\infty} \Big(\mathrm{tr}(a^*a)^n\Big)^{\frac{1}{2n}}.$$

(If $\dim V$ is even, this may be obtained by identifying the Clifford algebra with a matrix algebra.)

Suppose $x \in \Gamma(V^{\mathbb{C}}) \subset \mathrm{Cl}(V^{\mathbb{C}})^\times$, defining a transformation $A_x(v) = (-1)^{|x|}xvx^{-1}$ of $V^{\mathbb{C}}$ as before.

LEMMA 4.9. *The element $x \in \Gamma(V^{\mathbb{C}})$ satisfies $A_x(v)^* = A_x(v^*)$ for all $v \in V^{\mathbb{C}}$, if and only if x^*x is a positive real number.*

PROOF. For all $x \in \Gamma(V^{\mathbb{C}})$ and all $v \in V^{\mathbb{C}}$, we have

$$A_x(v)^* = (-1)^{|x|}(x^{-1})^* v^* x^* = A_{(x^{-1})^*}(v^*).$$

This coincides with $A_x(v^*)$ for all v if and only if $x = \lambda(x^{-1})^*$ for some $\lambda \in \mathbb{C}^\times$, i.e. if and only if $x^*x \in \mathbb{C}^\times$. Since x^*x is a positive element, this condition is equivalent to $x^*x \in \mathbb{R}_{>0}$. □

DEFINITION 4.10. We define

$$\Gamma_c(V) = \{x \in \Gamma(V^{\mathbb{C}}) |\ x^*x \in \mathbb{R}_{>0}\}$$
$$\mathrm{Pin}_c(V) = \{x \in \Gamma(V^{\mathbb{C}}) |\ x^*x = 1\}.$$

The group $\mathrm{Spin}_c(V)$ consists of the even elements in $\mathrm{Pin}_c(V)$.

By construction, $x \in \Gamma(V^{\mathbb{C}})$ lies in $\Gamma_c(V)$ if and only if the automorphism A_x of $V^{\mathbb{C}}$ preserves the real subspace V. That is, $\Gamma_c(V)$ is the inverse image of $\mathrm{O}(V) \subset \mathrm{O}(V^{\mathbb{C}})$ in $\Gamma(V^{\mathbb{C}})$. The exact sequence for $\Gamma(V^{\mathbb{C}})$ restricts to an exact sequence,

$$1 \to \mathbb{C}^\times \to \Gamma_c(V) \to \mathrm{O}(V) \to 1.$$

Similarly, using $\mathbb{C}^\times \cap \mathrm{Pin}_c(V) = \mathbb{C}^\times \cap \mathrm{Spin}_c(V) = \mathrm{U}(1)$, we have exact sequences

$$1 \to \mathrm{U}(1) \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \to 1,$$
$$1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \to 1.$$

REMARK 4.11. Of course, one could directly define these groups as the subgroup generated by $\mathrm{Pin}(V)$ resp. $\mathrm{Spin}(V)$ together with $\mathrm{U}(1)$. More precisely, $\mathrm{Spin}_c(V)$ is the quotient of $\mathrm{Spin}(V) \times \mathrm{U}(1)$ by the relation

$$(x, e^{i\psi}) \sim (-x, -e^{i\psi})$$

and similarly for $\mathrm{Pin}_c(V)$.

The norm homomorphism for $\Gamma(V^{\mathbb{C}})$ restricts to a group homomorphism,

$$\mathsf{N}\colon \mathrm{Pin}_c(V) \to \mathrm{U}(1),\ x \mapsto x^\top x.$$

Together with the map to $\mathrm{O}(V)$ this defines exact sequences,

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \times \mathrm{U}(1) \to 1,$$

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \times \mathrm{U}(1) \to 1$$

One of the motivations for the group $\mathrm{Spin}_c(V)$ is the following 'lifting problem'. Suppose J is an orthogonal complex structure on V, that is, $J \in \mathrm{O}(V)$ and $J^2 = -I$. Such a J exists if and only if $n = \dim V$ is even, and turns V into a vector space over $\mathbb{C}$, with scalar multiplication

$$(a + \sqrt{-1}b)x = ax + bJx.$$

Let $U_J(V) \subset \mathrm{SO}(V)$ be the corresponding unitary group (i.e. the elements of $\mathrm{SO}(V)$ preserving J).

THEOREM 4.12. *The inclusion* $U_J(V) \hookrightarrow \mathrm{SO}(V)$ *admits a unique lift to a group homomorphism* $U_J(V) \hookrightarrow \mathrm{Spin}_c(V)$*, in such a way that the composition with the map to* $\mathrm{U}(1)$ *is the map* $\mathrm{U}_J(V) \to \mathrm{U}(1),\ A \mapsto \det_J(A)$ *(complex determinant).*

PROOF. We may choose an orthogonal basis $e_1, \ldots, e_{2n}$ of V, with the property $J(e_i) = e_{n+i}$ for $i = 1, \ldots, n$. This identifies $V \cong \mathbb{C}^k$, and $U_J(V)$ with $U(k)$.

We are trying to construct a lift of the map

$$U(k) \to \mathrm{SO}(2k) \times \mathrm{U}(1),\ A \mapsto (A,\ \det_{\mathbb{C}}(A))$$

to the double cover. Since $\mathrm{U}(k)$ is connected, if such a lift exists then it is unique. To prove existence, it suffices to check that any loop representing a generator of $\pi_1(\mathrm{U}(k)) \cong \mathbb{Z}$ lifts to a loop in $\mathrm{Spin}_c(V)$. Since the inclusion $\mathrm{U}(1) \to \mathrm{U}(k)$ induces an isomorphism of fundamental groups, it is enough to check this for $k = 1$, i.e. $n = 2$. Hence, our task is to lift the map

$$\mathrm{U}(1) \to \mathrm{SO}(2) \times \mathrm{U}(1),\ e^{i\theta} \mapsto (R(\theta), e^{i\theta})$$

to the double cover, $\mathrm{Spin}(V) \times \mathrm{U}(1)/\mathbb{Z}_2$. We had found in example 4.6 that the curve $R(\theta)$ lifts to

$$x(\theta) = \exp(\theta/2 e_1 e_2) = \cos(\theta/2) + \sin(\theta/2)\, e_1 e_2$$

with property $x(\theta + 2\pi) = -x(\theta)$. The desired lift is explicitly given as,

$$e^{i\theta} \mapsto \big[\big(x(\theta), e^{i\theta/2}\big)\big].$$

where the brackets indicate the equivalence relation $(x, e^{i\psi}) \cong (-x, -e^{i\psi})$. ☐

REMARK 4.13. The two possible square roots of $\det_{\mathbb{C}}(A)$ for $A \in \mathrm{U}(k)$ define a double cover of $\mathrm{U}(k)$,

$$\tilde{U}(k) = \{(A, z) \in \mathrm{U}(k) \times \mathbb{C}^\times |\ z^2 = \det_{\mathbb{C}}(A)\}.$$

While the inclusion $\mathrm{U}(k) \hookrightarrow \mathrm{SO}(2k)$ does not live to the Spin group, the above proof shows that there exists a lift for this double cover (i.e. the double cover is identified with the pre-image of $\mathrm{U}(k)$).

5. Appendix: Graded algebras, graded derivations

A $\mathbb{Z}$*-graded vector space* is a vector space V with a direct sum decomposition $V = \bigoplus_{k\in\mathbb{Z}} V^k$. We will write $|v| = k$ for the degree of homogeneous elements $v \in V^k$. For $n \in \mathbb{Z}$ we denote by $V[n]$ the vector space V with the shifted grading $V[n]^k = V^{k+n}$. Thus, elements of degree k in V have degree $k - n$ in $V[n]$. Directs sums of graded vector spaces, and quotients by (compatibly) graded subspaces are graded vector spaces in the obvious way. The tensor product of two graded vector spaces $V \otimes W$ is a graded vector space, with $(V \otimes W)^k = \bigoplus_{l\in\mathbb{Z}} V^l \otimes W^{k-l}$.

A $\mathbb{Z}$*-graded algebra* is a graded vector space $\mathcal{A} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}^k$, with an associative algebra structure satisfying $\mathcal{A}^k\mathcal{A}^l \subset \mathcal{A}^{k+l}$ for all k, l. If $\mathcal{A} =$

$\bigoplus \mathcal{A}^k$ and $B = \bigoplus B^k$ are graded algebras, then the graded tensor product is a graded algebra for the product

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2 \otimes b_1 b_2).$$

A $\mathbb{Z}$-*graded Lie algebra* is a graded vector space $\mathfrak{g}$, equipped with a bilinear bracket $[\cdot,\cdot]\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ with $[\mathfrak{g}^k, \mathfrak{g}^l] \subset \mathfrak{g}^{k+l}$, such that the bracket is graded skew-symmetric and satisfies the Jacobi identity in the graded sense. That is,

$$[x, y] = -(-1)^{|x||y|}[y, x],$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]].$$

Given two graded Lie algebras $\mathfrak{g}, \mathfrak{k}$, their tensor product is a graded Lie algebra for the bracket

$$[x \otimes u,\ y \otimes v] = (-1)^{|u||y|}[x, y] \otimes [u, v].$$

For instance, any graded algebra $\mathcal{A}$ becomes a graded Lie algebra under the graded commutator, $[a, b] = ab - (-1)^{|a||b|}ba$. The graded algebra is called commutative is this bracket is trivial.

Remark 5.1 (Degree doubling). The signs in the definition of commutators, tensor products, etc. of $\mathbb{Z}$-graded objects are instances of the *super-sign convention*: Whenever an object of degree k moves past an object of degree l, a sign $(-1)^{kl}$ appears. For instance, the algebra $\wedge(V)$ is commutative in this graded sense, and $\wedge(V \oplus W) = \wedge(V) \otimes \wedge(W)$ as algebras only if one uses the graded tensor product.

On the other hand, the corresponding statements for the symmetric algebra do not involve signs. Nevertheless, this fits into the above framework after doubling degrees, i.e. putting $S(V)^{2k} = S^k(V),\ S(V)^{2k+1} = 0$. Similarly, if $\mathfrak{g}$ is an *ordinary* Lie algebra, with a grading such that $[\mathfrak{g}^k, \mathfrak{g}^l] \subset \mathfrak{g}^{k+l}$, it becomes a graded Lie algebra in our sense after doubling degrees.

Suppose $\mathcal{A}$ is a $\mathbb{Z}$-graded algebra. An endomorphism $D \in \mathrm{End}^n(\mathcal{A})$ of degree $n = |D|$ (i.e. $D(\mathcal{A}^\bullet) \subset \mathcal{A}^{\bullet+|D|}$) is called a derivation of degree n if

$$D(ab) = D(a)b + (-1)^{|a||D|}aD(b).$$

We denote by $\mathrm{Der}^n(\mathcal{A})$ the space of derivations of degree n. Some basic properties of derivations are

(1) Any $D \in \mathrm{Der}^k(\mathcal{A})$ vanishes on 1. This is immediate from the definition, applied to $a = b = 1$.
(2) Derivations are determined by their values on algebra generators.
(3) If D_1, D_2 are derivations of degree k_1, k_2 their (graded) commutator $[D_1, D_2]$ is a derivation of degree $k_1 + k_2$. Hence $\bigoplus_k \mathrm{Der}^k(\mathcal{A})$ is a graded Lie algebra.
(4) If $\mathcal{A}$ is graded commutative, then $\bigoplus_k \mathrm{Der}^k(\mathcal{A})$ is a left-module under $\mathcal{A}$.
(5) Any $a \in \mathcal{A}^k$ defines a graded derivation of degree k, by $\mathbb{Z}$-graded commutator: $D = [a, \cdot]$. Derivations of this type are called *inner*.

Given a graded Lie algebra $\mathfrak{g}$, one may similarly define a space $\mathrm{Der}^n(\mathfrak{g}) \subset \mathrm{End}^n(\mathfrak{g})$ of degree $n = |D|$ Lie algebra derivations by the requirement

$$D[x,y] = [Dx,y] + (-1)^{|D||x|}[x,Dy].$$

Again, $\bigoplus_{k\in\mathbb{Z}} \mathrm{Der}^k(\mathfrak{g})$ is a graded Lie algebra under graded commutator, and there is a natural homomorphism of graded Lie algebras $\mathfrak{g} \to \bigoplus_{k\in\mathbb{Z}} \mathrm{Der}^k(\mathfrak{g})$ given by $x \mapsto [x,\cdot]$.

In all of the above, one may replace the $\mathbb{Z}$-grading with a $\mathbb{Z}_2$-grading. We will use superscripts $\bar{0},\bar{1}$ to indicate the even, odd components. In particular, a $\mathbb{Z}_2$-graded vector space is a vector space with a direct sum decomposition $V = V^{\bar{0}} \oplus V^{\bar{1}}$. Replacing $\mathbb{Z}$ with $\mathbb{Z}_2$ in all of the above one defines $\mathbb{Z}_2$-graded algebras, Lie algebras, derivations and so on. It is common to refer to the $\mathbb{Z}_2$-graded objects as *super*-objects.

Any $\mathbb{Z}$-graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V^k$ inherits a $\mathbb{Z}_2$-grading by mod 2 reduction, i.e.

$$\mathcal{V}^{\bar{0}} = \bigoplus_{k\in\mathbb{Z}} V^{2k}, \quad \mathcal{V}^{\bar{1}} = \bigoplus_{k\in\mathbb{Z}} V^{2k+1}.$$

Definition 5.2. A *graded super-space* is a super-vector space V, together with a grading $V = \bigoplus_{k\in\mathbb{Z}}$ such that the $\mathbb{Z}_2$-grading is the mod 2 reduction of the $\mathbb{Z}$-grading. A *filtered super-space* is a super-graded vector space W, equipped with a filtration

$$W = \bigcup_{k\in\mathbb{Z}} W_{(k)}, \quad W_{(k)} \subset W_{(k+1)}$$

such that the associated graded space $\mathrm{gr}(W)$ with the induced $\mathbb{Z}_2$-grading is a graded super-space.

The definition of filtered super-space means that

$$W^{\bar{0}}_{(2k)} = W^{\bar{0}}_{(2k+1)}, \quad W^{\bar{1}}_{(2k+1)} = W^{\bar{1}}_{(2k+2)}$$

for all k.