Symplectic Geometry (Fall 2024)

Contents

1. Introduction

Symplectic geometry as its origins in physics, providing the mathematical framework for classical mechanics and geometrical optics. See Guillemin-Sternberg, *Symplectic Techniques in Physics* for an overview. For a rapid overview, recall Newton's equations of motion for a particle of mass m, moving in $\mathbb{R}^n$ under the force for a given potential $V(q_1, \ldots, q_n)$,

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i}.$$

This is a second order ordinary differential equation; the evolution of the system is determined once the initial position $q_i(0)$ and initial velocities $\dot{q}_i(0)$ are prescribed. One discovers that the *energy of the system*,

$$E = \frac{m}{2}\sum_i (\dot{q}_i)^2 + V(q)$$

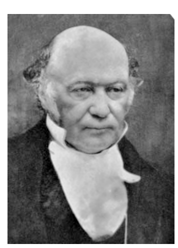
is *preserved*, i.e. it is constant along solution curves. (Verify by taking the t-derivative.) It is a standard tool in ODE theory to turn an n-th order ODE into a 1st-order ODE, by introducing the derivatives up to order $n-1$ as new variables. In the case at hand, this is done by considering the *linear momenta* $p_i = m\dot{q}_i$ as new variables. Thus, Newton's equations are turned into a system of equations

$$\dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad \dot{q}_i = \frac{1}{m}p_i;$$

regarded as a system of first order ODE's on *phase space* $\mathbb{R}^{2n}$; the energy function becomes the *Hamiltonian*

$$H(q,p) = \frac{1}{2m}\sum_i p_i^2 + V(q),$$

named after William Rowan Hamilton (1805-1865).



Actually, Newton's equations are expressed quite beautifully in terms of the Hamiltonian itself:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

One advantage of Hamilton's equations are their *symmetry properties*. By symmetry, we mean any (local) coordinate change from q, p to new coordinates $\tilde{q}, \tilde{p}$, in such a way that the differential equation in the new coordinates is

$$\dot{\tilde{p}}_i = -\frac{\partial \tilde{H}}{\partial \tilde{q}_i}, \quad \dot{\tilde{q}}_i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}.$$

with $\widetilde{H}(\widetilde{q}, \widetilde{p}) = H(q, p)$. Examples of such coordinate changes are

$$\tilde{p}_i = q_i, \quad \tilde{q}_i = -p_i,$$

but also

$$\tilde{q}_i = q_i, \; \tilde{p}_i = p_i + f_i(q_1, \ldots, q_n)$$

for arbitrary functions f_i. One may contrast this with the symmetry group of the gradient flow equation

$$\dot{x}_i = -\frac{\partial V}{\partial x_i}$$

here, we find that the symmetry group is finite-dimensional (the affine-linear transformations of $\mathbb{R}^n$)

In differential geometry, we prefer to think of first order autonomous ODE's as vector fields. In the case of Hamilton's equations, this is the *Hamiltonian vector field*

$$X_H = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \Big).$$

By introducing the *symplectic 2-form*

$$\omega = \sum_{i=1}^{n} \mathsf{d}q_i \wedge \mathsf{d}p_i$$

we may write Hamilton's equations concisely as

$$\iota(X_H)\omega = -\mathsf{d}H$$

Here ι denotes the *contraction* of a vector field with a 2-form, and

$$\mathsf{d}H = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial q_i} \mathsf{d}q_i + \frac{\partial H}{\partial p_i} \mathsf{d}p_i \Big).$$

We may define a *symplectic manifold* to be a manifold M equipped with a 2-form ω which, in suitable local coordinates, is given by $\omega = \sum_{i=1}^{n} \mathsf{d}q_i \wedge \mathsf{d}p_i$.

So much for our rapid overview. It should be said that symplectic geometry appears in many areas of mathematics and physics – Hamiltonian mechanics is only one of them. For example, they play an important role in the theory of partial differential equations – for example in the so-called *method of characteristics* and in the context of boundary conditions. They are used as tools in fields such as complex geometry or representation theory. By now, they are also studied in their own right – it turned out that the global topological aspects of symplectic manifolds are extremely interesting, leading to the field of symplectic topology.

A couple of historical remarks:[1] Symplectic geometry, as a subject of differential geometry, was developed in the 20th century. The 'symplectic group' was introduced by Hermann Weyl in his 1939 book [49] on the classical groups; the word symplectic is the Greek counterpart to the work 'complex' (which comes from Latin). Symplectic manifolds were first formally studied by Charles Ehresmann and his student Paulette Libermann, starting in in 1948.

Libermann's work included a first formulation and proof of Darboux's theorem in symplectic geometry [26] [2]). Other early contributions to symplectic geometry were made by various people in the early 1950s, such as Heinrich Guggenheimer [17] and André Lichnerowicz [28].

The theory blossomed in the late 1960s and early 1970s, with contributions by Souriau, Kostant, Arnold, Guillemin, Sternberg, Weinstein, Marsden and many others. Thurston [44] gave a first example of a symplectic manifold not admitting a Kähler structure, hence showing that the theory is really distinct from complex geometry. Symplectic geometry has since branched into various directions, ranging from Geometric Mechanics to Symplectic Topology.

These notes are an extended and corrected version of notes written in 1999/2000. My sources include the books by Abraham-Marsden [1], Guillemin-Sternberg [19], Liberman-Marle [27], and Weinstein [48], as well as an article by Sjamaar-Lerman [41]. Other references include the textbooks by Ana Canas da Silva [9], McDuff-Salamon [32], and Dwivedi-Herman-Jeffrey-van den Hurk [].

[1] I learned some of these details from the Ph.D. thesis of G. Ogando, `https://d-nb.info/1257915312/34`

[2] The classical Darboux theorem is a result on exterior differential systems.

2. Linear symplectic algebra

2.1. **Symplectic vector spaces.** Let E be a finite-dimensional, real vector space.

Definition 2.1. A *symplectic structure* on E is a skew-symmetric non-degenerate bilinear form

$$\omega\colon E \times E \to \mathbb{R}.$$

The pair (E, ω) is called a *symplectic vector space.*

Skew-symmetry of the symplectic form means that

$$\omega(v, w) = -\omega(w, v)$$

for all v, w; nondegeneracy means that the kernel

$$\ker \omega := \{v \in E |\, \omega(v, w) = 0 \text{ for all } w \in E\}$$

is trivial. One may rephrase these conditions in terms of the associated map

$$\omega^\flat\colon E \to E^*,\ \ v \mapsto \omega(v, \cdot).$$

The skew-symmetry of ω means that $\omega^\flat$ is equal to minus its adjoint, while nondegeneracy means that $\omega^\flat$ is an isomorphism (equivalently, injective).

Examples 2.2. (a) Let $E = \mathbb{R}^{2n}$ with basis vectors $e_1, \ldots, e_n, f_1, \ldots, f_n$. Then

(1) $$\omega(e_i, e_j) = 0,\ \ \omega(f_i, f_j) = 0,\ \ \omega(e_i, f_j) = \delta_{i,j}$$

defines a symplectic structure on $\mathbb{R}^{2n}$. This is the *standard symplectic structure* on $\mathbb{R}^{2n}$.

(b) Let V be a real vector space of dimension n, and V^* its dual space. Then

$$E = V \oplus V^*$$

has a natural symplectic structure given by

$$\omega((v, \alpha), (v', \alpha')) = \alpha'(v) - \alpha(v').$$

(c) Let E be a *complex* inner product space of (complex) dimension n, with inner product denoted $h\colon E \times E \to \mathbb{C}$. Then E, viewed as a real vector space, with bilinear form the imaginary part

$$\omega(v, w) = \mathrm{Im}(h(v, w))$$

is a symplectic vector space.

Actually, these three examples are really ‘the same’. Indeed, (b) turns into (a) once we choose a basis $e_1, \ldots, e_n$ of V, with dual basis $f_1, \ldots, f_n$ of V^*. Likewise, (c) turns into (a) once we choose a complex orthonormal basis $e_1, \ldots, e_n$ of E; together with $f_i = \sqrt{-1}e_i$ this defines a *real basis* of the underlying real vector space.

Definition 2.3. A *symplectomorphism* between symplectic vector spaces (E_1, ω_1) and (E_2, ω_2) is an isomorphism $A\colon E_1 \to E_2$ with

$$\omega_2(Av, Aw) = \omega_1(v, w)$$

for all $v, w \in W$. The group of symplectomorphisms from (E, ω) to itself is denoted

$$\mathrm{Sp}(E, \omega).$$

So, the claim is that the three examples above all are symplectomorphic. We shall soon see that in fact, all examples of $2n$-dimensional symplectic vector spaces are symplectomorphic to $\mathbb{R}^{2n}$ with the standard symplectic form.

2.2. **Subspaces of a symplectic vector space.** A subspace F of a symplectic vector space is called a *symplectic subspace* if the restriction of the 2-form $\omega\colon E \times E \to \mathbb{R}$ to $F \times F$ is still nondegenerate. In general, this need not be the case: the restriction may even by 0.

Definition 2.4. Let (E, ω) be a symplectic vector space. For any subspace $F \subseteq E$, we define the ω-orthogonal space F^ω by

$$F^\omega = \{v \in E,\ \omega(v, w) = 0 \text{ for all } w \in F\}$$

The isomorphism

$$\omega^\flat : E \to E^*,\ \langle \omega^\flat(v), w\rangle = \omega(v, w)$$

restricts to an isomorphism $F^\omega \to \mathrm{ann}(F)$, where

$$\mathrm{ann}(F) = \{\alpha \in E^* |\ \forall v \in F\colon \langle \alpha, v\rangle = 0\} \subseteq E^*$$

is the annihilator of F inside E^*.

Proposition 2.5. *We have that*

$$\dim F^\omega = \dim E - \dim F.$$

Furthermore,

$$(F^\omega)^\omega = F, \quad (F_1 + F_2)^\omega = F_1^\omega \cap F_2^\omega, \quad (F_1 \cap F_2)^\omega = F_1^\omega + F_2^\omega.$$

Exercise 2.6. Show that $F \subseteq E$ is symplectic if and only if $F \cap F^\omega = 0$.

Proof. The dimension formula follows from

$$\dim F^\omega = \dim(\mathrm{ann}(F)) = \dim E - \dim F.$$

Note next that all vectors in F are ω-orthogonal to all vectors in F^ω. Hence $F \subseteq (F^\omega)^\omega$; equality holds by the dimension formula. The other properties follow from the corresponding properties of annihilators. $\square$

Definition 2.7. A subspace $F \subseteq E$ of a symplectic vector space is called

(a) isotropic if $F \subseteq F^\omega$,
(b) coisotropic if $F^\omega \subseteq F$
(c) Lagrangian if $F = F^\omega$,

The set of all Lagrangian subspaces of E is called the *Lagrangian Grassmannian* and denoted $\mathrm{Lag}(E, \omega)$.

Note that F is isotropic if and only if the symplectic structure restricts to 0 on F. We will soon get a reasonably good understanding of the Lagrangian Grassmannian $\mathrm{Lag}(E, \omega)$. For now, it is a certain subset of the Grassmannian of n-dimensional subspaces of E. We shall soon see that it is, in fact, a submanifold.

Example 2.8 (Lagrangian coordinate subspaces). Let $E = \mathbb{R}^{2n}$ be the standard symplectic vector space. For any subset $I \subseteq \{1, \ldots, n\}$, the subspace

$$L_I = \mathrm{span}\{e_i | i \in I\} + \mathrm{span}\{f_i | i \notin I\}.$$

is a Lagrangian subspace.

Exercise 2.9. Let $E = \mathbb{R}^{2n}$ be the standard symplectic vector space. For subsets $I, J \subseteq \{1, \ldots, n\}$, consider the coordinate subspace

$$F_{IJ} = \mathrm{span}\{e_i | i \in I\} + \mathrm{span}\{f_j | j \in J\} \subseteq \mathbb{R}^{2n}.$$

What is $(F_{IJ})^\omega$, in this notation? What is the condition on I, J so that F_{IJ} is isotropic? Coisotropic? Neither?

Notice that F is isotropic if and only if F^ω is coisotropic. For example, every 1-dimensional subspace is isotropic and every codimension 1 subspace is coisotropic. Observe also that isotropic subspaces satisfy $\dim F \leq \dim F^\omega = \dim E - \dim F$. Hence

$$F \text{ isotropic} \quad \Rightarrow \quad \dim F \leq \frac{1}{2} \dim E,$$

with equality if and only if F is Lagrangian. By contrast,

$$F \text{ coisotropic} \quad \Rightarrow \quad \dim F \geq \frac{1}{2} \dim E$$

with equality if and only if F is Lagrangian.

Proposition 2.10. *For every symplectic vector space,* $\mathrm{Lag}(E, \omega) \neq \emptyset$*. In fact, one may choose two Lagrangian subspaces* $L, M \in \mathrm{Lag}(E, \omega)$ *with* $L \cap M = 0$*.*

Proof. (a) One constructs a Lagrangian subspace by induction, on dimension of isotropic subspaces. (We may start with the zero subspace.) Let L be an isotropic subspace of E. If L is not Lagrangian, pick any $v \in L^\omega - L$; then $L' = L + \mathrm{span}(v)$ is an isotropic subspaces of strictly larger dimension. The process ends once we obtain a Lagrangian subspace.

(b) Suppose $L \in \mathrm{Lag}(E, \omega)$ is a given Lagrangian subspace. Let $F \subseteq E$ be an isotropic subspace with $L \cap F = 0$. If F is not Lagrangian, we will show how to choose $v \in F^\omega - F$ so that $F' = F + \mathrm{span}(v)$ satisfies $F' \cap L = \{0\}$. (This process ends once we arrive at a Lagrangian subspace with $M \cap L = \{0\}$.) The subspace

$$F + (L \cap F^\omega) \subseteq F^\omega$$

is isotropic, since both F and $L \cap F^\omega$ are isotropic, and are ω-orthogonal to each other. Since F^ω is not isotropic, this must be a proper subspace. Choose

$$v \in F^\omega - (F + (L \cap F^\omega)).$$

To see that $F' = F + \mathrm{span}(v)$ still satisfies $F' \cap L = 0$, suppose $y \in F' \cap L$. Write $y = w + tv$ with $w \in F,\ t \in \mathbb{R}$. Then

$$tv = y - w \in (F + L) \cap L^\omega = F + (L \cap F^\omega).$$

By construction of v, this means $t = 0$, hence $y \in F \cap L = \{0\}$. □

An immediate consequence is that symplectic vector spaces are of even dimension: For any $L \in \mathrm{Lag}(E, \omega)$, we have that $\dim E = 2 \dim L$.

Remark 2.11. Lemma 2.10 can be strengthened: If $\mathcal{F}$ is a finite collection of subspaces $F \subseteq E$ (not necessarily isotropic) with the property $\dim F \leq \frac{1}{2} \dim E$, then there exists $L \in \mathrm{Lag}(E, \omega)$ such that $L \cap F = 0$ for all $F \in \mathcal{F}$.

Here is another way of constructing a Lagrangian complement L to a given Lagrangian subspace M:

Exercise 2.12. Let $M \in \mathrm{Lag}(E, \omega)$ be a Lagrangian subspace, and $F \subseteq M$ any vector space complement (so that $E = L \oplus F$). Show that there is a unique linear map $A \colon F \to L$ such that

$$F^\omega = \{v + Av |\ v \in F\}.$$

Show that all $F_t = \{v + tAv |\ v \in F\}$ for $t \in \mathbb{R}$ are vector space complements, and that $L = F_{1/2}$ is Lagrangian.

Some of you might prefer the following more conceptual version of the same exercise. Observe first that for a given Lagrangian subspace L, the map $E \to L^*$ (taking $w \in E$ to the linear functional $L \to \mathbb{R},\ v \mapsto \omega(w, v)$) vanishes exactly on L, and so descends to an isomorphism

$$E/L \to L^*.$$

Exercise 2.13. (a) Show that the space of vector space complements to $L \in \mathrm{Lag}(E, \omega)$ is canonically [3] an affine space[4] , with underlying linear space the vector space of

[3]Here, 'canonical' means, informally, that the constructions don't require any additional choices.

[4]An *affine space* may be defined as a manifold X with a free and transitive action of a vector space V. This means that for $v \in V$ and $x \in X$ there is an element $x' = v \cdot x$, in such a way that $0.x = x$ and

all linear maps $A\colon E/L = L^* \to L$. (This means that for any given complement F, we obtain a new complement $A \cdot F$, in such a way that $A_1 \cdot A_2 \cdot F = (A_1 + A_2) \cdot F$ and $0 \cdot F = F$.)

(b) Show that for any given complement F, the mid-point of the line segment from F to F^ω (as points of this affine space) is fixed under the involution, and hence is a Lagrangian complement.

Exercise 2.14. Let $L \in \mathrm{Lag}(E, \omega)$. Show that the set of *Lagrangian complements* to L (i.e., subspaces $M \in \mathrm{Lag}(E, \omega)$ with $L \cap M = \{0\}$) is canonically an affine space, with underlying vector space the self-adjoint linear maps $A\colon L^* \to L$.

This last exercise may be used to construct charts on the Lagrangian Grassmannian, making $\mathrm{Lag}(E, \omega)$ into a manifold of dimension equal to $n(n+1)/2$ where $n = \frac{1}{2} \dim E$.

2.3. **Symplectic bases.** Let (E, ω) be a symplectic vector space. By the results of the previous section, we may always choose a *Lagrangian splitting*

$$E = L \oplus M,$$

$L, M \in \mathrm{Lag}(E, \omega)$. Recall also that by Example 2.2 (b), the space $L \oplus L^*$ has a canonical symplectic structure.

Proposition 2.15. *The choice of a Lagrangian splitting $E = L \oplus M$ determines a symplectomorphism*

$$E \to L \oplus L^*,$$

where the symplectic form on $L \oplus L^*$ is given by the pairing.

Proof. Every $w \in M$ defines a linear functional $\alpha_w \in L^*$, by $\alpha_w(v) = \omega(v, w)$. If $\alpha_w = 0$, then $\omega(v, w) = 0$ for all $v \in L$, and hence $\omega(v, w) = 0$ for all $v \in E$, so $w = 0$. It follows that the map

$$M \to L^*,\ w \mapsto \alpha_w$$

is injective, and hence is an isomorphism. The resulting map

$$E = L \oplus M \to L \oplus L^*$$

is the desired symplectomorphism. □

In turn, this shows that any two symplectic vector spaces of a given dimension are symplectomorphic:

Theorem 2.16. *Every symplectic vector space (E, ω) of dimension $2n$ is symplectomorphic to $\mathbb{R}^{2n}$ with the standard symplectic form.*

$v_1.(v_2.x) = (v_1 + v_2).x$; moreover, any two elements x, x' are related in this way by a unique element $v \in V$. The choice of any base point $x_0 \in X$ gives an identification $X \cong V$.

Proof. Choose a Lagrangian splitting to identify $E = L \oplus L^*$. Let $e_1, \ldots, e_n$ be a basis for L and $f_1, \ldots, f_n$ the dual basis for $L^* \cong M$. By definition of the pairing,

$$\omega(e_i, f_j) = \alpha_{f_j}(e_i) = \delta_{ij}.$$

On the other hand, $\omega(e_i, e_j) = 0,\ \omega(f_i, f_j) = 0.$ □

Definition 2.17. A basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of (E, ω) for which ω has the standard form (1) is called a *symplectic basis*.

Remark 2.18. Let E_1, E_2 be two symplectic vector spaces of equal dimension, and with Lagrangian splittings $E_1 = L_1 \oplus M_1,\ E_2 = L_2 \oplus M_2$. Then any choice of isomorphism $L_1 \to L_2$ determines an isomorphism $M_1 = L_1^* \to M_2 = L_2^*$. Taken together, this defines a symplectomorphism $E_1 \to E_2$ taking the first splitting to the second splitting.

2.4. **Linear Reduction.** Let (E, ω) be a symplectic vector space.

Definition 2.19. A subspace $F \subseteq E$ is called *symplectic* if ω restricts to a nondegenerate bilinear form on F.

Equivalently, F is symplectic if and only if $F \cap F^\omega = \{0\}$ (since $F \cap F^\omega$ is the kernel of the restriction of ω to F). From this, we see that if F is symplectic, then so is F^ω, and furthermore $E = F \oplus F^\omega$.

A general subspace $F \subseteq E$ can be made into a symplectic vector space by taking the quotient by the kernel of $\omega|_{F \times F}$.

Proposition 2.20. *Given a subspace $F \subseteq E$, the quotient space*

$$E_F = F/(F \cap F^\omega)$$

inherits a symplectic form ω_F, where

$$\omega_F(\pi(v), \pi(w)) = \omega(v, w)$$

for all $v, w \in F$. Here $\pi : F \to F/(F \cap F^\omega)$ is the quotient map.

Proof. Observe first that ω_F is well-defined: For $v, w \in F$ the expression $\omega(v, w)$ depends only on $\pi(v), \pi(w)$. To see that it is symplectic, let $v \in F$ with $\pi(v) \in \ker(\omega_F)$. Then $\omega(v, w) = 0$ for all $w \in F$. That is, $v \in F \cap F^\omega$, that is, $\pi(v) = 0$. □

Definition 2.21. The space E_F with the symplectic form ω_F is called the *reduced space* or *symplectic quotient*.

In the special case that $F \cap F^\omega = 0$, the subspace F is called *symplectic*. Here, the restriction of ω to F is symplectic to begin with, and no quotient is needed. Another extreme case is when F is isotropic, in which case the symplectic quotient is $E_F = \{0\}$.

We are mainly interested in the case that F is coisotropic, with the symplectic quotient $E_F = F/F^\omega$.

Proposition 2.22. *Suppose $F \subseteq E$ is coisotropic and $L \in \mathrm{Lag}(E, \omega)$ Lagrangian. Let $L_F = \pi(L \cap F)$ be its image under the reduction map. Then $L_F \in \mathrm{Lag}(E_F)$.*

Proof. Since $L \cap F$ is isotropic, it is immediate that L_F is isotropic. To verify that L_F is Lagrangian we count dimensions:

$$\begin{aligned} \dim(L \cap F^\omega) &= \dim E - \dim(L \cap F^\omega)^\omega \\ &= \dim E - \dim(L + F) \\ &= \dim E - \dim L - \dim F + \dim(L \cap F) \\ &= \dim(L \cap F) - \dim F + \dim L. \end{aligned}$$

This shows that

$$\dim L_F = \dim(L \cap F) - \dim(L \cap F^\omega) = \dim F - \dim L,$$

on the other hand

$$\dim E_F = \dim F - \dim F^\omega = 2 \dim F - \dim E = 2 \dim F - 2 \dim L.$$

□

It may come as a surprise that the map

$$\mathrm{Lag}(E, \omega) \to \mathrm{Lag}(E_F, \omega_F)$$

constructed here is *not* continuous (unless the reduction is trivial). It is discontinuous exactly where L fails to be transverse to F. (Away from that subset, it is smooth as expected.)

Exercise 2.23. Let $E = \mathbb{R}^4$ with the standard symplectic basis, and $F = \mathrm{span}\{e_1, e_2, f_1\}$, so that $E_F \cong \mathbb{R}^2 = \mathrm{span}\{e_1, f_1\}$. For $t \in \mathbb{R}$ let

$$L_t = \mathrm{span}\{e_1 + t f_2,\ e_2 + t f_1\}.$$

Show that L_t is Lagrangian for all t, and compute $(L_t)_F$. You will find that the family $(L_t)_F$ is discontinuous at $t = 0$.

2.5. **Compatible complex structures.** Any complex vector space V may be regarded as a real vector space, by restricting the scalar multiplication from complex to real numbers. The $\mathbb{C}$-linear map $V \to V$ given by multiplication by $\sqrt{-1}$ becomes a real-linear map $J\colon V \to V$.

Conversely, a complex structure on a real vector space V is an automorphism $J : V \to V$ such that $J^2 = -\operatorname{Id}$. Given J, one may turn V into a complex vector space for the scalar multiplication $(a + \sqrt{-1}b)v = av + bJv$.

Definition 2.24. A complex structure J on a symplectic vector space (E, ω) is called ω-compatible if

$$g(v, w) = \omega(v, Jw)$$

defines a (real) inner product on V. We denote by

$$\mathcal{J}(E, \omega)$$

the set of compatible complex structures.

Notice that compatible complex structures J are symplectomorphisms:

$$\omega(Jv, Jw) = g(Jv, w) = g(w, Jv) = \omega(w, J^2 v) = -\omega(w, v) = \omega(v, w).$$

Similarly, J is orthogonal:

$$g(Jv, Jw) = \omega(Jv, J^2 w) = -\omega(Jv, w) = \omega(w, Jv) = g(w, v) = g(v, w).$$

Thus, $J^\top = J^{-1}$. Since $J^{-1} = -J$, we see that J is also skew-adjoint:

$$J^\top = -J.$$

We equip $\mathcal{J}(E, \omega)$ with the subset topology induced from $\operatorname{End}(E)$. Later we will see that it is in fact a smooth submanifold.

Example 2.25. For $E = \mathbb{R}^{2n}$ with standard symplectic structure, a compatible almost complex structure J is given by $Je_i = f_i,\ Jf_i = -e_i$. This identifies $(\mathbb{R}^{2n}, \omega, J)$ with $\mathbb{C}^n$.

Given $J \in \mathcal{J}(E, \omega)$, with corresponding real inner product $g \in \operatorname{Riem}(E)$, the space E becomes a complex inner product space, with inner product (Hermitian metric)

$$h(v, w) = g(v, w) + \sqrt{-1}\omega(v, w).$$

There is a corresponding unitary group $U(E)$ of complex-linear transformations of E preserving h. Note that in the equation $g(v, w) = \omega(v, Jw)$, any two of the structures ω, g, J determines the third. Hence, if a linear transformation preserves two of the structures, it also preserves the third. It follows that

$$\mathrm{U}(E) = \mathrm{Sp}(E, \omega) \cap \mathrm{O}(E, g)$$

where $\mathrm{O}(E, g)$ is the group of orthogonal transformations (preserving g).

The following theorem gives a convenient method for constructing compatible complex structures. For any vector space V let

$$\mathrm{Riem}(V) = \{ \text{ real inner products } g\colon V \times V \to \mathbb{R}\}$$

the set of inner products (Riemannian metrics) on V. It is a convex open cone inside the space $\mathrm{Sym}^2 V^*$ of symmetric bilinear forms on V. [5]

For a symplectic vector space (E, ω), we have a map

$$\psi\colon \mathcal{J}(E,\omega) \to \mathrm{Riem}(E)$$

taking J to its associated metric. This map has a left inverse:

Theorem 2.26. *Let (E,ω) be a symplectic vector space. There is a canonical continuous retraction*

$$\phi\colon\ \mathrm{Riem}(E) \to \mathcal{J}(E,\omega).$$

with $\phi \circ \psi(J) = J$.

Proof. Given $k \in \mathrm{Riem}(E)$ let A be defined by

$$k(v,w) = \omega(v, Aw).$$

Note that A is invertible, since $Aw = 0$ would imply that $k(v,w) = \omega(v,Aw) = 0$ for all v, hence $w = 0$. Writing $w = A^{-1}y$, we have we have

$$k(v, A^{-1}y) = \omega(v,y) = -\omega(y,v) = -k(y, A^{-1}v) = k(A^{-1}v, y).$$

This shows that A^{-1} is skew-adjoint (with respect to k), and hence A is skew-adjoint:

$$A^\top = -A.$$

Hence, $-A^2 = A^\top A$ is positive definite, and the absolute value

$$|A| = (A^\top A)^{1/2} = (-A^2)^{1/2},$$

is defined and commutes with A. The operator

$$J = A|A|^{-1}$$

satisfies $J^2 = -\mathrm{Id}$, hence defines a complex structure. The calculation

$$\omega(v, Jw) = \omega(v, A|A|^{-1}w) = k(v, |A|^{-1}w) = k(|A|^{-1/2}v, |A|^{-1/2}w)$$

shows that $g(v,w) = \omega(v, Jw)$ defines a (positive definite) inner product. We thus obtain a continuous map $\phi\colon\ \mathrm{Riem}(E) \to \mathcal{J}(E,\omega)$, taking k to J. Applying this procedure to the metric $\psi(J_0)$ for a compatible complex structure J_0, we obtain $A = J_0$, $|A| = I$, hence $J = J_0$. This shows $\phi \circ \psi = \mathrm{id}$. ☐

[5] For any vector space V, we denote by $\mathrm{Sym}^k(V)$ its k-th symmetric power. It may be identified with the space of k-multilinear maps $V^* \times \cdots \times V^* \to \mathbb{R}$ that are symmetric in all entries.

Corollary 2.27. *The space $\mathcal{J}(E,\omega)$ is contractible. In particular, any two compatible complex structures can be deformed into each other.*

Proof. Let $X = \mathrm{Riem}(E)$ and $Y = \mathcal{J}(E,\omega)$. The space X is contractible since it is a convex subset of a vector space. An explicit contraction $h^X\colon [0,1]\times X \to X,\ (t,x) \mapsto h^X_t$ is given by

$$h^X_t(g) = (1-t)g + tg_0$$

for any fixed choice of $g_0 \in \mathrm{Riem}(V)$. Since Y is a deformation retract of X, it too is contractible: $h^Y_t = \phi \circ h^X_t \circ \psi$ defines a retraction of Y. $\square$

The symplectic group $\mathrm{Sp}(E,\omega)$ acts on the space $\mathcal{J}(E,\omega)$ of compatible complex structures, by

$$A \cdot J = AJA^{-1}.$$

Indeed, it is clear that $(A\cdot J)^2 = -I$, so $A \cdot J$ is a complex structure. Furthermore, if J defines the inner product g, then

$$\omega(v,(A\cdot J)w) = \omega(v, AJA^{-1}w) = \omega(A^{-1}v, JA^{-1}w) = g(A^{-1}v, A^{-1}w) = (A\cdot g)(v,w)$$

where $A \cdot g$ (defined by the last equation) is again an inner product).

Proposition 2.28. *The action of the symplectic group on the space of compatible complex structures is transitive, with stabilizer at $J \in \mathcal{J}(E,\omega)$ equal to the unitary group $U(E)$ (with respect to J) That is, $\mathcal{J}(E,\omega)$ is a homogeneous space*

$$\mathcal{J}(E,\omega) = \mathrm{Sp}(E,\omega)/U(E). \tag{2}$$

Proof. Given $J, J' \in \mathcal{J}(E,\omega)$, let $e_1,\ldots,e_n$ be an orthonormal basis for the complex inner product defined by J, and $e'_1,\ldots,e'_n$ an orthonormal basis for the complex inner product defined by J'. We obtain symplectic bases $e_1,\ldots,e_n,f_1,\ldots,f_n$ by letting $f_i = Je_i$, and similarly $e'_1,\ldots,e'_n,f'_1,\ldots,f'_n$. The transformation A defined by

$$A(e_i) = e'_i,\ A(f_i) = f'_i$$

is symplectic, and it satisfies $J' = AJA^{-1}$, as one verifies on the basis:

$$(AJA^{-1})(e'_i) = (AJ)(e_i) = A(f_i) = f'_i = J'e'_i$$

and similarly $(AJA^{-1})(f'_i) = -e'_i = J'f'_i$. This shows that the action is transitive. We have already mentioned that the symplectic transformations preserving a given J are exactly the unitary transformations. $\square$

2.6. **The group $\mathrm{Sp}(E,\omega)$ of linear symplectomorphisms.** It is time to discuss the symplectic group of a symplectic vector space in some more detail.

Proposition 2.29. *The symplectic group* $\mathrm{Sp}(E,\omega)$ *is a connected Lie group of dimension*
$$\dim \mathrm{Sp}(E,\omega) = 2n^2 + n$$
where $\dim E = 2n$.

Proof. It is a standard result in Lie theory (Cartan's theorem) that any topologically closed subgroup of a Lie group is again a Lie group. Hence $\mathrm{Sp}(E,\omega) \subseteq \mathrm{GL}(E)$ is a Lie group. To determine its dimension, consider the action of the general linear group $\mathrm{GL}(E)$ acts transitively on the open subset $\mathcal{U} \subseteq \wedge^2 E^*$ consisting of non-degenerate 2-forms. From the fact that any two symplectic vector spaces of the same dimension are symplectomorphic, it follows that this action is transitive. The stabilizer at ω is $\mathrm{Sp}(E,\omega)$. It follows that $\mathcal{U} = \mathrm{GL}(E)/\,\mathrm{Sp}(E,\omega)$.

$$\dim \mathrm{Sp}(E,\omega) = \dim \mathrm{GL}(E) - \dim \mathcal{U} = (2n)^2 - \frac{(2n)(2n-1)}{2} = 2n^2 + n.$$

The fact that $\mathrm{Sp}(E,\omega)$ is connected may be seen, for example, from $\mathrm{Sp}(E,\omega)/\,\mathrm{U}(E) = \mathcal{J}(E,\omega)$, and the fact that $\mathrm{U}(E)$ and $\mathcal{J}(E,\omega)$ are both connected. ☐

Incidentally, this also gives $\dim \mathcal{J}(E,\omega) = \dim \mathrm{Sp}(E,\omega) - \dim U(E) = n^2 + n$. Below, we'll see another proof of this fact.

Remark 2.30. In the theory of Lie groups, there is another Lie group that is also called the symplectic group, and usually denoted $\mathrm{Sp}(n)$. (It is the unitary group of quaternionic space $\mathbb{H}^n$, if you know what that means.) This group is *compact*, whereas our symplectic group $\mathrm{Sp}(\mathbb{R}^{2n},\omega)$ is non-compact. (See below.) The two groups are closely related: working over $\mathbb{C}$, we can consider complex symplectic vector spaces and the the complex symplectic group $\mathrm{Sp}(\mathbb{C}^{2n},\omega) \subseteq \mathrm{GL}(2n,\mathbb{C})$; this contains both as real subgroups
$$\mathrm{Sp}(n) \subseteq \mathrm{Sp}(\mathbb{C}^{2n},\omega) \supseteq \mathrm{Sp}(\mathbb{R}^{2n},\omega).$$
In the terminology of (reductive) Lie groups, $\mathrm{Sp}(n)$ is the *compact real form* of while $\mathrm{Sp}(\mathbb{R}^{2n},\omega)$ is the *split real form*. Another example of such a correspondence is
$$\mathrm{U}(n) \subseteq \mathrm{GL}(n,\mathbb{C}) \supseteq \mathrm{GL}(n,\mathbb{R}).$$

Proposition 2.31. *The closed subgroup of* $\mathrm{Sp}(E,\omega)$ *preserving a given Lagrangian splitting* $E = L \oplus M$ *is canonically isomorphic to* $\mathrm{GL}(L)$*. In particular,* $\mathrm{Sp}(E,\omega)$ *is noncompact.*

Proof. Suppose $A \in \mathrm{Sp}(E,\omega)$ preserving a Lagrangian splitting $E = L \oplus M$, and let $B = A|_L$ be the restriction. Since ω gives a nondegenerate pairing between L, M, we have $M = L^*$; under this identification, $A|_M = (B^*)^{-1}$ (the conjugate transpose). Check: if $w \in W$, corresponding to $\alpha_w \in L^*$, and all $v \in L$ we have

$$\langle \alpha_{Aw}, v\rangle = \omega(v, Aw) = \omega(A^{-1}v, w) = \langle \alpha_w,\, A^{-1}v\rangle = \langle \alpha_w,\, B^{-1}v\rangle = \langle (B^{-1})^*\alpha_w, v\rangle.$$

Conversely, given $B \in \mathrm{GL}(L)$, the transformation given by $A|_L = B,\ A|_M = (B^*)^{-1}$ is symplectic. □

Fix $J \in \mathcal{J}(E,\omega)$. Let g be the inner product defined by J, and let $(\cdot)^\top$ denote the transpose of an endomorphism with respect to g.

Proposition 2.32. *An automorphism $A \in \mathrm{GL}(E)$ is in $\mathrm{Sp}(E,\omega)$ if and only if*

$$A^\top = J\,A^{-1}J^{-1}$$

where $A^\top$ is the transpose of A with respect to g. In particular, $\mathrm{Sp}(E,\omega)$ is invariant under transposition.

Proof. We have

$$\begin{aligned} A \in \mathrm{Sp}(E,\omega) &\Leftrightarrow \forall v,w \in E\colon \omega(Av,Aw) = \omega(v,w)\\ &\Leftrightarrow \forall v,w \in E\colon g(JAv,Aw) = g(Jv,w)\\ &\Leftrightarrow \forall v,w \in E\colon g(A^\top JAv,w) = g(Jv,w)\\ &\Leftrightarrow A^\top JA = J \qquad\qquad \Leftrightarrow A^\top = JA^{-1}J^{-1}\end{aligned}$$

□

Theorem 2.33 (Symplectic eigenvalue theorem)**.** *Let $A \in \mathrm{Sp}(E,\omega)$. Then $\det(A) = 1$, and all eigenvalues of A other than $1, -1$ come in either pairs*

$$\lambda, \overline{\lambda}, \quad |\lambda| = 1$$

or quadruples

$$\lambda, \overline{\lambda},\ \lambda^{-1},\ \overline{\lambda}^{-1}, \quad |\lambda| \neq 1.$$

The members of each multiplet all appear with the same multiplicity. The multiplicities of eigenvalues -1 and $+1$ are even.

Proof. Choose a compatible complex structure J. Since $\det(J) = 1$, Proposition 2.32 shows that

$$\det(A) = \det(A^\top) = \det(A^{-1}) = \det(A)^{-1};$$

hence $\det(A) = 1$ since $\mathrm{Sp}(E,\omega)$ is connected. For any $A \in \mathrm{GL}(E)$ the eigenvalues appear in complex-conjugate pairs of equal multiplicity (complex conjugation takes eigenvectors for λ to eigenvectors for $\overline{\lambda}$). For $A \in \mathrm{Sp}(E,\omega)$, the eigenvalues λ, λ^{-1} have equal multiplicity since by the matrices $A^\top$ and A^{-1} are similar: $A^\top = J\,A^{-1}J^{-1}$. The multiplicities of eigenvalues -1 and $+1$ have to be even since the product of all eigenvalues equals $\det A = 1$. □

Let us finally describe the Lie algebra of the symplectic group. For any matrix Lie group G, the Lie algebra $\mathfrak{g}$ consists of all $\xi \in \mathfrak{gl}(E)$ such that $\exp(t\xi) \in G$ for all t. In the case of the symplectic group, this gives:

Proposition 2.34. *The Lie algebra $\mathfrak{sp}(E, \omega)$ of the symplectic group consists of all $\xi \in \mathfrak{gl}(E)$ such that*

$$\omega(\xi v, w) + \omega(v, \xi w) = 0.$$

Given a compatible complex structure J, with corresponding inner product g and transposition operation $\top$, an endomorphism $\xi \in \mathfrak{gl}(E)$ is in $\mathfrak{sp}(E)$ if and only if

$$\xi^\top = J\xi J.$$

Proof. Taking the derivative of the equation $\omega(\exp(t\xi)v, \exp(t\xi)w) = \omega(v, w)$, we arrive at the condition $\omega(\xi v, w) + \omega(v, \xi w) = 0$. Similarly, taking the t-derivative of $\exp(t\xi)^\top J \exp(t\xi) = J$, we arrive at $\xi^\top J + J\xi = 0$, which is equivalent to $\xi^\top = J\xi J$. These are necessary conditions for ξ to be in $\mathfrak{sp}(E, \omega)$; by dimension count they are also sufficient. □

Exercise 2.35. For $E = \mathbb{R}^{2n}$ with the standard symplectic basis and the standard symplectic structure, J is given by a matrix in block form,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Verify that a matrix, also written in block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is symplectic if and only if

$$a^\top c = c^\top a, \quad b^\top d = d^\top b, \quad a^\top d - b^\top c = I.$$

Also find a similar description of the Lie algebra $\mathfrak{sp}(\mathbb{R}^{2n})$.

As a special case, for $n = 1$ we have $\mathrm{Sp}(\mathbb{R}^2, \omega) = \mathrm{SL}(2, \mathbb{R})$.

Exercise 2.36. Suppose $A \in \mathrm{Sp}(E, \omega)$ is symmetric, $A = A^\top$ so that A is diagonalizable and all eigenvalues are real. Let $E_\lambda = \ker(A - \lambda)$ denote the eigenspace. Then

$$E_\lambda^\omega = \oplus_{\lambda\mu \neq 1} E_\mu.$$

In particular, all E_λ for eigenvalues $\lambda \notin \{1, -1\}$ are isotropic while the eigenspaces for $\lambda \in \{1, -1\}$ are symplectic. Moreover, $E_\lambda \oplus E_{\lambda^{-1}}$ is symplectic.

2.7. **Polar decomposition of symplectomorphisms.** Recall that invertible matrices $A \in \mathrm{GL}(m)$ admits a unique polar decomposition

$$A = U|A|,$$

where U is orthogonal (i.e., $U^\top = U^{-1}$) and $|A|$ is positive definite. Here $B = (A^\top A)^{1/2}$, using functional calculus. Furthermore, the exponential map gives a diffeomorphism between symmetric (selfadjoint) matrices and positive definite matrices. This shows that as a manifold,

$$\mathrm{GL}(m) = \mathrm{O}(m) \times \{\xi |\ \xi^\top = \xi\}$$

where the second factor is a vector space.

For any matrix Lie group $G \subseteq \mathrm{GL}(m)$ that is invariant under transposition $A \mapsto A^\top$, the polar decomposition of matrices restricts to define a polar decomposition for G, called the *Cartan decomposition*

$$G = K \times P,$$

with $K = G \cap \mathrm{O}(m)$ and P the positive definite matrices in G; furthermore,

$$P = \exp(\mathfrak{p})$$

where $\mathfrak{p} = \mathfrak{g} \cap \{\xi |\ \xi = \xi^\top\}$. The exponential map for symmetric matrices restricts to a diffeomorphism $\mathfrak{p} \to P$. In conclusion, the map $(C, \xi) \mapsto C \exp(\xi)$ gives a global diffeomorphism

$$G \cong K \times \mathfrak{p}.$$

The Cartan decomposition shows that G is homotopy equivalent to its *maximal compact subgroup* K. That is, K captures all the topology. The Lie algebras satisfy $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as vector spaces, and

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

We specialize to the case of $\mathrm{Sp}(E, \omega)$. Fix a compatible complex structure J, and let $(\cdot)^\top$ be the corresponding transposition map. We had observed that $\mathrm{Sp}(E, \omega)$ is invariant under transposition; indeed, $A^\top = -JA^{-1}J$.

Theorem 2.37 (Polar decomposition)**.** *The symplectic group $G = \mathrm{Sp}(E, \omega)$ has maximal compact subgroup $K = \mathrm{U}(E)$. Its Cartan decomposition $G = KP$ is given by $P = \exp \mathfrak{p}$, with*

$$\mathfrak{p} = \{\xi \in \mathfrak{sp}(E, \omega) |\ J\xi = -\xi J\}.$$

Proof. We may pick a basis identifying $E \cong \mathbb{R}^{2n}$, with standard symplectic form and standard complex structure. We had already observed that $\mathrm{Sp}(\mathbb{R}^{2n}, \omega) \cap \mathrm{O}(2n) = U(n)$, the unitary group of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Hence, $\mathrm{U}(E) \cong \mathrm{U}(n)$ is the maximal compact subgroup. On the other hand, $\mathfrak{p} \subseteq \mathfrak{sp}(E, \omega)$ is defined by the condition $\xi = \xi^\top$. But recall that $\mathfrak{sp}(E, \omega)$ is characterized by the condition $\xi^\top = J\xi J$. This gives the alternative description in the theorem. □

Since $\mathcal{J}(E,\omega) = \mathrm{Sp}(E,\omega)/\mathrm{U}(E)$, this also shows:

Corollary 2.38. *Any fixed $J \in \mathcal{J}(E,\omega)$ defines a canonical diffeomorphism between $\mathcal{J}(E,\omega)$ and the vector space $\mathfrak{p}$.*

In particular, we see once again that $\mathcal{J}(E,\omega)$ is contractible.

Corollary 2.39. $\mathrm{Sp}(E,\omega)$ *deformation retracts onto* $\mathrm{U}(E)$. *In particular, it is connected and has fundamental group $\pi_1(\mathrm{Sp}(E,\omega)) = \mathbb{Z}$.*

Indeed, it is known that the determinant map $\det\colon \mathrm{U}(n) \to \mathrm{U}(1)$ induces an isomorphism on fundamental groups. Composing with the identification $\pi_1(\mathrm{U}(E)) \cong \pi_1(\mathrm{Sp}(E,\omega))$, we obtain an isomorphism

$$\mu:\ \pi_1(\mathrm{Sp}(E,\omega)) \to \mathbb{Z}.$$

This is known as the *Maslov index* of a loop of symplectomorphisms. It is independent of the choice of J, since any two choices are homotopic. There are other kinds of Maslov indices related to the topology of the Lagrangian Grassmannian, which we shall discuss next.

Exercise 2.40. Show that the Maslov index is a group morphism. That is, if A, B are two loops and AB their pointwise product, then $\mu(AB) = \mu(A) + \mu(B)$.

2.8. **The Lagrangian Grassmannian.** The manifold structure on the Grassmannian $\mathrm{Gr}(k,V)$ of k-planes in an n-dimensional vector space V can be defined in two equivalent ways. In the first approach, one constructs explicit charts: Given any fixed subspace $S \subseteq V$ of dimension $n-k$, the set of subspaces transverse to S is canonically an affine space, and serves as a chart domain. In the second approach, one takes any subgroup $G \subseteq \mathrm{GL}(V)$ whose action on $\mathrm{Gr}(k,V)$ is transitive (e.g. $\mathrm{GL}(V)$ itself, or the orthogonal group for some inner product), then $\mathrm{Gr}(k,V) = G/H$ where H is the stabilizer of any fixed subspace. By Cartan's theorem, H is a Lie subgroup, and by standard results the homogeneous space G/H is a manifold.

Similar approaches also work for the Lagrangian Grassmannian. Let (E,ω) be a symplectic vector space of dimension $2n$. Choose $J \in \mathcal{J}(E,\omega)$ as before, defining an inner product g. The symplectic group $\mathrm{Sp}(E,\omega)$ acts on the Lagrangian Grassmannian, by

$$A \cdot L = A(L).$$

This action is transitive. In fact, already the restriction to the unitary group (for given choice of J) is transitive:

Proposition 2.41. *The action of $U(E)$ on $\mathrm{Lag}(E,\omega)$ is transitive, with stabilizer at $L \in \mathrm{Lag}(E,\omega)$ the orthogonal group $\mathrm{O}(L)$ (for the restriction $g|_{L\times L}$). That is, $\mathrm{Lag}(E,\omega)$ is a homogeneous space*

$$\mathrm{Lag}(E,\omega) = \mathrm{U}(E)/\,\mathrm{O}(L).$$

In particular, it is a connected manifold of dimension $\frac{n(n+1)}{2}$.

Proof. Let $h = g + \sqrt{-1}\omega$ be the complex inner product defined by J, ω. Since $h|_{L\times L} = g|_{L\times L}$ any g-orthonormal basis $e_1, \ldots, e_n$ of L (as a real inner product space) is also an h-orthornormal basis for E (as a complex inner product space). Given another Lagrangian subspace $L' \in \mathrm{Lag}(E,\omega)$, with orthonormal basis $e'_1, \ldots, e'_n$, the linear map taking $e_1, \ldots, e_n$ to $e'_1, \ldots, e'_n$ is unitary. Hence $U(E)$ acts transitively on the set of Lagrangian subspaces. The stabilizer for this action are linear transformations taking $e_1, \ldots, e_n$ to another orthonormal basis of L; it is hence the orthogonal group $\mathrm{O}(L)$. The dimension is computed as

$$\dim \mathrm{U}(n) - \dim \mathrm{O}(n) = \frac{n(n+1)}{2}.$$

□

For $E = \mathbb{R}^{2n}$, with L the subspace spanned by $e_1, \ldots, e_n$, we obtain

$$\mathrm{Lag}(\mathbb{R}^{2n},\omega) = \mathrm{U}(n)/\,\mathrm{O}(n).$$

For $n = 1$, this reduces to $\mathrm{Lag}(\mathbb{R}^2,\omega) = \mathrm{U}(1)/\,\mathrm{O}(1) = S^1/\{\pm 1\} = \mathbb{R}P(1)$, which agrees with the fact that the Lagrangian subspaces in $\mathbb{R}^2$ are just the 1-dimensional subspaces.

On the other hand, let us consider the charts approach. Given $M \in \mathrm{Lag}(E,\omega)$, let

$$\mathrm{Lag}(E;M) = \{L \in \mathrm{Lag}(E,\omega)|\, L \cap M = \{0\}\}$$

be the space of *Lagrangian* complements.

Proposition 2.42. *The space $\mathrm{Lag}(E;M)$ is canonically an affine space, with the space $\mathrm{Sym}^2(M)$ of symmetric bilinear forms on M^* as its underlying vector space. Given $\beta \in \mathrm{Sym}^2(M)$ and $L, L' \in \mathrm{Lag}(E;M)$ with $\beta \cdot L = L'$, there is a canonical isomorphism*

$$\ker(\beta) \cong L \cap L'.$$

Proof. Note that we can identify the symmetric bilinear forms $\beta\colon M^* \times M^* \to \mathbb{R}$ with self-adjoint linear maps $\psi\colon M^* \to M$, via $\beta(\mu_1,\mu_2) = \langle \mu_1, \psi(\mu_2)\rangle$.

Let $\pi\colon E \to M^*$ be the linear map taking v to the restriction of the linear functional $\omega^\flat(v) \in E^*$ to $M \subseteq E$. The kernel of this linear map is the subspace M, defining an exact sequence

$$0 \to M \to E \to M^* \to 0.$$

Giving a Lagrangian complement L to M is the same as giving a Lagrangian splitting

$$j\colon M^* \to E$$

of this sequence; i.e., a splitting whose image $L = j(M^*)$ is isotropic (hence Lagrangian). The set of splittings of this sequence is an affine space over the space of linear maps $\mathrm{Hom}(M^*, M)$. [6] That is, if j is a given splitting, then every other splitting is of the form $j'(\mu) = j(\mu) + \psi(\mu)$ for a unique linear map $\psi\colon M^* \to M$. The condition that j' is isotropic means

$$\begin{aligned} 0 &= \omega(j'(\mu_1), j'(\mu_2)) \\ &= \omega(j(\mu_1) + \psi(\mu_1),\ j(\mu_1) + \psi(\mu_2)) \\ &= \omega(j(\mu_1), \psi(\mu_2)) - \omega(j(\mu_2), \psi(\mu_2)) \\ &= \langle \mu_1, \psi(\mu_2)\rangle - \langle \mu_2, \psi(\mu_1)\rangle. \end{aligned}$$

This is exactly the condition that ψ is self-adjoint; equivalently the associated bilinear form is symmetric. This gives a free and transitive action of $\mathrm{Sym}^2(M)$ on the space of isotropic splittings, or equivalently on $\mathrm{Lag}(E; M)$.

Given two subspace $L, L' \in \mathrm{Lag}(E; M)$, the condition $w \in L \cap L'$ means that $j_L(\mu) = j_{L'}(\mu)$ where $\mu = \pi(v)$. That is, $\psi(\mu) = 0$, which is the same as $\mu \in \ker(\beta)$. □

Remark 2.43. The symmetric bilinear form on M^* may be described directly as

$$\beta(\mu_1, \mu_2) = \omega(j_{L'}(\mu_1), j_L(\mu_2))$$

where $j_L\colon M^* \to L$ is the inverse map to $\pi|_L$. In other words, β is the composition of

$$\omega|_{L' \times L}\colon L' \times L \to \mathbb{R}$$

with the identifications $L' \cong M^*,\ L \cong M^*$ given by restriction of π.

Remark 2.44. The coordinate version of this proposition is as follows: Consider $\mathbb{R}^{2n}$ with standard symplectic basis, and let $M = \mathrm{span}\{f_1, \ldots, f_n\}$. Every $L \in \mathrm{Lag}(\mathbb{R}^{2n}; M)$ has a unique basis of the form $g_i = e_i + \sum_j S_{ij} f_j$. The condition $L^\omega = L$ translates into S being a symmetric matrix. If L, L' are two such Lagrangian subspaces and g_j, g_j' the corresponding bases, the pairing $\omega :\ L \times L' \to \mathbb{R}$ is given by

$$\omega(g_i, g_j') = S_{ij} - S_{ij}'.$$

That is, the dimension of the intersection $L \cap L' = L^\omega \cap L'$ equals the nullity of $S - S'$.

[6] Quite generally, the set of splittings of an exact sequence of vector spaces $0 \to A \to B \to C \to 0$ is an affine space over $\mathrm{Hom}(C, A)$; given one such splitting $j\colon C \to B$, all other splittings are obtained by adding a linear map from C with values in the kernel of the map $B \to C$, i.e., with values in A.

The sets $\mathrm{Lag}(E; M)$ for $M \in \mathrm{Lag}(E,\omega)$ form an open covering of the Lagrangian Grassmannian. One we fix $L \in \mathrm{Lag}(E; M)$, we obtain an identification of the affine space $\mathrm{Lag}(E; M)$ with the vector space $\mathrm{Sym}^2(M^*)$. One hence obtains charts for the Lagrangian Grassmannian $\mathrm{Lag}(E,\omega)$. In particular, we see again that the dimension is $n(n+1)/2$.

Exercise 2.45. Show that for $L \in \mathrm{Lag}(E,\omega)$, there is a canonical vector space isomorphism

$$T_L \,\mathrm{Lag}(E,\omega) \cong \mathrm{Sym}^2(L^*)$$

(the space of symmetyric bilinear forms on $L \times L \to \mathbb{R}$) (One way of constructing the isomorphism is as follows. Given a family of Lagrangian subspaces L_t representing a tangent vector at $L = L_0$, and letting $v, w \in L$, let $w_t \in L_t$ with $w_t \in L_t,\ w_0 = w$. Then

$$\frac{d}{dt}\Big|_{t=0}\omega(v, w_t)$$

depends only on v, w, and gives a symmetric bilinear form.)

2.9. **Maslov indices, I.** Since $\det :\ \mathrm{U}(E) \to S^1$ takes values ± 1 on $\mathrm{O}(L)$, its square descends to a function $\det^2 : \mathrm{Lag}(E,\omega) \to S^1$. This function depends on the choice of J and L, but any two such choices are homotopic, and hence the homotopy class of the map $\det^2$ is choice independent.

Theorem 2.46 (Arnold [2])**.** *The map* $\det^2 : \mathrm{Lag}(E,\omega) \to S^1$ *defines an isomorphism of fundamental groups,*

$$\mu :\ \pi_1(\mathrm{Lag}(E,\omega)) \cong \pi_1(S^1) = \mathbb{Z}$$

(independent of the choice of L or J). It is called the Maslov index of a loop of Lagrangian subspaces.

Proof. [7] Choose an orthonormal basis for L to identify $L = \mathbb{R}^n \subseteq \mathbb{C}^n$ and $E \cong \mathbb{C}^n$ so that $\mathrm{Lag}(E,\omega) = \mathrm{U}(n)/\,\mathrm{O}(n)$. For $t \in [0,1]$ let $A_k(t) \in \mathrm{U}(n)$ be the diagonal matrix

$$A_k(t) = \begin{pmatrix} e^{\sqrt{-1}k\pi t} & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

Since $A_k(1) \in \mathrm{O}(n)$, we obtain a loop $L_k(t) = A_k(t)\mathbb{R}^n$. This loop has Maslov index k, which shows that μ is surjective.

To show that μ is injective, it is enough to show that any loop $L(t),\ t \in [0,1]$ with $L(0) = L(1)$ the base point $\mathbb{R}^n \subseteq \mathbb{R}^{2n}$ can be deformed into one of the loops L_k. To see this, choose any lift of $L(t)$ to a path $A(t) \in \mathrm{U}(t)$ (not necessarily closed) with $A(0) = I$.

[7] We didn't present this in class; same for following proposition.

Then $A(1) \in \mathrm{O}(n)$. Since $\mathrm{O}(n)$ has two connected components distinguished by the sign of the determinant, we can arrange that $A(1)$ is a diagonal matrix

$$A(1) = \begin{pmatrix} \pm 1 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

We want to show that $A(t)$ can be deformed into one of the paths $A_k(t)$ while keeping the endpoints fixed. If the upper left corner of $A(1)$ is $+1$, so that $A(1) = I$; and $A(t)$ is actually a loop, this follows since the map $\pi_1(U(n)) \to \pi_1(S^1) = \mathbb{Z}$ is an isomorphism; in this case k must be even. If the upper left corner of $A(1)$ is -1, then the path $B(t) = A(t)\,A_{-1}(t)$ (pointwise product) is a loop, i.e. can be deformed into $A_{2l}(t)$ for some l. Thus $A(t) = B(t)A_1(t)$ can be deformed into $A_{2l}(t)\,A_1(t) = A_{2l+1}(t)$. □

Proposition 2.47. *Let $A : S^1 \to \mathrm{Sp}(E, \omega)$ and $L : S^1 \to \mathrm{Lag}(E, \omega)$ be loops of symplectomorphisms resp. of Lagrangian subspaces. Then*

$$\mu(A(L)) = \mu(L) + 2\mu(A).$$

Proof. Using the notation from the previous proof we may assume that A takes values in $\mathrm{U}(n)$ since $\mathrm{Sp}(E, \omega) = U(E) \times \mathfrak{p}$. Any such A is homotopic to a loop A_{2l} where $l = \mu(A)$. The proposition follows since $A_{2l} \circ A_k = A_{k+2l}$. □

Let $M \in \mathrm{Lag}(E, \omega)$ be given. The fact that $\mathrm{Lag}(E; M)$ is contractible can be used to generalize the Maslov index from loops to arbitrary paths

$$L : [0, 1] \to \mathrm{Lag}(E, \omega)$$

so long as $L(0),\ L(1) \in \mathrm{Lag}(E; M)$. Indeed, we can complete L to a *loop* $\tilde{L} \colon [0, 1] \to \mathrm{Lag}(E, \omega)$ with $\tilde{L}(t) = L(2t)$ for $0 \le t \le 1/2$ and $\tilde{L}(t) \in \mathrm{Lag}(E; M)$ for $1/2 \le t \le 1$ such that $\tilde{L}(1) = L(0)$. The *Maslov intersection index* is defined as

$$[L : M] = \mu(\tilde{L}) \in \mathbb{Z}$$

which is independent of completion to a loop (provided that $\tilde{L}(t) \in \mathrm{Lag}(E; M)$ for $1/2 \le t \le 1$). The definition requires transversality at the end points; below we will remove that restriction as well.

Remark 2.48. The Maslov index can be interpreted as a (signed) intersection number of L with the "singular cycle"

$$\Sigma = \mathrm{Lag}(E) \setminus \mathrm{Lag}(E; M)$$

(the set of Lagrangian subspaces that are *not* transverse to M). It was in this form that Maslov originally introduced his index. The difficulty of this approach is that the singular cycle is not a smooth submanifold of $\mathrm{Lag}(E, \omega)$. Given a path in $\mathrm{Lag}(E, \omega)$, one perturbs this path until it intersects only the smooth part of the singular cycle and all intersections are transverse. It is then necessary to prove that the index is independent of the choice of perturbation.

Maslov invented his index around 1965, in the context of geometrical optics ("high frequency asymptotics") and quantum mechanics "semi-classical approximation". It appears physically as a phase shift when a light ray passes through a focal point; a phenomenon discovered in the 19th century. On the more mathematical side, Maslov's theory gave rise to Hörmander's theory of Fourier integral operators in PDE.

Maslov's index can be generalized to paths L that are not necessarily transverse to M at the end points. This was first done by Dazord in a 1979 paper and re-discovered several times since then. We will describe one such construction in the following section.

2.10. **Maslov indices, II.** In this section we describe a different approach towards Maslov indices, using the Hörmander-Kashiwara index of a Lagrangian triple. As a motivation, consider the action of $\mathrm{Sp}(E, \omega)$ on $\mathrm{Lag}(E, \omega)$. We have seen that this action is transitive. Moreover, any two ordered pairs of transverse Lagrangian subspaces can be carried into each other by some symplectomorphism. An analogous statement is true if one fixes the dimension of the intersection $\dim(L_1 \cap L_2)$.

Exercise 2.49. Show that for any $L_1, L_2 \in \mathrm{Lag}(E, \omega)$ there exists a symplectic basis in which L_1 is spanned by the $e_1, \ldots, e_n$ and L_2 by $e_1, \ldots, e_k, f_{k+1}, \ldots, f_n$. where $k = \dim(L_1 \cap L_2)$. It follows that the action of $\mathrm{Sp}(E, \omega)$ on $\mathrm{Lag}(E, \omega) \times \mathrm{Lag}(E, \omega)$ has $n+1$ orbits, labeled by the dimension of intersections.

Is this true also for *triples* of Lagrangian sub-spaces?

Exercise 2.50. Let $E = \mathbb{R}^2$ with symplectic basis e, f. Let $L_1 = \mathrm{span}\{e\}$, $L_2 = \mathrm{span}\{f\}$. What is the form of a matrix of the most general symplectomorphism preserving L_1, L_2? Let $L_3 = \mathrm{span}\{e+f\}$, and (L_1', L_2', L_3') a second triple of Lagrangian subspaces with $L_1' = L_1, L_2' = L_2$. Show by direct computation that there exists a symplectic transformation $A \in \mathrm{Sp}(E, \omega)$ with $L_j' = A(L_j)$ for all $j = 1, 2, 3$, if and only if $L_3' = \mathrm{span}\{e + \lambda f\}$ with $\lambda > 0$.

Thus, specifying the dimensions of intersections is insufficient for describing the orbit of a Lagrangian triple L_1, L_2, L_3. There is another invariant called the Hörmander-Kashiwara index of a Lagrangian triple.

Before we define the index, let us recall:

Definition 2.51. The signature $\mathrm{Sig}(B) \in \mathbb{Z}$ of a real symmetric matrix B is the number of positive eigenvalues, minus the number of negative eigenvalues.

The signature has the property $\mathrm{Sig}(ABA^\top) = \mathrm{Sig}(B)$ for any invertible matrix A. If $\beta \in \mathrm{Sym}^2(V^*)$ is a symmetric bi-linear form (equivalently, a quadratic form) on a vector space V, one defines

$$\mathrm{Sig}(\beta) := \mathrm{Sig}(B)$$

where B is the matrix of β in a given basis of V. The signature and the nullity are the only invariants of a symmetric bilinear form: That is, the action of $\mathrm{GL}(V)$ on $\mathrm{Sym}^2(V^*)$ has a finite number of orbits labeled by $\dim(\ker(\beta))$ and $\mathrm{Sig}(\beta)$. Letting $n = \dim V$, the signature for given $\ell = \dim(\ker(\beta))$, may take on any of the values

$$-n+\ell,\ -n+\ell+2, \ldots, n-\ell-2, n-\ell.$$

Given three Lagrangian subspaces (not necessarily transverse) consider the symmetric bilinear form $\beta(L_1, L_2, L_3)$ on their direct sum $L_1 \oplus L_2 \oplus L_3$, given by

$$\begin{aligned}\beta(L_1, L_2, L_3)\big((v_1, v_2, v_3), (v_1', v_2', v_3')\big) = \frac{1}{2}\Big(&\omega(v_1, v_2') + \omega(v_2, v_3') + \omega(v_3, v_1') \\ &+ \omega(v_1', v_2) + \omega(v_2', v_3) + \omega(v_3', v_1)\Big).\end{aligned}$$

Definition 2.52. The *index of the the Lagrangian triple* (L_1, L_2, L_3) is the signature of this bilinear form,

$$s(L_1, L_2, L_3) := \mathrm{Sig}(\beta(L_1, L_2, L_3)) \in \mathbb{Z}.$$

The index of a triple was introduced by Hörmander (in his paper [21] on Fourier integral operators) in case one of the Lagrangians is transverse to the other two, and by Kashiwara in general (according to the book [29] by Lion-Vergne). Clearly s is invariant under the action of $\mathrm{Sp}(E, \omega)$ on triples of Lagrangian subspaces.

Choosing bases for L_1, L_2, L_3, the definition gives $\beta(L_1, L_2, L_3)$ as a symmetric $3n \times 3n$-matrix. One can reduce to signatures of $n \times n$-matrices as follows. Choose a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of E, such that L_1, L_2, L_3 are transverse to the span of $f_1, \ldots, f_n$. Let S_j denote the symmetric bilinear forms on the span of $e_1, \ldots, e_n$ corresponding to L_j. In terms of the basis, S_j is just a matrix, and $Q(L_1, L_2, L_3)$ is given by a symmetric matrix,

$$Q(L_1, L_2, L_3) = \frac{1}{2}\begin{pmatrix} 0 & S_1 - S_2 & S_3 - S_1 \\ S_1 - S_2 & 0 & S_2 - S_3 \\ S_3 - S_1 & S_2 - S_3 & 0 \end{pmatrix}.$$

Lemma 2.53. $s(L_1, L_2, L_3) = \mathrm{Sig}(S_1 - S_2) + \mathrm{Sig}(S_2 - S_3) + \mathrm{Sig}(S_3 - S_1)$.

Proof. (Brian Feldstein) An elementary calculation shows that the invertible matrix

$$T = \begin{pmatrix} 0 & I & I \\ I & 0 & I \\ I & I & 0 \end{pmatrix}.$$

(where I is the identity $n \times n$ matrix) satisfies

$$T\,Q(L_1, L_2, L_3)\,T^\top = \begin{pmatrix} S_3 - S_1 & 0 & 0 \\ 0 & S_2 - S_3 & 0 \\ 0 & 0 & S_1 - S_2. \end{pmatrix}.$$

From this the lemma is immediate. □

Remark 2.54. This may be formulated more intrinsically, as follows. We choose a 4th Lagrangian subspace M that is transverse to each of L_1, L_2, L_3. That is, $L_i \in \mathrm{Lag}(E; M)$ for $i = 1, 2.3$. Recall that $\mathrm{Lag}(E; M)$ is an affine space, with the space $\mathrm{Sym}^2(M)$ of bilinear forms on $M^* = E/M$ as its space of motions. To each pair i, j corresponds a bilinear form $S_{ij} \in \mathrm{Sym}^2(M)$, in such a way that $\beta_{ij} \cdot L_j = L_i$. In terms of these bilinear forms,

$$s(L_1, L_2, L_3) = \mathrm{Sig}(\beta_{21}) + \mathrm{Sig}(\beta_{32}) + \mathrm{Sig}(\beta_{13}).$$

Theorem 2.55. *The signature* $s : \mathrm{Lag}(E, \omega)^3 \to \mathbb{Z}$ *of a Lagrangian triple has the following properties:*

(a) *s is anti-symmetric under permutations of L_1, L_2, L_3:*

$$s(L_1, L_2, L_3) = s(L_2, L_3, L_4) = -s(L_2, L_1, L_3).$$

(b) *(Cocycle Identity) For all quadruples $L_1, L_2, L_3, L_4 \in \mathrm{Lag}(E, \omega)$,*

$$s(L_2, L_3, L_4) - s(L_1, L_3, L_4) + s(L_1, L_2, L_4) - s(L_1, L_2, L_3) = 0.$$

(c) *If $M(t)$ is a continuous path of Lagrangian subspaces such that $M(t)$ is always transverse to $L_1, L_2 \in \mathrm{Lag}(E, \omega)$, then*

$$t \mapsto s(L_1, L_2, M(t))$$

is constant as a function of t.

(d) *Any ordered triple of Lagrangian subspaces is determined up to symplectomorphism by the five numbers*

$$\dim(L_1 \cap L_2), \quad \dim(L_2 \cap L_3), \quad \dim(L_3 \cap L_1),$$
$$\dim(L_1 \cap L_2 \cap L_3), \quad s(L_1, L_2, L_3).$$

Proof. The first property is immediate from the definition, while the second and third property follow from Lemma 2.53. The fourth property is left as a non-trivial exercise. (Perhaps try it first for the case that the L_j are pairwise transverse.) □

Exercise 2.56. Let L_1, L_2, L_3 be three Lagrangian subspaces. Suppose L_1, L_2 are both transverse to L_3. Choose a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of E such that

$$L_1 = \mathrm{span}\{e_1, \ldots, e_n\}, \; L_3 = \mathrm{span}\{f_1, \ldots, f_n\}.$$

Thus $L_3 = \mathrm{span}\{g_1, \ldots, g_n\}$ where

$$g_i = e_i + \sum_j S_{ij} f_j$$

for a symmetric matrix $n \times n$-matrix S.

a) Prove that [8]

$$s(L_1, L_2, L_3) = \mathrm{Sig}(S).$$

b) Show that one may choose the symplectic basis so that for all i, we have $g_i = e_i + f_i$ for $i < k$, $g_i = e_i - f_i$ for $k \le i < \ell$, and $g_i = e_i$ for $i \ge \ell$. What is $s(L_1, L_2, L_3)$ in terms of k, ℓ?

Proposition 2.57. *Suppose $L_1(t), L_2(t) \in \mathrm{Lag}(E, \omega)$ are two paths of Lagrangian subspaces, $a \le t \le b$. Suppose there exists $M \in \mathrm{Lag}(E, \omega)$ transverse to $L_1(t)$ and $L_2(t)$ for all $t \in [a, b]$. then the difference*

$$[L_1 : L_2] := \frac{1}{2}\big(s(L_1(a), L_2(a), M) - s(L_1(b), L_2(b), M)\big)$$

is independent of the choice of such M.

Proof. Let M, M' be two choices. By the cocycle identity, the first term changes by

$$s(L_1(a), L_2(a), M) - s(L_1(a), L_2(a), M') = s(L_1(a), M, M') - s(L_2(a), M, M').$$

We have to show that this equals the change of the second term,

$$s(L_1(b), L_2(b), M) - s(L_1(b), L_2(b), M') = s(L_1(b), M, M') - s(L_2(b), M, M').$$

But $s(L_1(t), M, M')$ and $s(L_2(t), M, M')$ are independent of t, since L_i stay transverse to M, M'. □

This proposition tells us how to define the Maslov intersection index for 'short' paths. We define the Maslov intersection index for two arbitrary paths $L_1, L_2 : [a, b] \to \mathrm{Lag}(E, \omega)$ by requiring additivity under concatenation:

[8] Need to verify sign

Definition 2.58 (Maslov index). ([34], see also [33, 38]). Let $L_1, L_2 \colon [a, b] \to \mathrm{Lag}(E, \omega)$ be two continuous paths of Lagrangians. We define the Maslov index $[L_1 : L_2] \in \frac{1}{2}\mathbb{Z}$ by the formula

$$[L_1 : L_2] = \frac{1}{2}\sum_{j=1}^{k}(s(L_1(t_{j-1}), L_2(t_{j-1}), M_j) - s(L_1(t_j), L_2(t_j), M_j)).$$

Here $a = t_0 < t_1 < \cdots t_k = b$ is a subdivision, and $M_1, \ldots, M_k$ are Lagrangian subspaces, with the property that for all $j = 1, \ldots, k$, both $L_1(t), L_2(t) \in \mathrm{Lag}(E; M_j)$ for all $t \in [t_{j-1}, t_j]$.

By the proposition, this is independent of the choice of subdivision and of the choice of the M_j.

Note that this definition does not require transversality at the endpoints. The Maslov intersection index is additive under concatenation of paths, and is anti-symmetric $[L_1 : L_2] = -[L_2 : L_1]$.

Exercise 2.59. Show that for any path of symplectomorphisms $A : [a, b] \to \mathrm{Sp}(E, \omega)$, $[A(L_1) : A(L_2)] = [L_1 : L_2]$.

Exercise 2.60. Let $E = \mathbb{R}^2$, and let $L_1, L_2 : [a, b] \to \mathrm{Lag}(E, \omega)$ be defined by $L_1(t) = \mathrm{span}(f + te)$ and $L_2(t) = \mathrm{span}(f)$. Find $[L_1 : L_2]$. How does it depend on a, b?

Exercise 2.61. Let $L_1, L_2, L_3 : [a, b] \to \mathrm{Lag}(E, \omega)$ be three paths of Lagrangian subspaces. Show that

$$[L_1 : L_2] + [L_2 : L_3] + [L_3 : L_1] = \frac{1}{2}(s(L_1(a), L_2(a), L_3(a)) - s(L_1(b), L_2(b), L_3(b)).$$

Definition 2.58 may also be used to define Maslov indices of paths of symplectomorphisms. Let E^- denote E with minus the symplectic form, and let $E \oplus E^-$ be equipped with the symplectic form $\mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega$ where pr_i are the projections to the first and second factor.

Proposition 2.62. *For any symplectomorphism $A \in \mathrm{Sp}(E, \omega)$, the graph*

$$\Gamma_A := \{(Av, v) |\, v \in E\} \subseteq E \oplus E^-$$

is a Lagrangian subspace.

Proof. Let pr_1, pr_2 denote the projections from $E \oplus E^-$ to the respective factor. For all $v_1, v_2 \in E$, we have

$$(\mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega)((Av_1, v_1), (Av_2, v_2)) = -\omega(v_1, v_2) + \omega(Av_1, Av_2) = 0.$$

□

On the other hand, $E \oplus E^-$ has a distinguished Lagrangian subspace given by the diagonal $\Delta \subseteq E \oplus E^-$.

Definition 2.63. The *Maslov index of a path of symplectomorphisms* $A : [a, b] \to \mathrm{Sp}(E, \omega)$ is defined by

$$[\Gamma_A : \Delta] \in \frac{1}{2}Z.$$

For loops based at the identity this reduces (up to a factor of 2) to the index $\mu(A)$ introduced earlier.

Exercise 2.64. The set of paths $[a, b] \to \mathrm{Sp}(E, \omega)$ is a group under pointwise multiplication. Given two paths of symplectomorphisms $A_1, A_2 \colon [a, b] \to \mathrm{Sp}(E, \omega)$, give a formula for

$$\mu(A_1 A_2) - \mu(A_1) - \mu(A_2),$$

and use it to show that this difference is bounded, independently of the choice of A_1, A_2. (This shows that the Maslov index is a *quasi-morphism* of groups.)

3. Foundations of symplectic geometry

An *almost complex structure* on a manifold M is given by a family of complex structures on the tangent spaces

$$J_m : T_mM \to T_mM, \quad J_m^2 = -\,\mathrm{id}_{T_mM},$$

depending smoothly on m. A *complex structure* on a real manifold M may be defined in terms of an atlas with $\mathbb{C}^n$-valued charts with holomorphic transition functions. The Newlander-Nirenberg theorem gives a necessary and sufficient *integrability condition* for an almost complex structure $J = \{J_m\}$ to define a complex structure (in terms of the vanishing of the so-called *Nijenhuis torsion*).

Similarly, an *almost symplectic structure* on a manifold may be defined to be a family of symplectic structures on tangent spaces,

$$\omega_m : T_mM \times T_mM \to \mathbb{R}.$$

In other words, it is a 2-form ω whose restriction to every tangent space is nondegenerate. There is an integrability condition on such 2-forms, which is much easier to state than for almost complex structures.

3.1. **Definition of symplectic manifolds.**

Definition 3.1. A *symplectic structure* on a manifold M is a non-degenerate 2-form $\omega \in \Omega^2(M)$ which is *closed*:

$$\mathsf{d}\omega = 0.$$

Non-degeneracy means that for each $m \in M$, the form $\omega|_m$ is a symplectic form on T_mM, in particular $\dim M = 2n$ is even. The nondegeneracy may also be characterized as follows:

Proposition 3.2. *A 2-form $\omega \in \Omega^2(M)$ is nondegenerate (i.e., symplectic) if and only if the top exterior power*

$$\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$$

is non-zero everywhere.

Proof. This is a pointwise statement (we could have discussed in in the section on symplectic vector spaces); hence we check at any given $m \in M$.

"$\Leftarrow$" Suppose $\omega_m^n \neq 0$, and let $v \in T_mM$ be non-zero. Since ω_m^n is a volume element, we have $\iota(v)\omega_m^n \neq 0$. But $\iota(v)\omega_m^n = n(\iota(v)\omega_m) \wedge \omega_m^{n-1}$, so we must have $\iota(v)\omega_m \neq 0$. This shows $\ker(\omega_m) = 0$.

"⇒" Suppose $\ker(\omega_m) = 0$. Choose a symplectic basis $e_1, f_1, \ldots, e_n, f_n$ of the tangent space T_mM. We have

$$\begin{aligned}\iota(e_n)\cdots\iota(e_1)\omega_m^n &= n\iota(e_n)\cdots\iota(e_2)\Big(\omega_m^{n-1}\wedge\iota(e_1)\omega_m\Big)\\ &= n(n-1)\iota(e_n)\cdots\iota(e_3)\Big(\omega_m^{n-2}\wedge\iota(e_2)\omega_m\wedge\iota(e_1)\omega_m\Big)\\ &\cdots\\ &= n!\,\iota(e_n)\omega_m\wedge\cdots\wedge\iota(e_1)\omega_m\end{aligned}$$

(We are using that $\omega_m(e_i, e_j) = 0$ for all i, j.) Hence

$$\iota(f_n)\cdots\iota(f_1)\iota(e_n)\cdots\iota(e_1)\omega_m^n = \pm n!.$$

In particular $\omega_m^n \neq 0$. □

Definition 3.3. Let (M, ω) be a (almost) symplectic manifold. The volume form

$$\Lambda = \exp(\omega)_{[\dim M]} = \frac{1}{n!}\omega^n$$

is called the *Liouville form.*

The existence of a canonical volume form means, in particular, that (almost) symplectic manifolds come equipped with a natural orientation.

Remark 3.4. Similarly, every almost complex manifold (M, J) carries a natural orientation. In fact, we may pick a Riemannian metric g on M such that J is orthogonal[9]; then $\omega(v, w) = g(Jv, w)$ defines an almost symplectic structure, which, in turn, defines an orientation.

3.2. **Symplectomorphisms, Hamiltonian vector fields.** Before discussing first examples, let us consider the automorphisms of a symplectic manifold:

Definition 3.5. Let (M, ω) be a symplectic manifold.

(a) A *symplectomorphism* of M is a diffeomorphism $F \in \mathrm{Diff}(M)$ such that

$$F^*\omega = \omega.$$

The group of symplectomorphism of M onto itself is denoted $\mathrm{Diff}(M, \omega)$.

(b) A *symplectic vector field* on (M, ω) is a vector field X with the property

$$\mathcal{L}_X\omega = 0.$$

The Lie algebra of symplectic vector fields is denoted $\mathfrak{X}(M, \omega)$.

Remark 3.6. More generally a symplectomorphism between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a smooth map $F\colon M_1 \to M_2$ such that $F^*\omega_2 = \omega_1$.

---

[9] If g_0 is any choice of Riemannian metric, then $g(v, w) = g_0(v, w) + g_0(Jv, Jw)$ has this property.

It is immediate that a symplectomorphism F is volume preserving:

$$F^*\omega = \omega \Rightarrow F^*(\omega^n) = \omega^n.$$

(The converse is not true unless $\dim M = 2$.) Furthermore, the local flow F_t of a symplectic vector field is volume preserving: $F_t^*\omega = \omega$ follows by integrating the differential equation

$$\frac{d}{dt}F_t^*\omega = -F_t^*\mathcal{L}_X\omega = 0.$$

That is, $\mathfrak{X}(M,\omega)$ is informally the Lie algebra of the infinite-dimensional group $\mathrm{Diff}(M,\omega)$.

Definition 3.7. For any $H \in C^\infty(M,\mathbb{R})$, the corresponding *Hamiltonian vector field* X_H is the unique vector field satisfying

$$\iota_{X_H}\omega = -\mathsf{d}H$$

The space of Hamiltonian vector fields is denoted $\mathfrak{X}_{\mathrm{Ham}}(M,\omega)$.

Proposition 3.8. *Every Hamiltonian vector field is a symplectic vector field. That is,*

$$\mathfrak{X}_{\mathrm{Ham}}(M,\omega) \subseteq \mathfrak{X}(M,\omega).$$

Proof. Suppose $X = X_H$, that is $\iota_X\omega = -\mathsf{d}H$. Then

$$\mathcal{L}_X\omega = \mathsf{d}\iota_X\omega = -\mathsf{d}\mathsf{d}H = 0.$$ □

3.3. **Basic example: Open subsets of $\mathbb{R}^{2n}$.** The prototype of a symplectic manifold is an open subset $U \subseteq \mathbb{R}^{2n}$. Let $q_1,\ldots,q_n,p_1,\ldots,p_n$ be coordinates with respect to a symplectic basis $e_1,f_1,\ldots,e_n,f_n$ for $\mathbb{R}^{2n}$. This identifies $e_j = \frac{\partial}{\partial q_j}$ and $f_j = \frac{\partial}{\partial p_j}$. In terms of the dual 1-forms $\mathsf{d}q_1,\ldots,\mathsf{d}p_n$, the symplectic form is given by

$$\omega = \sum_{j=1}^n \mathsf{d}q_j \wedge \mathsf{d}p_j.$$

Darboux's theorem, to be discussed later, shows that every symplectic structure is locally of this form, in suitable coordinates. The Liouville volume form $\omega^n/n!$ is

$$\Lambda = \mathsf{d}q_1 \wedge \mathsf{d}p_1 \wedge \ldots \wedge \mathsf{d}q_n \wedge \mathsf{d}p_n.$$

Given a smooth function H on U, we have

$$X_H = \sum_{j=1}^n \Big(\frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j}\Big).$$

Hence the ordinary differential equation defined by X_H is

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

Example 3.9. The Hamiltonian $H(q,p) = \frac{1}{2}(q^2+p^2)$ gives the vector field $X_H = q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}$, generating (with our sign conventions) the flow given by clockwise rotation. (This agrees with the physics interpretation as a harmonic oscillator.)

3.4. **Cotangent bundles.** Let $M = T^*Q$ be the cotangent bundle of a manifold Q, i.e., the dual of the tangent bundle. Let

$$\pi: \ T^*Q \to Q$$

be the bundle projection, $T\pi\colon T(T^*Q) \to TQ$ its tangent map.

Definition 3.10. The *canonical 1-form*

$$\theta \in \Omega^1(T^*Q)$$

is given, at any $\mu \in T^*Q$, in terms of its pairings with tangent vectors $v \in T_\mu(T^*Q)$ by

$$\langle \theta_\mu, v\rangle = \langle \mu, T\pi(v)\rangle.$$

Let us verify that the definition makes sense: The element $\mu \in T^*Q$ is a covector at $\pi(\mu) \in Q$, while $T\pi(v) \in T_{\pi(\mu)}Q$ is a tangent vector at $\pi(\mu)$; hence we may pair them.

Remark 3.11. To rephrase the definition: By dualizing the tangent map

$$T_\mu\pi\colon T_\mu(T^*Q) \to T_{\pi(\mu)}Q,$$

we obtain a map

$$(T_\mu\pi)^*\colon T^*_{\pi(\mu)}Q \to T^*_\mu(T^*Q).$$

But $T^*_{\pi(\mu)}Q$ contains the element μ itself. Hence, we obtain a covector

$$\theta_\mu = (T_\mu\pi)^*(\mu).$$

Another characterization of the form θ is as follows. Given a 1-form $\alpha \in \Omega^1(Q)$ (regarded, e.g., as a $C^\infty(Q)$-linear map $\mathfrak{X}(Q) \to C^\infty(Q)$), let $\sigma_\alpha\colon Q \to T^*Q$ be the corresponding section of the cotangent bundle. (Usually one would just denote it by α.)

Proposition 3.12. *The canonical 1-form is the unique 1-form $\theta \in \Omega^1(T^*Q)$ with the property that for every 1-form $\alpha \in \Omega^1(Q)$ on the base,*

$$\alpha = \sigma_\alpha^*\theta$$

where $\sigma_\alpha\colon Q \to T^*Q$ defined by α.

Proof. Let us evaluate the right hand side on a tangent vector $w \in T_qQ$ with base point $q \in Q$. Let $\mu = \alpha_q = \sigma_\alpha(q) \in T^*Q$. Since $\pi \circ \sigma_\alpha = \mathrm{id}_Q$, we have

$$\begin{aligned}\langle(\sigma_\alpha^*\theta)_q, w\rangle &= \langle\theta_\mu, (T_q\alpha)(w)\rangle\\ &= \langle\mu, (T_\mu\pi)((T_q\alpha)(w))\rangle\\ &= \langle\mu, w\rangle\\ &= \langle\alpha_q,\, w\rangle.\end{aligned}$$

For the uniqueness part, let us call a tangent vector $v \in T_\mu(T^*Q)$ *vertical* if it is in the kernel of $T_\mu\pi$. Every non-vertical tangent vector is tangent to the range of σ_α for some 1-form $\alpha \in \Omega^1(Q)$. Thus, we may write $v = (T_q\sigma_\alpha)(w)$, for with $\alpha(q) = \mu$ and $w \in T_qQ$. Hence, the defining identity $\langle\theta_\mu, v\rangle = \langle\mu, T\pi(v)\rangle$ holds on non-vertical vectors. The general case follows by continuity (since non-vertical vectors are dense) or because every tangent vector to T^*Q may be written as a sum of two non-vertical ones. □

Remark 3.13. One usually doesn't make notational distinction between a 1-form as C^∞-linear map $\alpha\colon \mathfrak{X}(M) \to C^\infty(M)$ (by contraction with vector fields) as a function on the tangent bundle $\alpha\colon TM \to \mathbb{R}$ (by evaluating on tangent vectors), or as a section $\alpha\colon M \to T^*M$. Thus, the formula just proved is often written as

$$\alpha^*\theta = \alpha.$$

We shall need the expression for the 1-form θ in local coordinates. Let $q_1, \ldots, q_n$ be local coordinates on some open subset $U \subseteq Q$. Then $\mathsf{d}q_1, \ldots, \mathsf{d}q_n$ are sections of $T^*Q|_U$, and span the cotangent bundle over U. (That is, the differentials form a *frame.*) A general covector $\mu \in T_q^*Q$ can be written as $\mu = \sum_i p_i \mathsf{d}q_i|_q$. Thus, the coordinates $q_1, \ldots, q_n$ of q, together with the coordinates $p_1, \ldots, p_n$, for coordinates on $T^*Q|_U$. The coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ on $T^*Q|_U$ are called *cotangent coordinates.*

Proposition 3.14. *In local cotangent coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ on T^*Q, the canonical 1-form θ is given over $T^*Q|_U$ by*

$$\theta = \sum_j p_j\, \mathsf{d}q_j.$$

Proof. Let $\alpha = \sum_j \alpha_j \mathsf{d}q_j$ be a 1-form on U. Viewed as a section, it is the map

$$\sigma_\alpha(q_1, \ldots, q_n) = (q_1, \ldots, q_n, \alpha_1(q_1, \ldots, q_n), \ldots, \alpha_n(q_1, \ldots, q_n)).$$

That is, $\sigma_\alpha^* p_j = \alpha_j, \quad \sigma_\alpha^* q_j = q_j$. It follows that

$$\sigma_\alpha^*\Big(\sum_j p_j\, \mathsf{d}q_j\Big) = \sum_j \alpha_j \mathsf{d}q_j = \alpha.$$ □

Exercise 3.15. It is instructive to give another proof, directly in terms of the definition. That is, verify the property $\langle \theta_\mu, v\rangle = \langle \mu, T\pi(v)\rangle$ for the coordinate tangent vectors $\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$.

Theorem 3.16. *Let T^*Q be a cotangent bundle and θ its canonical 1-form. Then $\omega = -\mathsf{d}\theta$ is a symplectic structure on M.*

Proof. In local cotangent coordinates, $-\mathsf{d}\theta = \sum_j \mathsf{d}q_j \wedge \mathsf{d}p_j$. □

This 2-form is called the *canonical symplectic structure* of the cotangent bundle.

We will now describe some natural symplectomorphisms and Hamiltonian vector fields on $M = T^*Q$, taking into account its structure as a fiber bundle $\pi\colon T^*Q \to Q$.

Definition 3.17. Let $\pi\colon P \to Q$ be a surjective submersion from a manifold P to a manifold Q.

(a) A diffeomorphism $F \in \mathrm{Diff}(P)$ is called a *lift* of a diffeomorphism $f \in \mathrm{Diff}(Q)$ if it satisfies $\pi \circ F = f \circ \pi$.
(b) A vector field $X \in \mathfrak{X}(P)$ is called a *lift* of a vector field $Y \in \mathfrak{X}(Q)$ if it satisfies $X \sim_\pi Y$, that is, $(T_p\pi)(X_p) = Y_{\pi(p)}$ for all $p \in P$.

The diffeomorphisms of P that are lifts under π may be called *fibration-preserving*. The form a group $\mathrm{Diff}(P,\pi)$. Similarly, the vector fields on P that are lifts of some vector fields under π form a Lie algebra $\mathfrak{X}(P,\pi)$.

Remark 3.18. Lifts of vector fields may be constructed locally, and then patched with a partition of unity. The resulting lift is unique up to vertical vector fields.

Example 3.19 (Tangent lifts). For $P = TQ$, every diffeomorphism $f \in \mathrm{Diff}(Q)$ has a distinguished *tangent lift*

$$f_T = Tf \in \mathrm{Diff}(TQ,\pi).$$

Similarly, every vector field $Y \in \mathfrak{X}(Q)$ has a distinguished *tangent lift*

$$Y_T \in \mathfrak{X}(TQ,\pi),$$

in such a way that the local flow of X is the tangent lift of the local flow of Y. Another way of describing it: for $f \in C^\infty(Q)$, let $f_T \in C^\infty(TQ)$ be the function $v \mapsto v(f)$. (In other words, f_T is just the exterior differential df, viewed as a function on TQ.) Then Y_T is uniquely characterized by the properties

$$Y_T(\pi^* f) = 0, \quad Y_T(f_T) = (Yf)_T.$$

for all f.

We are interested in the case $P = T^*Q$. The *cotangent lift* of a diffeomorphism $f \in \mathrm{Diff}(Q)$ is defined by

$$f_{T^*} = (Tf^{-1})^*.$$

The *cotangent lift of a vector field* $Y \in \mathfrak{X}(Q)$ is the vector field Y_{T^*} whose local flow is the cotangent lift of the local flow of Y.

Exercise 3.20. (a) Show that both tangent and cotangent of vector fields are Lie algebra morphisms.
(b) Find coordinate expressions for the tangent and cotangent lifts. (The solution for cotangent lifts will be given below.)

For all $\alpha \in \Omega^1(Q)$ one has a commutative diagram,

$$\begin{array}{ccc} T^*Q & \xrightarrow{f_{T^*}} & T^*Q \\ \uparrow{\scriptstyle \sigma_\alpha} & & \uparrow{\scriptstyle \sigma_{(f^{-1})^*\alpha}} \\ Q & \xrightarrow[f]{} & Q \end{array}$$

(This is really how the pullback of a 1-form is defined.)

Proposition 3.21. *Let $f \in \mathrm{Diff}(Q)$ be a diffeomorphism. Then the cotangent lift of f preserves the canonical 1-form,*

$$(f_{T^*})^*\theta = \theta.$$

Conversely, the cotangent lifts of diffeomorphisms are the unique *fibration preserving diffeomorphisms $F \in \mathrm{Diff}(T^*Q)$ satisfying $F^*\theta = \theta$.*

Proof: Next time.

Proof. This is 'essentially clear' since our definition of the canonical 1-form was coordinate-free. For the skeptic, check the property $\sigma_\alpha^*(f_{T^*})^*\theta = \alpha$ from Proposition 3.12 : We have

$$\begin{aligned} \sigma_\alpha^*(f_{T^*})^*\theta &= ((f_{T^*}) \circ \sigma_\alpha)^*\theta \\ &= (\sigma_{(f^{-1})^*\alpha} \circ f)^*\theta \\ &= f^*(\sigma_{(f^{-1})^*\alpha})^*\theta \\ &= f^*(f^{-1})^*\alpha \\ &= \alpha. \end{aligned}$$

For the uniqueness claim, suppose F is a fiber-preserving diffeomorphism with $F^*\theta = \theta$. Let f be its base map. By composing F with the inverse of f_{T^*}, we may assume $f = \mathrm{id}_Q$. We have to show $F = \mathrm{id}_{T^*Q}$. Since f fixed base points, we may use local coordinates near any given point of Q, and corresponding cotangent coordinates q_i, p_i on T^*Q, so $\theta = \sum_i p_i \mathsf{d}q_i$. Since $F^*q_i = q_i$, the property $F^*\theta = \theta$ implies that also $F^*p_i = p_i$, that is F is the identity map. $\square$

Since cotangent lifts preserve the canonical 1-form, they also preserve the symplectic form ω. It follows that the cotangent lift of diffeomorphisms gives a group morphism

$$\mathrm{Diff}(Q) \to \mathrm{Diff}(T^*Q, \omega), \quad f \mapsto f_{T^*}. \tag{3}$$

Given $\alpha \in \Omega^1(Q)$ let $G_\alpha \colon T^*Q \to T^*Q$ be the diffeomorphism obtained by adding α fiberwise; that is, $\mu \mapsto \mu + \alpha_{\pi(\mu)}$. Note that G_α is fibration-preserving, with base map the identity map on Q. The map

$$G_\alpha \in \mathrm{Diff}(T^*Q, \pi)$$

may also be described through its property $G_\alpha \circ \sigma_\beta = \sigma_{\alpha+\beta}$.

Proposition 3.22. *For all $\alpha \in \Omega^1(Q)$,*

$$G_\alpha^*\theta - \theta = \pi^*\alpha$$

Thus G_α is a symplectomorphism if and only if $\mathsf{d}\alpha = 0$, that is $\alpha \in \Omega^1_{\mathrm{cl}}(Q)$. Conversely, every fibration-preserving symplectomorphism of T^*Q for which the base map on Q is trivial is obtained in this way from a closed 1-form.

Proof. Let $\beta \in \Omega^1(Q)$. Then

$$\sigma_\beta^*(G_\alpha^*\theta - \pi^*\alpha) = (G_\alpha \circ \sigma_\beta)^*\theta - (\pi \circ \sigma_\beta)^*\alpha = \sigma_{\alpha+\beta}^*\theta - \alpha = \alpha + \beta - \alpha = \beta$$

By the characterizing property of θ this proves $G_\alpha^*\theta - \pi^*\alpha = \theta$.

For the converse, suppose F is a fibration-preserving symplectomorphism of T^*Q with trivial base map. Consider the 1-form

$$\widetilde{\alpha} = F^*\theta - \theta.$$

This 1-form is closed, since $\mathsf{d}(F^*\theta - \theta) = F^*\mathsf{d}\theta - \mathsf{d}\theta = -F^*\omega + \omega = 0$. We claim that $\widetilde{\alpha}$ is 'basic', i.e., $\widetilde{\alpha} = \pi^*\alpha$ for a 1-form $\alpha \in \Omega^1(Q)$ (necessarily closed). This is equivalent to the statement that $\iota_Z\widetilde{\alpha} = 0,\ \mathcal{L}_Z\widetilde{\alpha} = 0$ for all vertical vector fields $Z \in \mathfrak{X}(T^*Q)$. [10] But

$$\iota_Z\widetilde{\alpha} = \iota_Z(F^*\theta - \theta) = F^*(\iota_{F_*(Z)}\theta) - \iota_Z\theta = 0$$

since θ vanishes on vertical vector fields, and $Z,\ F_*(Z)$ are vertical. Moreover,

$$\mathcal{L}_Z\widetilde{\alpha} = \mathcal{L}_Z(F^*\theta - \theta) = \iota_Z\mathsf{d}(F^*\theta - \theta) + \mathsf{d}\iota_Z(F^*\theta - \theta) = 0$$

(both terms vanish). We have thus shown that $F^*\theta - \theta = \pi^*\alpha$. But this means $F = G_\alpha$. ∎

[10] It is a general fact for surjective submersions $\pi\colon P \to Q$ with connected fibers that a differential form $\beta \in \Omega(P)$ is of the form $\beta = \pi^*\alpha$ if and only if it satisfies $\iota_Z\beta = 0,\ \mathcal{L}_Z\beta = 0$ for all vertical vector fields Z. This may be proved, for example, in coordinates adapted to the submersion.

We have thus constructed a group morphism

$$\Omega^1_{\mathrm{cl}}(Q) \to \mathrm{Diff}(T^*Q, \omega) \tag{4}$$

For any representation of a group G on a vector space V, one defines the semi-direct product $V \rtimes G$ to be the group whose underlying set is $V \times G$ and with product structure,

$$(v_1, g_1)(v_2, g_2) = (v_1 + g_1 \cdot v_2, g_1 g_2).$$

In our case, we let $\mathrm{Diff}(Q)$ act on closed 1-forms by $f \cdot \alpha = (f^{-1})^*\alpha$. Summarizing our discussion, we have shown:

Theorem 3.23. *The group* $\mathrm{Diff}(T^*Q, \omega) \cap \mathrm{Diff}(T^*Q, \pi)$ *of fibration preserving symplectomorphisms of* T^*Q *is a semidirect product*

$$\Omega^1_{\mathrm{cl}}(Q) \rtimes \mathrm{Diff}(Q) \hookrightarrow \mathrm{Diff}(T^*Q, \omega), \quad (\alpha, f) \mapsto G_\alpha \circ f_{T^*}$$

There is a similar description for the infinitesimal setting: The Lie algebra of fibration-preserving symplectic vector fields is a semidirect product

$$\mathfrak{X}(T^*Q, \omega) \cap \mathfrak{X}(T^*Q, \pi) = \Omega^1_{\mathrm{cl}}(Q) \rtimes \mathfrak{X}(Q)$$

(where the second factor is mapped to vector fields on T^*Q via cotangent lift. We will now show that these cotangent lifts are, in fact, Hamiltonian vector fields.

Proposition 3.24. *The cotangent lift* $X = Y_{T^*}$ *is the unique vector field with the properties*

$$X \sim_\pi Y, \quad \mathcal{L}_X \theta = 0.$$

It is a Hamiltonian vector field X_H *for* $H = -\iota_X \theta$.

Proof. Since the flow of the cotangent lift Y_{T^*} preserves θ, we have $\mathcal{L}_{Y_{T^*}} \theta = 0$. For the uniqueness part, note that other lifts are of the form $X = Y_{T^*} + Z$ where Z is a vertical vector field ($Z \sim \pi 0$). If $\mathcal{L}_X \theta = 0$ then $\mathcal{L}_Z \theta = 0$. But for a vertical vector field, $\iota_Z \theta = 0$. Hence the property $\mathcal{L}_Z \theta = 0$ gives $\iota_Z \omega = -\iota_Z d\theta = -\mathcal{L}_Z \theta = 0$, hence $Z = 0$.

Letting $H = -\iota_X \theta$, we obtain

$$\mathsf{d}H = -\mathsf{d}\iota_X \theta = \iota_X \mathsf{d}\theta - L_X \theta = -\iota_X \omega.$$

This shows $X = X_H$. □

Exercise 3.25. Suppose $X \in \mathfrak{X}(T^*Q, \omega)$ is a symplectic vector field, which is vertical with respect to π. Show that

$$\iota_X \omega = -\pi^* \alpha$$

for a closed 1-form $\alpha \in \Omega^1(Q)$.

Let us express some of these results in cotangent coordinates. Suppose

$$Y = \sum_j Y_j \frac{\partial}{\partial q_j}.$$

Every lift (in particular the cotangent lift) is of the form

$$X = \sum_j Y_j \frac{\partial}{\partial q_j} + \sum_j f_j(q,p) \frac{\partial}{\partial p_j}.$$

The vertical part does not contribute to contraction with θ. Hence

$$-\iota_X \theta = -\sum_j Y_j(q) \iota_{\frac{\partial}{\partial q_j}} \theta = -\sum_j Y_j(q) p_j.$$

The function $H = -\sum_j Y_j(q) p_j$ has exterior differential

$$\mathsf{d}H = \sum_j Y_j(q) \mathsf{d}p_j - \sum_{j,k=1}^n p_j \frac{\partial Y_j}{\partial q_k} \mathsf{d}q_j;$$

so, the Hamiltonian vector field is

$$Y_{T^*} = X_H = \sum_{j=1}^n Y_j \frac{\partial}{\partial q_j} - \sum_{j,k=1}^n p_j \frac{\partial Y_j}{\partial q_k} \frac{\partial}{\partial p_k}.$$

Remarks 3.26. Note that the Hamiltonians corresponding to cotangent lifts are those which are *linear along the fibers* of T^*Q. Other interesting flows are generated by Hamiltonians that are *constant along the fibers* of T^*Q, i.e. of the form $H = \pi^* f$, with $f \in C^\infty(Q)$. The flow generated by such an H is given, in terms of the notation G_α introduced above, by $\phi_t = G_{-t\ df}$. The Hamiltonian vector field corresponding to H is, in local cotangent coordinates,

$$X_{\pi^* f} = -\sum_j \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j}.$$

Exercise 3.27. Verify these claims!

On the total space of any vector bundle $E \to Q$ there is a canonical distinguished vector field $\mathcal{E} \in \mathfrak{X}(E)$, called the Euler vector field; its flow ϕ_t is fiberwise multiplication by e^{-t}. In our case $E = T^*Q$, we have in local cotangent coordinates

$$\mathcal{E} = \sum_j p_j \frac{\partial}{\partial p_j}.$$

On the other hand, we have $\theta = \sum_j p_j\ dq_j$. We hence see that

$$\mathcal{L}_{\mathcal{E}} \theta = \theta, \quad \iota_{\mathcal{E}} \theta = 0. \tag{5}$$

(The first formula says that θ is homogeneous of degree 1).

Proposition 3.28. *The Euler vector field satisfies*

$$\mathcal{L}_{\mathcal{E}}\omega = \omega,\ \ \iota_{\mathcal{E}}\omega = -\theta.$$

Proof. The first identity follows by applying $-\mathsf{d}$ to (5). The second formula is obtained using the Cartan formula for the Lie derivative, $\iota_{\mathcal{E}}\omega = -\iota_{\mathcal{E}}\mathsf{d}\theta = -\mathcal{L}_{\mathcal{E}}\theta = -\theta$. □

Remark 3.29. A symplectic manifold M, together with a free $\mathbb{R}$-action whose generating vector field $\mathcal{E}$ satisfies such that $L_{\mathcal{E}}\omega = \omega$ is called a *symplectic cone.* Thus, cotangent bundles minus their zero section are examples of symplectic cones. Another example is $\mathbb{R}^{2n} - \{0\}$, for the action given as multiplication by $e^{-t/2}$.

Proposition 3.30. *For every closed 2-form $\sigma \in \Omega^2(Q)$, the sum $\omega + \pi^*\sigma$ is a symplectic form on T^*Q. The Liouville form of $\omega + \pi^*\sigma$ equals that for ω.*

We leave the proof as an exercise.

This has the following somewhat silly corollary: *Every closed 2-form $\sigma \in \Omega^2_{\mathrm{cl}}(Q)$ arises as the pullback of a symplectic 2-form under some embedding.* (Proof: Consider Q as the zero section of $M = T^*Q$ with symplectic form $\omega = -\mathsf{d}\theta + \pi^*\sigma$.)

3.5. **Kähler manifolds.** An *almost complex manifold* is a manifold M together with a smoothly varying complex structure J_m on each tangent space T_mM; i.e. a smooth section

$$J\colon\ M \to \mathrm{End}(TM) = \sqcup_m \mathrm{End}(T_mM)$$

satisfying

$$J^2 = -\,\mathrm{id}\,.$$

A *complex manifold* is a manifold, together with an atlas with charts taking values in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, in such a way that the transition functions are holomorphic maps. Every complex manifold is almost complex, the automorphism J given by multiplication by $\sqrt{-1}$ in complex coordinate charts. The *Newlander-Nirenberg theorem* states that an almost complex structure J is integrable, i.e. comes from a complex manifold, if and only if the so-called *Nijenhuis tensor*[11] Nij_J vanishes. Moreover, the complex structure (defined in terms of holomorphic charts) is uniquely determined by J. Here Nij_J is defined by

$$\mathrm{Nij}_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y]$$

for all vector fields X, Y. (One may check that it the expression is C^∞-linear in both entries.) We won't make use of this criterion, but let us note one consequence: If (M, J) is a complex manifold, and $N \subseteq M$ is a (real) submanifold such that $J(TN) \subseteq TN$, then N is a complex submanifold. Similarly, a C^∞ map between complex manifolds is holomorphic if and only if its tangent map intertwines the corresponding J's.

[11]Please consult the internet for proper pronounciation.

An almost complex structure J on a symplectic manifold (M, ω) is called ω-compatible if

$$g(X, Y) = \omega(X, JY)$$

defines a Riemannian metric. In other words, J_m is ω_m-compatible for all $m \in M$.

Definition 3.31. We denote by $\mathcal{J}(M, \omega)$ the set of ω-compatible almost complex structures J on M, and by $\mathrm{Riem}(M)$ the set of all Riemannian metrics g on M.

One may regard these sets as the smooth sections of fiber bundle $\sqcup_m \mathcal{J}(T_m M, \omega_m)$, respectively $\sqcup_m \mathrm{Riem}(T_m M)$. The constructions from linear symplectic algebra (Section 2.5) can be carried out fiberwise:

- We have a map
$$\psi \colon \mathcal{J}(M, \omega) \to \mathrm{Riem}(M)$$
associating to $\{J_m\}$ the corresponding inner products $\{g_m\}$ on $\{T_m M\}$.
- This map has a canonically defined left inverse,
$$\phi \colon \mathrm{Riem}(M) \to \mathcal{J}(M, \omega);$$
In particular $\mathcal{J}(M, \omega)$ is non-empty.
- Any two compatible almost complex structures $J_0, J_1 \in \mathcal{J}(M, \omega)$ can be smoothly deformed within $\mathcal{J}(M, \omega)$: There exists a family of complex structures $J_t \in \mathcal{J}(M, \omega)$ taking on the given values for $t = 0, 1$, and such that the map
$$J : [0, 1] \times M \to \mathrm{End}(TM), (t, m) \mapsto J_t(m)$$
is smooth. (More strongly, given a manifold S, any map $S \to \mathcal{J}(M, \omega)$, which is smooth in the sense that the associated maps $S \times M \to \mathrm{End}(TM)$ are smooth, is smoothly homotopic to a constant map through compatible almost complex structures.)

Definition 3.32. A *Kähler manifold* is a triple (M, ω, J) where ω is a symplectic structure and J is an ω-compatible complex structure.

Thus, for a Kähler manifold both the 2-form and the complex structure must be integrable. A first example of a Kähler manifold is

$$\mathbb{C}^n = \mathbb{R}^{2n},$$

with the standard symplectic and complex structure. More generally, every open subset of $\mathbb{C}^n$, or in fact any complex inner product space, is a Kähler manifold.

Remark 3.33. As usual, the data (ω, J) determine a Riemannian metric g, and any two of (ω, J, g) determine the third.

Proposition 3.34. *Every complex complex submanifold of a Kähler manifold is again a Kähler manifold, and in particular is symplectic.*

Proof. Let (M, ω, J) be a Kähler manifold, and $N \subseteq M$ a complex submanifold. This means that the tangent bundle TN is J-invariant, and $J|_{TN} = J_N$ is the complex structure on N. Let $\iota \colon N \to M$ be the inclusion, and

$$\omega_N = \iota^*\omega.$$

To show that ω_N is symplectic, we have to show that for nonzero $v \in T_nN$ there exists $w \in T_nN$ with $\omega_N(v, w) \neq 0$. But $w = J_N v$ has this property, since $\omega_N(v, Jv) = \omega(v, Jv) = g(v, v) > 0$. A similar calculation also shows that J_N is ω_N-compatible. □

This gives many new examples: All complex submanifolds of $\mathbb{C}^n$ are Kähler manifolds, and in particular are symplectic.

Remark 3.35. More generally, we have a similar result for complex immersions rather than just embeddings.

We next consider complex projective space,

$$\mathbb{C}P(n) = (\mathbb{C}^{n+1}\backslash\{0\})/(\mathbb{C}\backslash\{0\}) = S^{2n+1}/\,\mathrm{U}(1). \tag{6}$$

By construction, it has a complex structure in such a way that the quotient map $\mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P(n)$ is holomorphic. Let

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{\ \iota\ } & \mathbb{C}^n \\ {\scriptstyle\pi}\big\downarrow & & \\ \mathbb{C}P(n) & & \end{array} \tag{7}$$

denote the embedding and projection. Let $\omega \in \Omega^2(\mathbb{R}^{2n})$ be the standard symplectic structure.

Proposition 3.36. *The manifold $\mathbb{C}P(n)$ has a unique 2-form ω_{FS} (called the* Fubini-Study form*) such that*

$$\pi^*\omega_{FS} = \iota^*\omega.$$

This 2-form is symplectic, and the standard complex structure on $\mathbb{C}P(n)$ is ω_{FS}-compatible. That is, $\mathbb{C}P(n)$ is naturally a Kähler manifold, and so are all complex submanifolds of $\mathbb{C}P(n)$.

Proof. To show that $\iota^*\omega$ descends under the submersion π, we have to show

$$\iota_Z(\iota^*\omega) = 0,\ \mathcal{L}_Z(\iota^*\omega) = 0$$

for all vertical vector fields $Z \in \mathfrak{X}(S^{2n+1})$ for the submersion π. Using Cartan's formula for $\mathcal{L}_Z$, and since $\iota^*\omega$ is closed, the second condition follows from the first. For the first condition, it suffices to check at any $z \in S^{2n+1}$. Consider the complex quotient map $q\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P(n)$ and its tangent map

$$T_z q\colon T_z\mathbb{C}^{n+1} \to T_{\pi(z)}\mathbb{C}P(n).$$

Identify $T_z\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$. Then $\ker(T_z q) = \mathbb{C}\cdot z$ contains the real line $\mathbb{R}\cdot z = T_z(S^{2n+1})^\perp$, hence it also contains $J(\mathbb{R}\cdot z)$. The latter is orthogonal to $\mathbb{R}\cdot z$, hence tangent to S^{2n+1}. This shows

$$J(\mathbb{R}\cdot z) = \ker(T_z q) \cap T_z(S^{2n+1}) = \ker(T_z\pi).$$

We arrive at a g-orthogonal direct sum decomposition

$$T_z\mathbb{C}^{n+1} = \underbrace{(T_zS^{2n+1} \cap J(T_zS^{2n+1})) \oplus \ker(T_z\pi)}_{T_zS^{2n+1}} \oplus \mathbb{R}\cdot z$$

where the last two summands add to $\ker(T_z q)$.

From this decomposition, we see that $\iota^*\omega|_z$ (the restriction of ω to T_zS^{2n+1}) has kernel exactly $\ker(T_z\pi)$. This shows that $\iota^*\omega$ descends to a 2-form on $\mathbb{C}P(n)$. Moreover, the quotient map gives an isomorphism of complex vector spaces

$$T_zS^{2n+1} \cap J(T_zS^{2n+1}) \to T_{\pi(z)}\mathbb{C}P(n),$$

and the 2-form on $T_{\pi(z)}\mathbb{C}P(n)$ is induced by the 2-form on this complex subspace. It is hence symplectic, and compatible with the symplectic structure. □

Later we will see this construction of the Fubini-Study form ω_{FS} more systematically as a symplectic reduction.

Combining with Proposition 3.34, we see that every complex submanifold of $\mathbb{C}P(n)$ (e.g., nonsingular projective variety) is in particular a symplectic manifold, with 2-form given by pullback of the Fubini-Study form.

Remark 3.37. It is natural to ask if every symplectic manifold admits a compatible complex structure, i.e. whether every symplectic manifold is Kähler. A counterexample was given by Thurston [44], apparently the example had previously been found by Kodaira (but nobody seems to give a reference). By now there are many counterexamples. In particular, there are examples due to Geiges of symplectic manifolds not admitting any complex structure at all, and also simply connected examples due to McDuff. See the book by McDuff-Salamon [32] for references and further information.

4. Basic properties of symplectic manifolds

Now that we have some examples of symplectic manifolds, let us develop the general theory.

4.1. **Hamiltonian and symplectic vector fields.**

We begin with a study of the Lie algebras of Hamiltonian and symplectic vector fields. Let (M, ω) be a symplectic manifold.

Proposition 4.1. *There is a short exact sequence*

$$0 \to \mathfrak{X}_{Ham}(M, \omega) \to \mathfrak{X}(M, \omega) \to H^1(M) \to 0. \tag{8}$$

where $H^1(M)$ is the first de Rham cohomology group.

Proof. The map $\omega^\flat \colon TM \to T^*M$ is a vector bundle isomorphism; on the level of sections it gives an isomorphism

$$\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M),\ X \mapsto \iota_X\omega = \omega(X, \cdot)$$

between vector fields and 1-forms. By definition, a vector field X is symplectic if and only if $\mathcal{L}_X\omega = 0$. Since $\mathsf{d}\omega = 0$, Cartan's identity shows that this is equivalent to $\omega^\flat(X)$ being *closed*. Similarly, X is Hamiltonian if $\iota_X\omega = -\mathsf{d}H$ for some smooth function H, if and only if $\omega^\flat(X)$ is exact. That is, we have isomorphisms

$$\mathfrak{X}(M, \omega) \xrightarrow{\omega^\flat} \Omega^1_{\mathrm{cl}}(M), \quad \mathfrak{X}_{\mathrm{Ham}}(M, \omega) \xrightarrow{\omega^\flat} \Omega^1_{\mathrm{ex}}(M).$$

Taking the quotient, we obtain a canonical isomorphism with $H^1(M) = \Omega^1_{\mathrm{cl}}(M)/\Omega^1_{\mathrm{ex}}(M)$. □

We conclude that if $H^1(M) = \{0\}$ (e.g. for simply connected spaces such as $M = \mathbb{C}^n$ or $M = \mathbb{C}P(n)$) then *every* symplectic vector field is Hamiltonian.

Proposition 4.2. *The Lie bracket of two symplectic vector fields is a Hamiltonian vector field:*

$$[\mathfrak{X}(M, \omega), \mathfrak{X}(M, \omega)] \subseteq \mathfrak{X}_{Ham}(M, \omega).$$

In fact, for $Y_1, Y_2 \in \mathfrak{X}(M, \omega)$,

$$[Y_1, Y_2] = X_{\omega(Y_1, Y_2)}.$$

Proof. Let $Y_1, Y_2 \in \mathfrak{X}(M, \omega)$. Then

$$\begin{aligned}
\mathsf{d}(\omega(Y_1, Y_2)) &= \mathsf{d}\iota_{Y_2}\iota_{Y_1}\omega \\
&= L_{Y_2}\iota_{Y_1}\omega - \iota_{Y_2}\mathsf{d}\iota_{Y_1}\omega \\
&= \iota([Y_2, Y_1])\omega + \iota_{Y_1}L_{Y_2}\omega + \iota_{Y_2}\iota_{Y_1}\mathsf{d}\omega - \iota_{Y_2}\mathcal{L}_{Y_1}\omega \\
&= \iota([Y_2, Y_1])\omega \\
&= -\iota([Y_1, Y_2])\omega.
\end{aligned}$$

□

In particular, $\mathfrak{X}_{Ham}(M, \omega)$ is an *ideal* in the Lie algebra $\mathfrak{X}(M, \omega)$ and the quotient Lie algebra $\mathfrak{X}(M, \omega)/\mathfrak{X}_{Ham}(M, \omega)$ is abelian. It follows that (8) is an exact sequence of Lie algebras, where $H^1(M)$ carries the trivial Lie algebra structure.

4.2. **Poisson brackets.** Consider next the surjective map

$$C^\infty(M) \to \mathfrak{X}_{Ham}(M, \omega), \quad H \mapsto X_H.$$

Its kernel is the space $H^0(M)$ of locally constant functions. (If M is connected then $H^0(M) = \mathbb{R}$.) We thus have an exact sequence of vector spaces

$$0 \longrightarrow H^0(M) \longrightarrow C^\infty(M) \longrightarrow \mathfrak{X}_{Ham}(M, \omega) \longrightarrow 0. \tag{9}$$

We shall define a Lie algebra structure on $C^\infty(M)$ to make this into an exact sequence of Lie algebras. Proposition 4.2 indicates what the right definition of the Lie bracket should be.

Definition 4.3. Let (M, ω) be a symplectic manifold. The Poisson bracket of two functions $F, G \in C^\infty(M, \mathbb{R})$ is defined as

$$\{F, G\} = \omega(X_F, X_G).$$

From the definition, it is immediate that the Poisson bracket is skew-symmetric. Using that $\iota(X_F)\omega = -\mathsf{d}F$ (by definition), together with Cartan's identity, one has the alternative formulas

$$\{F, G\} = L_{X_F}G = -L_{X_G}F. \tag{10}$$

Proposition 4.4. *The Poisson bracket defines a Lie algebra structure on* $C^\infty(M,\mathbb{R})$*: That is, it is anti-symmetric and satisfies the Jacobi identity*

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0 \tag{11}$$

for all F,G,H*. The Poisson bracket has the following compatibility with the algebra structure on* $C^\infty(M)$*:*

$$\{F,GH\}=\{F,G\}H+G\{F,H\}. \tag{12}$$

. The map $C^\infty(M)\to\mathfrak{X}(M)$, $F\mapsto X_F$ *is a Lie algebra morphism:*

$$X_{\{F,G\}}=[X_F,X_G]. \tag{13}$$

Proof. Equation (13) is just a special case of Proposition 4.2. The first statement follows from (13) and the calculation,

$$\begin{aligned}
\{F,\{G,H\}\} &= L_{X_F}\{G,H\}\\
&= L_{X_F}(\omega(X_G,X_H))\\
&= \omega([X_F,X_G],X_H)+\omega(X_G,[X_F,X_H])\\
&= \omega(X_{\{F,G\}},X_H)+\omega(X_G,X_{\{F,H\}})\\
&= \{\{F,G\},H\}+\{G,\{F,H\}\}.
\end{aligned}$$

Finally, (12) may be written as

$$L_{X_F}(GH)=(L_{X_F}G)H+G(L_{X_F}H),$$

and this holds because vector fields are derivations. □

Remark 4.5. This result motivates a generalization of symplectic structures: A Poisson structure on a manifold is a Lie bracket $\{\cdot,\cdot\}$ on the algebra of functions satisfying (12). It is also relevant for ideas of 'quantization'; the basic observation is that for any algebra $\mathcal{A}$, the commutator $[a,b]=ab-ba,\ a,b\in\mathcal{A}$ is a Lie bracket satisfying a property similar to (12), $[a,bc]=[a,b]c+b[a,c]$. The idea is, then, to pass from the commutative algebra $C^\infty(M)$ to a noncommutative algebra $\mathcal{A}$ in such a way that Poisson brackets become commutators. (It turns out that this naive idea of quantization doesn't really work, but it's nonetheless a good guiding principle.)

Corollary 4.6. *Suppose* $F,G\in C^\infty(M)$ *Poisson-commute. Then:*

(a) *The function* G *is constant along the integral curves of* X_F*, while* F *is constant along the integral curves of* X_G*.*
(b) *The flows of the Hamiltonian vector fields* X_F,X_G *commute.*

Proof. The first claim is immediate from (10), the second claim is immediate from (13). □

Remark 4.7. Despite its simplicity, this Corollary is of crucial importance in applications of symplectic geometry in physics, e.g., in classical mechanics. There, one is given a 'Hamiltonian' H, the vector field X_H generates the dynamics of the system, and the functions that Poisson commute with H are 'conserved quantities' or 'integrals of motion'.

Definition 4.8. An algebra $\mathcal{A}$ together with a Lie bracket $[\cdot,\cdot]$ is called a Poisson algebra if

$$[FG, H] = F[G, H] + [F, H]G$$

for all $F, G, H \in \mathcal{A}$.

For any algebra A, the canonical Lie bracket $[F, G] = FG - GF$ satisfies this property.

Proposition 4.9. *The algebra* $(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ *is a Poisson algebra.*

Proof.

$$\{FG, H\} = L_{X_H}(FG) = (L_{X_H}F)G + F(L_{X_H}G) = \{F, H\}\,G + F\{G, H\}.$$

□

Proposition 4.10. *For any compact connected symplectic manifold, Lie algebra extension (9) has a canonical splitting. That is, there exists a canonical Lie algebra morphism*

$$\mathfrak{X}_{Ham}(M,\omega) \to C^\infty(M,\mathbb{R})$$

that is a right inverse to the map $F \mapsto X_F$.

Proof. The required map associates to every $X \in \mathfrak{X}_{Ham}(M,\omega)$ the unique H such that $X_H = X$ and $\int_M H\Lambda = 0$ (where Λ is the Liouville form). The equality

$$\int_M \{F, G\}\,\Lambda = \int_M (\mathcal{L}_{X_F}G)\Lambda = \int_M \mathcal{L}_{X_F}(G\Lambda) = 0$$

shows that this is indeed a Lie algebra morphism. □

Let us give the expression for the Poisson bracket for open subsets $U \subseteq \mathbb{R}^{2n}$, with symplectic coordinates $q_1,\ldots,q_n,p_1,\ldots,p_n$. We have

$$X_F = \sum_{j=1}^{n}\Big(\frac{\partial F}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial}{\partial p_j}\Big),$$

hence $\{F, G\} = X_F(G)$ is given by

$$\{F, G\} = \sum_{j=1}^{n}\Big(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}\Big).$$

Exercise 4.11. Verify directly, in local coordinates, that the right hand side of this formula defines a Lie bracket.

4.3. **Lagrangian submanifolds.** The notions of isotropic, coisotropic, Lagrangian, and symplectic subspaces of symplectic vector spaces carry over to submanifolds; one simply requires the defining property on each tangent spaces.

Definition 4.12. A submanifold (or, more generally, an immersion) $\iota : N \hookrightarrow M$ is called *coisotropic* (resp. *isotropic, Lagrangian, symplectic*) if at any point $m \in N$, T_mN is a coisotropic (resp. isotropic, Lagrangian, symplectic) subspace of T_mM.

Thus, $\iota\colon N \hookrightarrow M$ is isotropic if $\iota^*\omega = 0$, and is Lagrangian if furthermore $\dim N = \frac{1}{2}\dim M$. All 1-dimensional submanifolds of M are isotropic, all codimension 1 submanifolds are coisotropic. We'll give a long list of examples of Lagrangian submanifolds:

Example 4.13. The fibers $\pi^{-1}(q)$ of a cotangent bundle $\pi\colon T^*Q \to Q$ are Lagrangian.

Example 4.14. The zero section of a cotangent bundle is Lagrangian. More generally, if $\alpha \in \Omega^1(Q)$ is a 1-form and $\sigma_\alpha\colon\ Q \to T^*Q$ the corresponding section, then the range of σ_α is Lagrangian if and only if α is closed. This follows from

$$\sigma_\alpha^*\omega = -\sigma_\alpha^*\mathsf{d}\theta = -\mathsf{d}\sigma_\alpha^*\theta = -\mathsf{d}\alpha.$$

Example 4.15. The submanifold $\mathbb{R}^n \subseteq \mathbb{C}^n$ is Lagrangian, as is $\mathbb{R}P(n) \subseteq \mathbb{C}P(n)$. More generally, if a Kähler manifold has a *complex conjugation map*, by which we mean an isometric anti-linear involution $F\colon M \to M$, i.e.

$$F^*g = g, \quad F^*J = -J, \quad F \circ F = \mathrm{id}_M$$

then the corresponding set of real points (fixed points under complex conjugation) is a Lagrangian submanifold. This follows as a special case of the next example.

Example 4.16. If (M, ω) is a symplectic manifold, and $F\colon M \to M$ is an anti-symplectic involution, i.e,

$$F^*\omega = -\omega, \quad F \circ F = \mathrm{id}_M$$

then the fixed point set

$$N = \{m \in M |\ F(m) = m\}$$

is Lagrangian. Indeed, the fixed point set of an action of a compact group (here $\mathbb{Z}_2$) is always a submanifold. This reduces the problem to a symplectic vector space (V, ω) with an anti-symplectic linear involution $F\colon V \to V$. Decompose into eigenspaces

$$V = V_+ \oplus V_-$$

for the involution. Then ω restricts to zero on both $V_\pm$, hence both are Lagrangian. (By the way, a similar reasoning shows that the fixed point set of a *symplectic* involution is a symplectic submanifold.)

Example 4.17. For any symplectic manifold (M,ω), we denote by $\overline{M}$ the same manifold with the symplectic structure $-\omega$. Then the diagonal

$$\Delta \subseteq M \times \overline{M}$$

is a Lagrangian submanifold. More generally, a diffeomorphism $\Phi \in \mathrm{Diff}(M)$ between symplectic manifolds is a symplectomorphism if and only if its graph

$$\mathrm{gr}(\Phi) = \{(\Phi(m), m)|\ \mathsf{m} \in M\} \subseteq M \times \overline{M}$$

is Lagrangian. (Exercise).

Example 4.18. Consider a smooth map $F\colon Q_1 \to Q_2$. Then

$$\mathrm{gr}(T^*F) \subseteq T^*Q_2 \times \overline{T^*Q_1}$$

is a Lagrangian submanifold, where we define

$$\mathrm{gr}(T^*F) = \{(\mu_2, \mu_1)|\ \exists q \in Q_1\colon \mu_1 \in T^*_q Q_1,\ \mu_2 = (T^*_q F)(\mu_1)\}.$$

The proof is left as an exercise. (As usual, one mainly has to understand the linear version.)

For our next example, recall that the *conormal bundle* of a submanifold $j\colon S \hookrightarrow Q$ is the subbundle of T^*Q along S given as the annihilator bundle of the tangent bundle to S: $\nu(Q,S)^* = \mathrm{ann}(TS)$. In other words, letting $j^* = (T^*j)\colon T^*Q|_S \to T^*S$ be the pullback map,

$$\nu(Q,S)^* = \{\mu \in T^*Q|_S|\ j^*\mu = 0\}.$$

The conormal bundle is the dual bundle to the normal bundle $\nu(Q,S) = TQ|_S/TS$; hence the name.

Proposition 4.19. *The conormal bundle to any submanifold $j\colon S \hookrightarrow Q$ is a Lagrangian submanifold of T^*Q. In fact, letting $\iota\colon \nu(Q,S) \to T^*Q$ be the inclusion, we already have $\iota^*\theta = 0$.*

Proof. Coordinate proof: Around any point of S we can choose submanifold coordinates $q_1, \ldots, q_n$ on Q such that S is given by equations $q_{k+1} = 0, \ldots, q_n = 0$. In the corresponding cotangent coordinates q_j, p_j on T^*Q, the conormal bundle $\mathrm{ann}(TS)$ is given by equations $q_{k+1} = 0, \ldots, q_n = 0$, $p_1, \ldots, p_k = 0$. Clearly each summand in $\theta = \sum_j p_j \mathsf{d} q_j$ vanishes after pullback to this submanifold. Hence also $\iota^*\omega = -\iota^*\mathsf{d}\theta = 0$. □

The examples of 'conormal bundle' and 'closed 1-form' can be combined:

Proposition 4.20. *Let $S \subseteq Q$ be a submanifold, and $\alpha \in \Omega^1(S)$ be a closed 1-form on S. Then the subset*

$$N = \{\mu \in T^*Q|_S|\ j^*\mu = \alpha|_{\pi(\mu)}\}$$

is a Lagrangian submanifold.

Proof. In submanifold coordinates for $S \subseteq Q$, the subset from the proposition is given by equations

$$q_{k+1} = 0, \ldots, q_n = 0, p_1 = \alpha_1(q_1, \ldots, q_k), \ldots, p_k = \alpha_k(q_1, \ldots, q_k).$$

Hence the pull-back of ω to this submanifold is given by

$$\sum_{i=1}^{k} \mathsf{d}q_i \wedge \mathsf{d}\alpha_j = \sum_{i,j=1}^{k} \frac{\partial \alpha_i}{\partial q_j} \mathsf{d}q_i \wedge \mathsf{d}q_j$$

so that ω vanishes on this submanifold if and only if α is closed. □

As a special case, we see that any submanifold $S \subseteq Q$ together with a function $f \in C^\infty(S)$ determines a Lagrangian submanifold of T^*Q. Indeed, we may just take $\alpha = \mathsf{d}f$ in the proposition.

4.4. **Lagrangian relations.** The description of symplectic diffeomorphisms as Lagrangian submanifolds (via their graphs), suggests the concept of a *Lagrangian relation.* Recall that a relation between two sets M_1, M_2 is simply a subset

$$R \subseteq M_2 \times M_1;$$

we write $m_1 \sim_R m_2$ if $(m_2, m_1) \in R$. Maps $F \colon M_1 \to M_2$ may be regarded as relations via their graphs $\mathrm{gr}(F) \subseteq M_2 \times M_1$. We think of R as a 'generalized map'

$$R \colon M_1 \dashrightarrow M_2.$$

Composition of relations $R' \circ R$ is defined as

$$R' \circ R = \{(m_3, m_1) \in M_3 \times M_1 |\ \exists m_2 \in M_2 \colon m_1 \sim_R m_2,\ m_2 \sim_{R'} m_3\}.$$

If M_1, M_2 are manifolds, then we can speak of *smooth* relations, meaning that R is a submanifold. The composition of smooth relations need not be smooth, in general. There is a notion of 'clean composition' of relations R', R, ensuring that the composition is again a smooth submanifold (sometimes immersed, i.e. with self-intersections). Here are simpler 'transverse composition' conditions.

Definition 4.21. A composition of smooth relations $R' \circ R$ is called *transverse* if:

(i) The submanifolds $R' \times R$ and $M_3 \times \Delta_{M_2} \times M_1$ of $M_3 \times M_2 \times M_2 \times M_1$ are transverse.

(ii)
$$(TR' \times TR) \cap (0_{M_3} \times T\Delta_{M_2} \times 0_{M_1}) = \{0\}.$$

The first condition guarantees that the intersection is a smooth submanifold,

$$R' \diamond R \subseteq M_3 \times M_2 \times M_2 \times M_1.$$

The second condition means that the projection $q \colon M_3 \times M_2 \times M_2 \times M_1 \to M_3 \times M_1$ satisfies $\ker(Tq) \cap T(R' \diamond R) = 0$, hence it restricts to an immersion on $R' \diamond R$. The image of the latter is $R' \circ R \subseteq M_3 \times M_1$ which hence is an immersed submanifold.

A *Lagrangian relation* between symplectic manifolds $R\colon M_1 \dashrightarrow M_2$ is a Lagrangian submanifold

$$R \subseteq M_2 \times \overline{M_1}.$$

For Lagrangian relations $R'\colon M_2 \dashrightarrow M_3,\ R\colon M_1 \dashrightarrow M_2$, the transversality conditions (i), (ii) are actually equivalent to each other (as one finds by writing the condition on tangent spaces and taking ω-orthogonals). The composition of Lagrangian relations satisfying the transversality condition is again a Lagrangian relation. (The proof in the linear case is a homework exercise.)

Example 4.22. For every smooth map $F\colon Q_1 \to Q_2$, the cotangent relation $T^*F\colon T^*Q_1 \dashrightarrow T^*Q_2$ is a Lagrangian relation. Given maps $F\colon Q_1 \to Q_2$ and $F'\colon Q_2 \to Q_3$, one has

$$\mathrm{gr}(T^*F') \circ \mathrm{gr}(T^*F) = \mathrm{gr}(T^*(F' \circ F)).$$

Example 4.23. A Lagrangian relation $\mathrm{pt} \dashrightarrow M$ is the same thing as a Lagrangian submanifold $N \subseteq M$. (Same for Lagrangian relations $M \dashrightarrow \mathrm{pt}$.)

Example 4.24. Let $j\colon S \to Q$ be a submanifold. This defines a cotangent relation

$$T^*j\colon T^*S \dashrightarrow T^*Q$$

If $\alpha \in \Omega^1(S)$ is a closed 1-form, the range of σ_α is a Lagrangian submanifold of T^*S, which we may think of as a Lagrangian relation

$$\mathrm{pt} \dashrightarrow T^*S.$$

Composing, we obtain a Lagrangian relation $\mathrm{pt} \to T^*Q$ which we may think of as a Lagrangian relation $\{\mathrm{pt}\} \dashrightarrow T^*S$. This recovers Proposition 4.20. (The transversality condition (i) holds because the map $T^*Q|_S \to T^*S$ is surjective.)

Here is a nice application of these considerations.

Theorem 4.25 (Tulczyjew)**.** *Let $E \to B$ be a vector bundle, $E^* \to B$ its dual bundle. There is a canonical symplectomorphism $T^*E \cong T^*E^*$.*

Proof. Consider first the case that $B = \mathrm{pt}$, thus $E = V$ is a vector space. Consider the pairing

$$f\colon V^* \oplus V \to \mathbb{R}, (\mu, v) \mapsto \langle \mu, v\rangle.$$

The range of differential $\mathsf{d}f$ is Lagrangian submanifold $T^*(V^* \oplus V)$, given by

$$\{(\mu, v; v, \mu)|\ v \in V, \mu \in V^*\} \subseteq T^*(V^* \oplus V) = (V^* \oplus V) \times (V \oplus V^*).$$

After the identification $T^*(V^* \oplus V) \cong T^*V^* \times T^*V$, followed by sign change in T^*V (given by map $(v_1, \mu_1) \mapsto (v_1, -\mu_1)$, this becomes the Lagrangian submanifold

$$\{((\mu, v), (v, -\mu)|\ v \in V, \mu \in V^*\} \in T^*V^* \times \overline{T^*V}$$

which we may regard as the graph of the symplectomorphism $(v, -\mu) \mapsto (\mu, v)$. For the general case, consider $E^* \oplus E$ as a submanifold

$$E^* \oplus E \subseteq E^* \times E.$$

The pairing defines a function

$$f:\ E^* \oplus E \to \mathbb{R},\quad (\mu, v) \mapsto \langle \mu, v\rangle.$$

By Proposition 4.20, the closed 1-form $\mathsf{d}f$ defines a Lagrangian submanifold

$$L \subseteq T^*(E^* \times E) = T^*E^* \times T^*E.$$

One checks that the projections from L onto both factors T^*E and T^*E^* are diffeomorphisms. After sign change in the second factor (i.e., mapping a covector to minus the covector) it becomes a Lagrangian submanifold of $T^*E \times \overline{T^*E^*}$, which is the graph of a symplectomorphism. ☐

Remark 4.26. Tulczyjew [45] considered the case $E = TQ$; his aim was to give a geometric interpretation of the Legendre transform. I learned the argument above from the thesis of Roytenberg [39, page 33], who also pointed out that it works for arbitrary vector bundles.

4.5. **Coisotropic submanifolds.** For any submanifold $N \subseteq M$, let

$$C^\infty(M)_N = \{F \in C^\infty(M) |\, F|_N = 0\}$$

be its *vanishing ideal*. The tangent bundle TN has the characterization

$$TN = \{v \in TM |\, v(F) = 0\ \text{ for all } F \in C^\infty(M)_N\},$$

On the other hand, the annihilator $\mathrm{ann}(TN)$ is spanned by all differentials $\mathsf{d}F|_N$ with $F \in C^\infty(M)_N$. For symplectic manifolds, this translates to the following fact:

Proposition 4.27. *For any submanifold N of a symplectic manifold (M, ω), the subbundle TN^ω is spanned by restrictions of Hamiltonian vector fields $X_F|_N$ with $F \in C^\infty(M)_N$.*

Proof. The map $\omega^\flat : TM \to T^*M$ restricts to an isomorphism $TN^\omega \to \mathrm{ann}(TN)$, and identifies the Hamiltonian vector field $X_F \in \mathfrak{X}(M)$ with $\mathsf{d}F$. ☐

Recall that a submanifold N of a symplectic manifold (M, ω) is coisotropic if its tangent spaces are coisotropic, that is,

$$TN^\omega \subseteq TN.$$

Theorem 4.28. *The following three statements are equivalent:*

(a) *N is a coisotropic submanifold of M.*
(b) *For all $F \in C^\infty(M)_N$, the Hamiltonian vector field X_F is tangent to N.*
(c) *The space $C^\infty(M)_N$ is a Poisson subalgebra of $C^\infty(M)$.*

Proof. **(a) ⇔ (b).** By the proposition, TN^ω is spanned by restrictions of Hamiltonian vector fields $X_F|_N$ with $F \in C^\infty(M)_N$. Hence N is coisotropic ($TN^\omega \subseteq TN$) if and only if every such vector field is tangent to N.

(b) ⇔ (c). X_F is tangent to N if and only if $X_F(G) \in C^\infty(M)_N$ for all $G \in C^\infty(M)_N$. Since $X_F(G) = \{F, G\}$ this shows that (b) and (c) are equivalent. □

Remark 4.29. Note that in order to verify condition (c) around a given point $m \in N$, it is not necessary to check on all functions vanishing on N. It suffices to check $\{F_i, F_j\} = 0$ for any finite collection of functions vanishing on N near m and such that $dF_1, \ldots, dF_k$ span the conormal bundle at m.

Proposition 4.30. *Let (M, ω) be a symplectic manifold, and $\pi\colon M \to Q$ a submersion to another manifold. Then the fibers of π are coisotropic if and only if the functions in*

$$\pi^* C^\infty(Q) \subseteq C^\infty(M)$$

all Poisson commute.

Proof. **"⇐".** Suppose all fibers of π are coisotropic. Given $q \in Q$, suppose $N = \pi^{-1}(q)$. For every $f \in C^\infty(Q)$, the function $\pi^* f - f(q)$ lies in the vanishing ideal of N. Hence, given $f_1, f_2 \in C^\infty(Q)$, the Poisson bracket

$$\{\pi^* f_1, \pi_2^* f_2\} = \{\pi^* f_1 - f_1(q), \pi_2^* f_2 - f_2(q)\}$$

must vanish when restricted to N. Since q was arbitrary, the restriction to all fibers must vanish. That is, $\{\pi^* f_1, \pi_2^* f_2\} = 0$. **"⇒".** Suppose the functions in $\pi^* C^\infty(Q) \subseteq C^\infty(M)$ all Poisson commute. Given $q \in Q$, choose functions $f_1, \ldots, f_k$ with $f_i(q) = 0$, and such that the differentials $\mathsf{d} f_i|_q$ span $T_q^* Q$. (In other words, choose local coordinates around q.) Then the differentials of $F_i = \pi^* f_i$ span the conormal bundle to $N = \pi^{-1}(q)$ everywhere. By the preceding remark, since the F_i Poisson commute, it follows that N is coisotropic. □

The fibers of this submersion have codimension equal to the dimension of Q. In particular, if $\dim Q = \frac{1}{2} \dim M$, the fibers of F define a Lagrangian foliation of M. This is the setting for completely integrable systems. We will discuss this case later in more detail.

4.6. **Constant rank submanifolds.** A 2-form $\sigma \in \Omega^2(N)$ is said to have *constant rank* if the rank of the map $\sigma_n^\flat \colon T_nN \to T_nN^*,\ n \in N$ is independent of n. Equivalently, the kernel of this map has constant dimension.

Definition 4.31. A submanifold $\iota\colon N \hookrightarrow M$ of a symplectic manifold is called a *constant rank submanifold* if $\iota^*\omega$ has constant rank.

Isotropic, Lagrangian, coisotropic, and symplectic submanifolds are all examples of constant rank submanifolds.

Proposition 4.32. *Let N be a manifold together with a closed 2-form $\sigma \in \Omega^2(N)$ of constant rank. Then the subbundle* $\ker(\sigma)$ *is integrable, i.e. defines a foliation.*

Proof. Suppose $X_1, X_2 \in \mathfrak{X}(N)$ with $\iota_{X_j}\sigma = 0$. Since σ is closed, this implies $L_{X_j}\sigma = \mathsf{d}\iota_{X_j}\sigma = 0$. Hence

$$\iota_{[X_1,X_2]}\sigma = L_{X_1}\iota_{X_2}\sigma - \iota_{X_2}L_{X_1}\sigma = 0.$$

By Frobenius' theorem, this shows that $\ker\sigma$ is integrable. □

A foliation of N is called *fibrating* if the leaves of the foliation are the fibers of a submersion $\pi : N \to B$. (In this case, B is the space of leaves of the foliation.) The closed form σ is basic for this fibration since $\iota_X\sigma = 0$ and $L_X\sigma = 0$ for all vertical vector fields. It follows that B inherits a unique 2-form ω_B such that

$$\pi^*\omega_B = \sigma$$

Lemma 4.33. *The 2-form ω_B is symplectic.*

Proof. The 2-form ω_B is closed since $\pi^*\mathsf{d}\omega_B = \mathsf{d}\sigma = 0$ implies $\mathsf{d}\omega_B = 0$. It is also nondegenerate: For suppose $v \in T_bB$ is in the kernel of ω_B. Let $\tilde{v} \in T_nN$ be a lift (for some $n \in \pi^{-1}(b)$. Then $\iota(\tilde{v})\sigma_n = 0$, so $\tilde{v} \in \ker(\sigma_n) = \ker(T_n\pi)$. It follows that $v = (T_n\pi)(\tilde{v}) = 0$. □

Definition 4.34. The symplectic manifold (B, ω_B) is called the symplectic reduction of (N, σ).

For a constant rank submanifold $j\colon N \hookrightarrow M$ of a symplectic manifold, one refers to the foliation defined by $\sigma = j^*\omega$ as the *null foliation* of N. Thus, if the null foliation is fibrating then the symplectic reduction $N/\sim$ is defined. Of course, in practice it rarely happens that the null foliation is fibrating, unless additional symmetries are at work.

5. Normal Form Theorems

We shall next prove various versions of the so-called *Darboux theorem*. In its simplest form (due to Libermann), the theorem says that every symplectic form ω is locally given by $\omega = \sum_i \mathsf{d}q_i \wedge \mathsf{d}p_i$ in suitable *symplectic coordinates*. More generally, one has various normal form theorems around constant rank submanifolds.

5.1. Moser's argument.

Moser's argument (also known as Moser's trick) was used by Moser [37] to show that on a compact oriented manifold any two normalized volume forms are diffeomorphism equivalent. His proof involves the flows of time-dependent vector fields.

Definition 5.1. The flow of a time-dependent vector field $X_t \in \mathfrak{X}(Q)$ $(t \in \mathbb{R})$ is the smooth family of diffeomorphisms ϕ_t such that $\phi_0 = \mathrm{id}_Q$ and such that

$$\phi_t^*\,(L_{X_t} f) = -\frac{\mathsf{d}}{\mathsf{d}t}\,\phi_t^* f \tag{14}$$

for all $f \in C^\infty(Q)$.

Written in local coordinates, the flow is described as the solution of a time-dependent system of ODE's: With $X_t = \sum_i a_i(x,t)\frac{\partial}{\partial x_i}$, this is the system

$$\frac{\mathsf{d}}{\mathsf{d}t} x_i(t) + a_i(x(t),t) = 0.$$

Hence, we have *local* existence and uniqueness for such equations. (Given $m \in M$ and any $\epsilon > 0$, there exists an open neighborhood U such that $\phi_t \colon U \to M$ is defined for $|t| < \epsilon$ and satisfies the differential equation.) In general it is only defined on a certain domain of definition. If X_t is supported in some fixed compact subset (in particular, if Q is compact), the flow exists globally, for all t. The definition implies a similar identity for differential forms $\alpha \in \Omega^*(Q)$,

$$\phi_t^*\, L_{X_t}\alpha = -\frac{\mathsf{d}}{\mathsf{d}t}\,\phi_t^*\alpha.$$

Theorem 5.2 (Moser)**.** *Let Q be a compact, oriented manifold, and Λ_0, Λ_1 two volume forms such that*

$$\int_M \Lambda_0 = \int_M \Lambda_1.$$

Then there exists a smooth family of diffeomorphisms $\phi_t \in \mathrm{Diff}(Q)$ with $\phi_0 = \mathrm{id}_M$, such that

$$\phi_1^*\Lambda_1 = \Lambda_0.$$

Proof. Moser's argument is as follows. First, note that every $\Lambda_t = (1-t)\Lambda_0 + t\Lambda_t$ is a volume form. Second, since Λ_0 and Λ_1 have the same integral they define the same cohomology class: $\Lambda_1 = \Lambda_0 + \mathsf{d}\beta$ for some $n-1$-form β. Thus

$$\Lambda_t = \Lambda_0 + t\,\mathsf{d}\beta.$$

We would like to construct a family of diffeomorphisms ϕ_t defined on some open neighborhood of $[0,1]$, with the property

$$\phi_0 = \mathrm{id}_M, \quad \phi_t^*\Lambda_t = \Lambda_0. \tag{15}$$

This is equivalent to the differentiated version, $\frac{d}{dt}(\phi_t^*\Lambda_t) = 0$. Let X_t be the vector field for which the sought-after ϕ_t is the corresponding time-dependent flow, defined by (14). Then

$$\begin{aligned} -\frac{\mathsf{d}}{\mathsf{d}t}\,\phi_t^*\Lambda_t &= \phi_t^*(\mathcal{L}_{X_t}\Lambda_t - \frac{\mathsf{d}}{\mathsf{d}t}\Lambda_t) \\ &= \phi_t^*\Big(\mathsf{d}\iota_{X_t}\Lambda_t - \mathsf{d}\beta\Big) \\ &= \phi_t^*\mathsf{d}\Big(\iota_{X_t}\Lambda_t - \beta\Big) \end{aligned}$$

This expression will vanish for all t, provided that

$$\iota_{X_t}\Lambda_t = \beta \tag{16}$$

for all t. Since each Λ_t is a volume form, the map $X \mapsto \iota_X\Lambda_t$ from vector fields to $(n-1)$-forms ($n = \dim Q$) is an isomorphism. It follows that (16) has a unique solution, given by a time dependent vector field X_t, and its flow satisfies (15). □

Moser's theorem shows that volume forms on a given compact oriented manifold Q are classified up to diffeomorphism by their integral.

Remark 5.3. Let us also note the following variant: If Λ_t is any family of volume forms on a manifold Q (not necessarily compact), with

$$\frac{\mathsf{d}}{\mathsf{d}t}\Lambda_t = \mathsf{d}\beta_t \tag{17}$$

for a smooth family of forms β_t with support in some compact subset $K \subseteq Q$, then there exists a family of diffeomorphisms ϕ_t (equal to the identity outside K) such that $\phi_t^*\Lambda_t = \Lambda_0$ for all t. These diffeomorphisms are obtained as the flow of the time dependent vector field X_t such that $\iota(X_t)\Lambda_t = \beta_t$. (Actually, the existence of β_t satisfying (17) is equivalent to saying that the Λ_t's coincide outside a compact set K, and $\int_Q(\Lambda_t - \Lambda_0) = 0$ for all t. In fact, one may construct primitives β_t with the help of a Riemannian metric, using Hodge theory.)

Moser [37] remarks that his argument also applies to prove Darboux's theorem in symplectic geometry. This was used extensively in the work of Weinstein to obtain many related normal form theorems in symplectic geometry.

Theorem 5.4 (Moser stability theorem)**.** *Let ω_t be a smooth family of symplectic 2-forms on a compact manifold M, with t varying in some interval around 0, and such that*

$$\frac{\mathsf{d}}{\mathsf{d}t}\omega_t = \mathsf{d}\beta_t$$

for some smooth family of 1-forms $\beta_t \in \Omega^1(M)$. Then there exists a smooth family of diffeomorphisms ϕ_t such that $\phi_0 = \mathrm{id}_M$ and

$$\phi_t^* \omega_t = \omega_0$$

for all t.

Proof. The conditions $\phi_0 = \mathrm{id}_M$ and $\phi_t^*\omega_t = \omega_0$ are equivalent to the differentiated version

$$\frac{\mathsf{d}}{\mathsf{d}t}\, \phi_t^*\omega_t = 0.$$

Letting X_t be the time dependent vector field corresponding to the flow ϕ_t, we have

$$\begin{aligned} -\frac{\mathsf{d}}{\mathsf{d}t}\, \phi_t^*\omega_t &= \phi_t^*\Big(\mathcal{L}_{X_t}\omega_t - \frac{\mathsf{d}}{\mathsf{d}t}\omega_t\Big) \\ &= \phi_t^*\mathsf{d}(\iota_{X_t}\omega_t - \beta_t) \end{aligned}$$

Thus, the flow of the time-dependent vector field X_t defined by

$$\iota_{X_t}\omega_t = \beta_t$$

has the desired property. □

Remark 5.5. The assumption of the theorem holds whenever $\frac{d}{dt}\omega_t$ is exact for all t. In other words, the forms ω_t must be cohomologous. Using a Riemannian metric and Hodge theory, one can always pick primitives depending smoothly on t.

Remark 5.6. A bit more generally, the argument works for noncompact manifolds and any family of symplectic forms ω_t such that $\frac{d}{dt}\omega_t = \mathsf{d}\beta_t$ for a family of 1-forms β_t supported in some fixed compact set.

5.2. **Homotopy operators.** Let Q_1, Q_2 be manifolds. A *smooth homotopy* between smooth maps $F_0,\ F_1 :\ Q_1 \to Q_2$ is a smooth map

$$F :\ [0,1] \times Q_1 \to Q_2,\ \ (t,q) \mapsto F_t(q),$$

having F_0, F_1 as its boundary values. If such an F exists, we call F_0, F_1 *(smoothly) homotopic*. (One can show that this is equivalent to the two maps being continuously homotopic.) Given a smooth homotopy, define the *homotopy operator*

$$\mathsf{h} :\ \Omega^k(Q_2) \to \Omega^{k-1}(Q_1)$$

by pullback followed by *fiber integration*:

$$\mathsf{h}(\alpha) = \int_{[0,1]} F^*\alpha.$$

Here fiber integration

$$\int_{[0,1]} : \Omega^k(\,[0,1] \times Q_1) \to \Omega^{k-1}(Q_1)$$

integrates over the $[0,1]$ factor. In detail, write a given k-form on $[0,1] \times Q$ as

$$\mathsf{d}s \wedge \beta_s + \gamma_s$$

where $\beta_s \in \Omega^{k-1}(Q),\ \gamma_s \in \Omega^k(Q)$ are forms on Q depending smoothly on s. Then

$$\int_{[0,1]} \Big(\mathsf{d}s \wedge \beta_s + \gamma_s) = \int_0^1 \beta_s\ |ds|$$

where the right hand side is the usual Riemannian integral over s.

Exercise 5.7. Verify that for any form β on $[0,1] \times Q$,

$$\int_{[0,1]} \mathsf{d}\beta + \mathsf{d}\int_{[0,1]} \beta = \iota_1^*\beta - \iota_0^*\beta$$

where $\iota_j : Q \to [0,1] \times Q$ are the two inclusions of boundary components. (Hint: fundamental theorem of calculus!)

As a consequence:

Proposition 5.8. *The map* h *has the property,*

$$\mathsf{d} \circ \mathsf{h} + \mathsf{h} \circ \mathsf{d} = F_1^* - F_0^* : \Omega^k(Q_2) \to \Omega^k(Q_1)$$

If $\alpha \in \Omega^k(Q_2)$ with $\mathsf{d}\alpha = 0$, then $\beta = \mathsf{h}(\alpha)$ satisfies

$$F_1^*\alpha - F_0^*\alpha = \mathsf{d}\beta.$$

In particular, F_0^* and F_1^* induce the same map in cohomology. A typical application is the following

Proposition 5.9. *Suppose* $U \subseteq M$ *is an open neighborhood with a smooth deformation retraction*

$$F\colon [0,1] \times U \to U$$

onto a submanifold $N \subseteq M$. *(That is,* $F_1 = \mathrm{id},\ F_0 = \iota \circ \pi$ *for some* $\pi\colon U \to N$.*) Then the inclusion* $\iota\colon N \hookrightarrow U$ *induces an isomorphism in cohomology,* $\iota^*\colon H(U) \to H(N)$.

Proof. The deformation retraction defines a homotopy operator with

$$\mathsf{dh} + \mathsf{hd} = \mathrm{id} - \iota^*\pi^*.$$

This means that ι^* defines an isomorphism in cohomology, with inverse induced by π^*. ☐

A special case of this result is for $U \subseteq \mathbb{R}^n$ any open ball around 0, with the deformation retraction onto $N = \{0\}$ given as scalar multiplication by t. The corresponding homotopy operator is called the *de Rham homotopy operator*. Similarly, vector bundles $\pi: E \to B$ have a canonical homotopy operator, defined by the linear retraction onto the base.

5.3. **Darboux-Weinstein theorems.** The following result is commonly known as Darboux's theorem. The classical Darboux theorem [10] is a local normal form theorem for 1-forms α such that the exterior differential $\mathsf{d}\alpha$ has constant rank k. Its version for symplectic manifolds was first proved by Paulette Liberman [26] (using the classical Darboux theorem).

Theorem 5.10 (Darboux's theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $\dim M = 2n$ and $m \in M$. Then there exist a coordinate chart (U, ϕ) around m, defining coordinates $q_1, p_1, \ldots, q_n, p_n$ such that*

$$\omega|_U = \phi^*\Big(\sum_j \mathsf{d}q_j \wedge \mathsf{d}p_j\Big).$$

Coordinate charts of this type are called *Darboux charts*, the coordinates are called *Darboux coordinates*.

Proof. Let $\omega_0 = \mathsf{d}q_j \wedge \mathsf{d}p_j$ the standard symplectic form on $\mathbb{R}^{2n}$. Using any coordinate chart centered at m, we may assume that M is an open neighborhood $U \subseteq \mathbb{R}^{2n}$ of $m = 0$, with ω some possibly non-standard symplectic form. Since any two symplectic forms on the vector space $T_0\mathbb{R}^{2n}$ are related by a linear transformation, we may assume that ω agrees with ω_0 on $T_0\mathbb{R}^{2n}$. All the forms

$$\omega_t = t\omega_1 + (1-t)\omega_0$$

are standard at $0 \in \mathbb{R}^{2n}$, and in particular are nondegenerate at 0. They remain nondegenerate on some neighborhood of 0. Replacing U with a small open ball, we may assume that ω_t are symplectic on all of U. Using the de Rham homotopy operator for the open ball put

$$\beta := \mathsf{h}(\omega_1 - \omega_0) \in \Omega^1(U).$$

Then $\frac{d}{dt}\omega_t = \mathsf{d}\beta$. Define a time-dependent vector field X_t on U by

$$\iota_{X_t}\omega_t = \beta.$$

The flow of this vector field will not be complete in general. But since $\omega_1 - \omega_0$ vanishes at $\{0\}$, the 1-form β and therefore the vector field X_t also vanish at $\{0\}$. Hence we can find a smaller neighborhood U' of 0 such that the flow $\phi_t : U' \to U$ is defined for all $t \in [0,1]$. The flow satisfies

$$-\frac{\mathsf{d}}{\mathsf{d}t}\phi_t^*\omega_t = \phi_t^*(L_{X_t}\omega_t - \mathsf{d}\beta) = \phi_t^*(\mathsf{d}\iota_{X_t}\omega_t - \mathsf{d}\beta) = 0,$$

hence by integration (with initial condition $\phi_0 = \mathrm{id}$) we have $\phi_t^*\omega_t = \omega_0$. In particular, $\phi_1^*\omega_1 = \omega_0$. Darboux's theorem follows by setting $\phi = (\phi_1)^{-1}$. □

Darboux's theorem shows that *symplectic manifolds have no local invariants*, in contrast to Riemannian geometry where curvature provides such invariants.

Darboux's theorem can be strengthened in many ways. The following results involve *tubular neighborhood embeddings*. For any manifold M and (topologically) closed submanifold N, let

$$\nu(M,N) = TM|_N/TN \to N$$

be the *normal bundle*. Some basic facts about this construction:

- $\dim \nu(M,N) = \dim M$.
- Any smooth map $\phi\colon M_1 \to M_2$ taking the closed submanifold $N_1 \subseteq M_1$ into a closed submanifold $N_2 \subseteq M_2$ induces a vector bundle morphism $\nu(\phi)\colon \nu(M_1,N_1) \to \nu(M_2,N_2)$.
- For a vector bundle $E \to N$, we have that $(TE)|_N = E \oplus TN$, and hence $\nu(E,N) = E$ canonically.

Definition 5.11. A *tubular neighborhood embedding* for a closed submanifold $N \subseteq M$ is an open neighborhood $U \subseteq \nu(M,N)$ of the zero section of the normal bundle, together with an embedding $\phi\colon U \to M$, taking $N \subseteq U$ to $N \subseteq M$, such that the induced map on normal bundles

$$\nu(\phi)\colon \nu(U,N) \to \nu(M,N)$$

is the identity map.

It is a basic result in differential geometry that tubular neighborhood embeddings always exist. They may be constructed, for example, with the help of a Riemannian metric, or with the help of *Euler-like vector fields*. One may always take $U \subseteq \nu(M,N)$ to be a bundle of open balls: for N compact, fix an inner product and take U to be elements of length $< \epsilon$ for ϵ sufficiently small. By rescaling along the fibers, we may even arrange that $U = \nu(M,N)$; we shall call this a *complete* tubular neighborhood embedding.

The following result says that for a closed submanifold N of a symplectic manifold (M,ω), a sufficiently small open neighborhood of $N \subseteq M$ is uniquely determined by the restriction (not to be confused with pullback)

$$\omega|_N \in \Gamma(\wedge^2 T^*M|_N).$$

Theorem 5.12. [48] *Let (M_0, ω_0) and (M_1, ω_1) be symplectic manifolds, and let*

$$N_j \subseteq M_j$$

be closed submanifolds. Suppose furthermore that $\psi: N_0 \to N_1$ a diffeomorphism, with a lift to an isomorphism of symplectic vector bundles

$$\hat{\psi}: TM_0|_{N_0} \to TM_1|_{N_1},$$

such that $\hat{\psi}$ restricts to the tangent map $T\psi: TN_0 \to TN_1$. Then ψ extends to a symplectomorphism $\phi: U_1 \to U_2$ between open neighborhoods U_i of N_i, in such a way that $\hat{\psi}$ is the restriction of $T\phi$.

Here, a *symplectic vector bundle*

$$E \to B$$

is a real vector bundle together with a smooth family of linear symplectic structures $\omega_b: E_b \times E_b \to \mathbb{R}$ on the fibers. (No integrability condition is imposed.) Here smoothness is the condition that for any two sections $\sigma, \tau \in \Gamma(E)$ the function $\omega(\sigma, \tau)$ on the base is smooth, or equivalently that ω define a smooth section of $\wedge^2 E^*$. Examples of symplectic vector bundles include the tangent bundle TM of an (almost) symplectic manifold, and also its restriction to any submanifold.

Proof. The map $\widehat{\psi}$ induces a morphism of normal bundles $\nu(M_0, N_0) \to \nu(M_1, N_1)$. Using complete tubular neighborhood embeddings, we may assume that $M_0 = M_1 =: M$ is the total space of a vector bundle

$$\pi: M = E \to N$$

over a given manifold $N_0 = N_1 = N$, with two given symplectic forms $\omega_0, \omega_1 \in \Omega^2(M)$ that agree along N. Let $\mathsf{h}: \Omega^k(M) \to \Omega^{k-1}(M)$ be the standard (de Rham) homotopy operator for the vector bundle $\pi: M \to N$, and put

$$\beta = \mathsf{h}(\omega_1 - \omega_0)$$

and $\omega_t = \omega_0 + t\mathsf{d}\beta$. Since ω_t agrees with ω_0 along N, it is in particular symplectic on a neighborhood of N in M. On that neighborhood we can define a time-dependent vector field X_t with $\iota_{X_t}\omega_t = \beta$. Let ϕ_t be its flow (defined on some neighborhood $U \subseteq M$ of N for all $t \in [0, 1]$), and put $\phi_1 =: \phi$. By Moser's argument $\phi^*\omega_1 = \omega_0$. ☐

We now specialize this master theorem to various interesting settings, starting with Lagrangian submanifolds.

Lemma 5.13. *For a Lagrangian submanifold $N \subseteq M$, there is a canonical isomorphism of vector bundles*

$$\nu(M, N) \to T^*N.$$

Proof. The map $\omega^\flat\colon TM|_N \to T^*M|_N$ takes $TN = TN^\omega$ to $\mathrm{ann}(N)$, and so descends to a vector bundle isomorphism $\nu(M,N) \to T^*N$. □

In particular, the total space of the normal bundle of a Lagrangian submanifold has a symplectic structure.

Theorem 5.14 (Lagrangian neighborhood theorem)**.** [46] *Let (M,ω) be a symplectic manifold, and $N \subseteq M$ a closed Lagrangian submanifold. Then there exists a tubular neighborhood embedding*

$$\psi\colon U \to M$$

of an open neighborhood $U \subseteq \nu(M,N) \cong T^*N$, intertwining the symplectic structures.

Proof. By the master theorem (Theorem 5.12) it suffices to construct an isomorphism of symplectic vector bundles $TM|_N \cong T(T^*N)|_N$, restricting to the identity map on TN. The subbundle $TN \subseteq TM|_N$ is Lagrangian; choose a a Lagrangian complement $L \subseteq TM|_N$. (E.g., take $L = J(TM|_N)$ for some compatible almost complex structure J on M.) The symplectic form gives an identification $L \cong (TN)^*$, and so

$$TM|_N \cong TN \oplus T^*N.$$

as a symplectic vector bundle (where the symplectic structure on the right hand side is given by the pairing). The same argument applies to M replaced with T^*N. Thus

$$TM|_N \cong TN \oplus T^*N \cong T(T^*N)|_N$$

is the desired symplectic isomorphism. □

Remark 5.15. One may look at this result as follows. For any k-form α whose pullback to a submanifold N vanishes, one obtains a *linear approximation* $\alpha_{[1]}$, which is a k-form on $\nu(M,N)$ which is homogeneous of degree 1 in the fiber direction. If M is symplectic and N is Lagrangian, then $\omega_{[1]}$ is again symplectic. The isomorphism $\nu(M,N) \to T^*N$ takes $\omega_{[1]}$ to the standard symplectic form. The essence of Theorem 5.14 is that there exists a tubular neighborhood embedding $\nu(M,N) \supseteq U \to M$ under which ω pulls back to its linearization.

The result generalizes to constant rank submanifolds. We shall need the following notion.

Definition 5.16. [41] Let N be a constant rank submanifold of a symplectic manifold (M,ω). The *symplectic normal bundle* of N is the symplectic vector bundle

$$TN^\omega/(TN \cap TN^\omega)$$

Examples 5.17.

(a) For coisotropic submanifolds the symplectic normal bundle is just 0.
(b) For an isotropic submanifold of dimension k, the symplectic normal bundle has rank $2(n-k)$ where $2n = \dim M$.
(c) For a symplectic submanifold, the symplectic normal bundle is just TN^ω.

The following theorem is due to Marle [30] (see also Sjamaar-Lerman [41]), extending earlier results of Weinstein [47] (for the cases N isotropic) and Gotay [16] (for the case N coisotropic). It says that a neighborhood of a constant rank submanifold $\iota\colon N \hookrightarrow M$ is characterized up to symplectomorphism by the pullback of the symplectic form $\iota^*\omega$, together with the symplectic normal bundle. In particular, if N is coisotropic, then a neighborhood is completely determined by $\iota^*\omega$.

Theorem 5.18 (Constant rank embedding theorem)**.** *For $j = 0, 1$, let $\iota_j : N_j \hookrightarrow M_j$ be closed constant rank submanifolds of symplectic manifolds (M_j, ω_j). Denote by*

$$F_j = TN_j^{\omega_j}/(TN_j \cap TN_j^{\omega_j})$$

their symplectic normal bundles. Suppose there exists a symplectic bundle isomorphism

$$\hat{\psi} :\ F_0 \to F_1,$$

with base map a diffeomorphism $\psi :\ N_0 \to N_1$ such that

$$\psi^*\iota_1^*\omega_1 = \iota_0^*\omega_0.$$

Then ψ extends to a symplectomorphism ϕ of neighborhoods of N_j in M_j, such that ϕ induces $\hat{\psi}$.

Proof. Given a closed constant rank submanifold N is a of a symplectic manifold (M, ω), consider the following three symplectic vector bundles over N:

$$\begin{aligned} E &= TN/(TN \cap TN^\omega), \\ F &= TN^\omega/(TN \cap TN^\omega), \\ G &= (TN \cap TN^\omega) \oplus (TN \cap TN^\omega)^*. \end{aligned}$$

Choose complementary subbundles $E' \subseteq TN$ and $F' \subseteq TN^\omega$ to $TN \cap TN^\omega$ in TN, TN^ω, respectively. Then

$$TN \cong E' \oplus (TN \cap TN^\omega),\ TN^\omega = F' \oplus (TN \cap TN^\omega).$$

Then E', F' are symplectic subbundles of $TM|_N$, and are symplectically orthogonal to each other. The quotient maps gives isomorphisms $E' \cong E,\ F' \cong F$. The symplectic vector bundle $G' = (E' \oplus F')^\omega$ contains $TN \cap TN^\omega$ as a Lagrangian subbundle; after choice of a complementary Lagrangian subbundle we obtain an isomorphism $G' \cong G$. This gives an isomorphism

$$TM|_N \cong E \oplus F \oplus G$$

as symplectic vector bundles. To prove the constant rank embedding theorem, choose isomorphisms of this type for both $TM_i|_{N_i}$. Let $\widehat{\psi}$ as in the statement of the theorem, with base map ψ. Then $T\psi$ induces an isomorphism of symplectic vector bundles $E_0 \cong E_1$ since $\psi : N_0 \to N_1$ preserves two-forms. We also obtain a unique isomorphism of symplectic vector bundles $G_0 \cong G_1$ preserving the given splitting and coinciding with $T\psi$ on $TN_i \cap TN_i^\omega$. Furthermore, $F_0 \cong F_1$ by assumption of the theorem. Putting all these together, we obtain an isomorphism $TM_0|_{N_0} \to TM_1|_{N_1}$ as in Theorem 5.12. □

We note the special cases:

(a) Given two coisotropic submanifolds $\iota_j \colon N_j \hookrightarrow M_j$, a diffeomorphism $\psi \colon N_0 \to N_1$ extends to a symplectomorphism of neighborhoods if and only if $\psi^*\iota_1^*\omega_1 = \iota_0^*\omega_0$.

(b) Given isotropic submanifolds $\iota_j \colon N_j \hookrightarrow M_j$, a diffeomorphism $\psi \colon N_0 \to N_1$ extends to a symplectomorphism of open neighborhoods if and only if $\psi^*(TN_1^\omega/TN_1)$ is isomorphic to TN_0^ω/TN as a symplectic vector bundle.

Remark 5.19 (Equivariant versions). The normal form theorems discussed in this section generalize to a setting with symmetries, under actions of compact Lie groups G. For example, the equivariant version of Darboux's theorem states that if G acts symplectically on (M,ω), and $m_0 \in M$ is a fixed point, then there is a G-equivariant symplectomorphism between G-invariant open neighborhoods of m_0 in M and 0 in the symplectic vector space $T_{m_0}M$. Similarly, the G-equivariant version of Weinstein's Lagrangian embedding theorem says that for a G-invariant Lagrangian submanifold $N \subseteq M$, there exists a G-equivariant symplectomorphism between invariant open neighbourhoods of N inside M and inside T^*N.

They key fact needed for the proof is the existence of *G-equivariant tubular neighbourhoods*. Suppose M is a manifold with an action of a compact group G, and $N \subseteq M$ is a G-invariant submanifold. Then we obtain a G-action on the vector bundle $TM|_N$, preserving TN, and hence a G-action on the normal bundle $\nu(M,N) = TM|_N/TN$. A G-equivariant tubular neighborhood embedding is given by a G-invariant neighborhood $U \subseteq \nu(M,N)$ of the zero section, and a tubular neighborhood embedding $\phi \colon U \to M$ intertwining the G-actions. If G is compact (more generally, if the G-action is proper), these may be constructed by using G-invariant Riemannian metrics. The proofs of the various normal form theorems also involved Moser's argument and homotopy operators, but these are all compatible with the G-actions.

6. Lagrangian fibrations and action-angle variables

Recall that a function $F \in C^\infty(M)$ on a symplectic manifold (M, ω) Poisson commutes with a given function $H \in C^\infty(M)$ if and only if F is constant along the integral curves of X_H. In the theory of completely integrable systems, one is looking for a large number of Poisson commuting functions. If $F_1, \ldots, F_k \in C^\infty(M)$ all Poisson commute, and $b \in \mathbb{R}^k$ is a regular value of $(F_1, \ldots, F_k)$, then the fiber $F^{-1}(b)$ is a coisotropic submanifold of codimension k. The maximum number of such a collection of Poisson commuting functions admitting regular values is given by $k = n = \frac{1}{2}\dim M$; in this case the regular fibers are Lagrangian. Letting $B \subseteq \mathbb{R}^n$ be the set of regular values, and replacing M with $M' = F^{-1}(B)$, we obtain a submersion with Lagrangian fibers. It turns out that the geometry of such Lagrangian submersions is quite interesting and restrictive. The following discussion is based mainly on the paper *On global action-angle variables* by Duistermaat [12].

6.1. Lagrangian submersions.

Let (M, ω) be a symplectic manifold, and B a manifold of dimension $\frac{1}{2}\dim M$. We had seen that a submersion $\pi\colon M \to B$ has Lagrangian fibers if and only if the subalgebra $\pi^* C^\infty(B)$ has trivial Poisson bracket: that is,

$$\{\pi^* f, \pi^* g\} = 0$$

for all $f, g \in C^\infty(B)$.

Definition 6.1. Let (M, ω) be a symplectic manifold. A Lagrangian submersion is a submersion $\pi : M \to B$ such that every fiber $\pi^{-1}(b)$ is a Lagrangian submanifold of M. It is called a Lagrangian fibration if $\pi\colon M \to B$ is furthermore a fiber bundle (i.e., locally trivial).

This implies in particular $\dim B = n = \frac{1}{2}\dim M$.

Examples 6.2.

(a) The fibers of a cotangent bundle $\pi : M = T^*Q \to Q$ define a Lagrangian fibration. The restriction of π to an open subset of M is a Lagrangian submersion (but not a fibration, in general).

(b) If $Q = (\mathbb{R}/\mathbb{Z})^n = T^n$ is an n-torus, we have a natural trivialization $T^*Q = Q \times \mathbb{R}^n$. Projection to the second factor defines a Lagrangian fibration $T^*Q \to \mathbb{R}^n$.

Is it possible to generalize the second example to compact manifolds Q other than a torus? That is, is it possible to find a Lagrangian fibration of T^*Q such that the zero section $Q \subseteq T^*Q$ is one of its leaves? We will find that the answer is **no**: A compact leaf of a Lagrangian fibration is *always* diffeomorphic to a torus.

For any submersion $\pi\colon M \to B$, we have the exact sequence of vector bundles over M

$$0 \to \ker(T\pi) \to TM \to \pi^*(TB) \to 0$$

where $\pi^*(TB)$ is the pullback bundle (with fiber at $m \in M$ given by $T_{\pi(m)}B$). Hence,

$$TM/\ker(T\pi) \cong \pi^*(TB).$$

In particular, for each $b \in B$ the restriction to the submanifold $\pi^{-1}(b)$ is a trivial vector bundle

$$(TM/\ker(T\pi))|_{\pi^{-1}(b)} = \pi^{-1}(b) \times T_bB.$$

Suppose now that (M, ω) is a symplectic manifold, and that $\pi\colon M \to B$ is a Lagrangian submersion. Then the vertical subbundle

$$\ker(T\pi) \subseteq TM$$

is a Lagrangian subbundle, and ω gives a nondegenerate pairing with the quotient bundle $TM/\ker(T\pi) \cong \pi^*TB$. (Recall that for every Lagrangian subspace L of a symplectic vector space, the symplectic form gives a pairing between L and V/L. The pairing involves a choice of sign, which we wil specify in the lemma below.) In summary:

Lemma 6.3. *If (M, ω) is symplectic, and $\pi\colon M \to B$ is a submersion with Lagrangian fibers, there is a canonical vector bundle isomorphism*

$$\ker(T\pi) \cong \pi^*(T^*B),$$

taking $v \in \ker(T\pi)_m$ to the unique $\mu \in T^*_{\pi(b)}B$ such that

$$\iota(v)\omega_m = -(T_m\pi)^*\mu.$$

Hence, for each $b \in B$ the tangent bundle of the fiber $T(\pi^{-1}(b)) = \ker(T\pi)|_{\pi^{-1}(b)}$ is trivial:

$$T(\pi^{-1}(b)) = \pi^{-1}(b) \times T^*_bB.$$

Remark 6.4. The fact that the tangent bundle of any fiber of a Lagrangian submersion is trivial already restricts the geometry: For instance, a fiber cannot be a 2-sphere. On the other hand 3-spheres are not yet ruled out.

For $\mu \in T^*_bB$, let $X_\mu \in \mathfrak{X}(\pi^{-1}(b))$ be the corresponding vector field. Similarly, for $\alpha \in \Omega^1(B)$ let $X_\alpha \in \mathfrak{X}(M)$ be the resulting vertical vector field. By definition,

$$\iota(X_\alpha)\omega = -\pi^*\alpha.$$

If $\mu = \alpha|_b$, then X_α restricts to the vector field X_μ.

Lemma 6.5. *For all $\mu_1, \mu_2 \in T^*_bB$ we have that*

$$[X_{\mu_1}, X_{\mu_2}] = 0.$$

Proof. Choose extensions of the covectors μ_1, μ_2 to 1-forms $\alpha_1, \alpha_2 \in \Omega^1(B)$ and $X_j = X_{\alpha_j}$. (The vector fields X_{μ_i} are just the restrictions.) We have

$$\begin{aligned} \iota_{[X_1,X_2]}\omega &= (L_{X_1}\iota_{X_2} - \iota_{X_2}L_{X_1})\omega \\ &= -L_{X_1}\pi^*\alpha_2 - \iota_{X_2}\mathsf{d}\iota_{X_1}\omega \\ &= -L_{X_1}\pi^*\alpha_2 + \iota_{X_2}\pi^*\mathsf{d}\alpha_1 \\ &= 0 \end{aligned}$$

since $\pi^*\alpha_j$ and $\pi^*\mathsf{d}\alpha_j$ are basic forms on $\pi : M \to B$. Since ω is non-degenerate this verifies $[X_1, X_2] = 0$. □

Remark 6.6. Given $m \in M$ with base point $b = \pi(m)$, a choice of basis $\mu_1, \ldots, \mu_n$ of T_b^*B defines local coordinates on $\pi^{-1}(b)$, by using the (commuting flows) of the vector fields X_{μ_i}. Changing the basis will change the coordinates by a linear transformation, changing m to a nearby point changes the coordinates by translation. Hence, the coordinates are canonically defined up to an affine transformation. This means that the fibers $\pi^{-1}(b)$ acquire an *affine structure*. (An affine structure on a manifold is given by an atlas whose transition functions are affine-linear transformations.)

Definition 6.7. A Lagrangian submersion $\pi \colon M \to B$ is *complete* if the vector fields X_μ on $\pi^{-1}(b)$ are complete, for all $b \in B$ and $\mu \in T_b^*B$.

Equivalently, for all $\alpha \in \Omega^1(B)$ the flow of X_α is complete, defining diffeomorphisms[12]

$$F_\alpha^t \colon M \to M.$$

Since the vector fields commute, the flows commute also; furthermore $F_\alpha^t \circ F_\beta^t = F_{\alpha+\beta}^t$ (since the flow of a sum of commuting vector fields is the composition of the flows). Let $F_\alpha = F_\alpha^{-1} \colon M \to M$ be the time flow for time (-1). We have

$$F_\alpha \circ F_\beta = F_{\alpha+\beta}, \quad F_0 = \mathrm{id}_M .$$

Hence, on each fiber $\pi^{-1}(b)$ we obtain an action of the vector space T_b^*B (regarded as an abelian group).

Remark 6.8. Taken together, this defines an action of the vector bundle T^*B on M, i.e. a map $T^*B \times_B M \to M$ (where the subscript means fiber product), satifying the usual axioms of an action.

Remark 6.9. As a special case, for $M = T^*Q \to B = Q$, we find that F_α agrees with the diffeomorphism G_α from Section 3.4, given by fiberwise addition of α. This is our motivation for using the $t = -1$ flow rather than the $t = 1$ flow.

[12]Our notion of complete Lagrangian submersion is a special case of the notion of *complete symplectic realizations* in Poisson geometry.

From now on, we shall assume that $\pi\colon M \to B$ is a complete Lagrangian submersion, and that *all of its fibers are connected.*

Proposition 6.10. *Let $\pi\colon M \to B$ be a complete Lagrangian submersion with connected fibers. Then each $\pi^{-1}(b)$ has a transitive and locally free action of the cotangent space T_b^*B. In particular,*

$$\pi^{-1}(b) \cong (\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k}$$

for some k, where $n = \frac{1}{2}\dim M$.

Here, a group action is *locally free* of its stabilizers are discrete (i.e., 0-dimensional).

Proof. For $m \in \pi^{-1}(b)$, since the vector fields X_μ, $\mu \in T_b^*B$ span the tangent space $T_m\pi^{-1}(b)$, the orbit $T_b^*B \cdot m$ is open in $\pi^{-1}(b)$. Since $\pi^{-1}(b)$ is a union of such orbits, and is connected, it follows that $\pi^{-1}(b)$ is a single orbit: That is, the action is transitive. For dimension reasons, since $\dim T_b^*B = \dim \pi^{-1}(b)$, the action has discrete stabilizers Λ_b. Hence,

$$\pi^{-1}(b) \cong T_b^*B/\Lambda_b,$$

the quotient of a vector space by a discrete subgroup. It is well-known (but not quite obvious) that every such lattice is generated over $\mathbb{Z}$ by some linearly independent vectors $e_1, \ldots, e_k$; extending to a basis $e_1, \ldots, e_n$ gives the identification with $(\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k}$. $\square$

We emphasize that much more than being diffeomorphic to $(\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k}$, the fibers are principal homogeneous spaces for the abelian group T_b^*B/Λ_b, and are identified with that group once a base point $m \in \pi^{-1}(b)$ is chosen. Observe also that Λ_b (the stabilizer of $m \in \pi^{-1}(b)$) does not depend on the choice of m in the fiber, since T_b^*B is an *abelian* group.

Let us next consider the compatibility of the T^*B-action with the symplectic structure.

Lemma 6.11. *The flow of X_α satisfies*

$$(F_\alpha^t)^*\omega = \omega + t\,\pi^*\mathsf{d}\alpha.$$

In particular, $F_\alpha = F_\alpha^{-1}$ is a symplectomorphism if and only if α is closed.

Proof. The lemma follows by integrating

$$\frac{\mathsf{d}}{\mathsf{d}t}(F_\alpha^t)^*\omega = -(F_\alpha^t)^*L_X\omega = (F_\alpha^t)^*\pi^*\mathsf{d}\alpha = \pi^*\mathsf{d}\alpha$$

(using $\pi \circ F_\alpha^t = \pi$) from 0 to t. $\square$

Taking the union of stabilizers over all $b \in B$, we obtain a subset

$$\Lambda = \bigsqcup_{b\in B} \Lambda_b \subseteq T^*B.$$

Proposition 6.12. *The subset $\Lambda \subseteq T^*B$ is a Lagrangian submanifold, transverse to the fibers of π.*

Proof. We check near a given fiber $\pi^{-1}(b_0)$, $b_0 \in B$. Choose an open neighborhood $U \subseteq B$ of b_0 over which the submersion admits a section $\sigma\colon U \to M$, i.e. $\pi \circ \sigma = \mathrm{id}|_U$. Restricting the action of T^*B to the range of σ, we obtain a smooth map

$$\phi\colon T^*B|_U \to M|_U = \pi^{-1}(U), \quad \mu \mapsto \mu \cdot \sigma(b) \text{ for } \mu \in T^*_bB$$

The map ϕ intertwines the cotangent projection $T^*B \to B$ with the map $\pi\colon M \to B$, and intertwines the vector bundle actions of $T^*B|_U$ on both sides. From this, it follows that the map has maximal rank, and so is a local diffeomorphism. Since $\Lambda_b = \{\mu |\ \mu{\cdot}\sigma(b) = \mu\}$, we see that

$$\Lambda|_U = \phi^{-1}(\sigma(U))$$

is a submanifold. Since $\sigma(U)$ is transverse to the fibers of M, its pre-image is transverse to the fibers of T^*B. This proves that Λ is a submanifold transverse to the fibers.

We next show that Λ is Lagrangian, near any given $\mu_0 \in \Lambda_{b_0}$. Since Λ is transverse to the fibers, there is an open neighborhood $U \subseteq B$ of b_0 and a 1-form $\alpha \in \Omega^1(U)$ such that Λ is given on some neighborhood of μ_0 as the range of α. Since Λ describes stabilizers of the action, this 1-form satisfies $F_\alpha = \mathrm{id}_{\pi^{-1}(U)}$. By the previous lemma, this means $\pi^*\mathsf{d}\alpha = 0$. Thus $\mathsf{d}\alpha = 0$. Since the range of a closed 1-form is a Lagrangian submanifold, this concludes the proof. □

Proposition 6.13. *The quotient*

$$\mathcal{T} = T^*B/\Lambda = \sqcup_{b\in B} T^*_bB/\Lambda_b$$

is a symplectic manifold, and the projection $\mathcal{T} \to B$ is a complete Lagrangian submersion with connected fibers.

Proof. As in the previous proof, choose a local section $\sigma\colon U \to M|_U$ of π. The choice determines a bijection

$$T^*B|_U/\Lambda_U \to M_U.$$

It is a standard fact from manifold theory that a quotient under an equivalence relation admits at most one smooth structure for which the quotient map is a submersion. Since $T^*B|_U \to M|_U$ is a submersion (even a local diffeomorphism), this shows that $T^*B|_U/\Lambda_U$ is a manifold.

Recall that the action of closed 1-forms $\alpha \in T^*B$ by fiberwise addition is a symplectic transformation of T^*B. Since Λ is *locally* described as the graph of closed 1-forms, it follows that the symplectic structure descends. □

Remark 6.14. By construction, Λ is a family of lattices $\Lambda_b \subseteq T_b^*B$. This family need not be locally trivial. For example, let $B = \mathbb{R}$, so that $T^*B = \mathbb{R} \times \mathbb{R}$. Let

$$\Lambda = \{(x,y) \in T^*B|\ x \neq 0, xy \in \mathbb{Z}\} \cup \{(0,0)\}.$$

This is a Lagrangian submanifold. For fixed $x \in \mathbb{R}$, the set Λ_x is given by $x^{-1}\mathbb{Z}$, for $x = \{0\}$ it is just $\{0\}$. The quotient T^*B/Λ is a well-defined manifold.

The choice of a local section $\sigma\colon U \to M$ determines a diffeomorphism $\psi\colon \mathcal{T}|_U \to M|_U$ (induced by the map ϕ above. In general, this won't be a symplectomorphism: Note that the pre-image of $\sigma(U)$ under the diffeomorphism is the identity section of $\mathcal{T}$ (image of the zero section in $T^*B|_U$), and so is *Lagrangian*. Hence, ψ can only be a symplectomorphism if $\sigma(U)$ is Lagrangian.

Proposition 6.15. *Suppose $U \subseteq B$ is such that $\pi\colon M \to B$ admits a section σ over U. If $H^2(U) = 0$ (e.g., if U is contractible), then we may take this section to be Lagrangian. For any choice of a Lagrangian section, the map*

$$\psi\colon \mathcal{T}|_U \to M|_U$$

is a symplectomorphism.

Proof. Let $\sigma\colon U \to M|_U$ be a given section. By assumption, the 2-form $\sigma^*\omega \in \Omega^2(U)$ is exact: $\sigma^*\omega = \mathsf{d}\beta$. The new section $\tilde{\sigma} = F_\beta \circ \sigma$ satisfies

$$\tilde{\sigma}^*\omega = \sigma^* F_\beta^*\omega = \sigma^*(\omega - \pi^*\mathsf{d}\beta) = \sigma^*\omega - \mathsf{d}\beta = 0.$$

In this way, we may arrange that $\sigma^*\omega = 0$, i.e. the graph of σ is a Lagrangian submanifold. Let ψ be the resulting diffeomorphism, and let $\tilde{\psi} : T^*B|_U \to M|_U$ be the map covering ψ. We want to show that

$$\tilde{\psi}^*\omega = \omega_{\mathrm{can}}$$

where $\omega_{\mathrm{can}} = -\mathsf{d}\theta$ is the canonical symplectic form on the cotangent bundle. The map $\tilde{\psi}$ is uniquely defined by is property

$$\tilde{\psi} \circ \alpha = F_\alpha \circ \sigma$$

for every 1-form $\alpha \in \Omega^1(U)$. Equivalently, it takes the zero section to σ, and satisfies

$$\tilde{\psi} \circ G_\alpha = F_\alpha \circ \tilde{\psi}$$

for all α, where G_α is the diffeomorphism of $T^*B|_U$ given by fiberwise addition of α. Let Y_α be the vector field on T^*B defined by $\iota(Y_\alpha)\omega_{\mathrm{can}} = -\pi_{T^*B}^*\alpha$. Letting G_α^t be its time-1-flow, we have

$$\tilde{\psi} \circ G_\alpha^t = F_\alpha^t \circ \tilde{\psi}.$$

Hence, the local diffeomorphism $\widetilde{\psi}$ takes Y_α on T^*B to X_α on M. We have

$$\iota(Y_\alpha)\widetilde{\psi}^*\omega = \widetilde{\psi}^*\iota(X_\alpha)\omega = -\widetilde{\psi}^*\pi^*\alpha = -\pi_{T^*B}^*\alpha = \iota(Y_\alpha)\omega_{\mathrm{can}}$$

for all α, which shows that the 2-forms agree on vertical vectors. Since the 2-forms are closed, this shows that

$$\widetilde{\psi}^*\omega - \omega_{\mathrm{can}} \in \Omega^2(T^*B)$$

is basic. To show that it is zero, it is enough to show that its pullback under any section $\tau\colon U \to T^*B|_U$ vanishes. We may take this section to be the zero section of T^*B. Then $\widetilde{\psi} \circ \tau = \sigma$, and

$$\tau^*(\widetilde{\psi}^*\omega - \omega_{\mathrm{can}}) = \sigma^*\omega - \tau^*\omega_{\mathrm{can}} = 0.$$

We conclude $\tilde{\psi}^*\omega = \omega_{\mathrm{can}}$. □

To summarize the discussion: Let $\pi : M \to B$ be a complete Lagrangian fibrations with connected fibers. Then:

- There is an action of the vector bundle $T^*B \to B$ on M (i.e., actions of the vector spaces T_b^*B on $\pi^{-1}(b)$).
- The stabilizers
$$\Lambda = \bigsqcup_{b\in B} \Lambda_b$$
for this action define a Lagrangian submanifold $\Lambda \subseteq T^*B$.
- The quotient
$$T^*B/\Lambda = \bigsqcup_{b\in B} T_b^*B/\Lambda_b$$
is a symplectic manifold, with a Lagrangian fibration $\tau\colon \mathcal{T} \to B$.
- The choice of local *Lagrangian* sections $\sigma\colon U \to M$ determines symplectomorphisms $\mathcal{T}|_U \to M|_U$.

Although the Lagrangian fibrations $\mathcal{T} \to B$ and $M \to B$ 'look the same' locally, they can be different globally. For example, the image of the zero section in T^*B defines a global Lagrangian section of $\mathcal{T} = T^*/\Lambda$, but $\pi\colon M \to B$ need not admit a global section.

6.2. **Action-angle coordinates.** Let us now make the additional assumption that the fibers are also compact. Thus, we consider a Lagrangian submersion $\pi : M \to B$ with compact, connected fibers. Choose a covering of B by contractible open subsets U (i.e., each U is diffeomorphic to an open ball). Since U is simply connected, the group bundle $\Lambda|_U \to U$ is trivial:

$$\Lambda|_U = U \times \mathbb{Z}^n,$$

The isomorphism is unique up to an action of

$$\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Z}),$$

the group of invertible matrices A such that both A and A^{-1} have integer coefficients (this implies $\det A = \pm 1$). The trivialization of $\Lambda|_U$ also trivializes the cotangent bundle (since a lattice basis of Λ_b is a vector space basis for T_b^*B). Hence,

$$T^*B|_U = U \times \mathbb{R}^n.$$

Let $\beta_1, \dots, \beta_n \in \Omega^1(U)$ the closed 1-forms corresponding to the standard basis vectors of $\mathbb{Z}^n$. Since U is simply connected, we may write

$$\beta_i = \mathsf{d}I_i,$$

where the $I_i \in C^\infty(U)$ are coordinates on U.

Definition 6.16. The coordinates $I_i \in C^\infty(U)$ are called *action coordinates* for the Lagrangian fibration.

Let us consider the uniqueness of the action coordinates. Given a trivialization of $\Lambda|_U$, the action coordinates are determined up to a constant. Changing the trivialization of $\Lambda|_U$ by $A \in \mathrm{GL}(n, \mathbb{Z})$ will change the action coordinates by the same transformation. We hence see that the action coordinates are uniquely determines up to an affine transformation hose linear part has integral entries. This is called an *integral affine structure.* To summarize:

Proposition 6.17. *If $\pi\colon M \to B$ is a Lagrangian fibration with compact, connected fibers, then the base B acquires a canonical integral affine structure.*

So, we see that the geometry of the base is quite restricted; e.g., if B is compact an connected it must itself be a torus. (In concrete examples, B is rarely compact.)

As usual, having local coordinates on $U \subseteq B$ defines cotangent coordinates on $T^*B|_U$. Rather than $q_1, p_1, \dots . q_n, p_n$, we will use notation

$$I_1, s_1, \dots, I_n, s_n \in C^\infty(T^*B|_U)$$

for these coordinates. The s_i descend to $\mathbb{R}/\mathbb{Z}$-valued functions on

$$(T^*B/\Lambda)|_U = U \times (\mathbb{R}/\mathbb{Z})^n.$$

Definition 6.18. The coordinates $I_1, s_1, \dots, I_n, s_n$ on $(T^*B/\Lambda)|_U$ are called the action-angle coordinates.

By construction, the symplectic form on $(T^*B/\Lambda)|_U$ is given in action-angle coordinates by

$$\sum_i \mathsf{d}I_i \wedge \mathsf{d}s_i.$$

Until now, our action-angle coordinates are coordinates on $(T^*B/\Lambda||_U$ rather than on $M|_U$. The choice of a Lagrangian section $\sigma\colon U \to M|_U$ gives a (symplectic) identification

$$M|_U \cong \mathcal{T}|_U = (T^*B/\Lambda)|_U.$$

In particular, we see that $M \to B$ is itself a fiber bundle with typical fiber a torus of dimension $n = \frac{1}{2}\dim M$. Using this identification, the action-angle coordinates becomes coordinates on $M|_U$. The identification depends on the choice of σ, which in turn is unique up to the action of a closed exact 1-form. Since we assume U is simply connected, this 1-form is of the form $\mathsf{d}f$. The action coordinates are not affected by this change, but the angle coordinates change by addition of (the pullback of) this function f.

In his paper [12], Duistermaat discusses the existence of global action-angle coordinates. A necessary condition for the existence of global action coordinates is that the bundle Λ be trivial. This is automatic if B is simply connected; one obtains the trivialization by parallel transport. In general, after trivializing a fiber $\Lambda_b \cong \mathbb{Z}^n$, the parallel transport defines a monodromy map

$$\pi_1(B) \to \mathrm{GL}(n, \mathbb{Z}).$$

If this monodromy is trivial, we obtain a trivialization of Λ, and hence of T^*B and of T^*B/Λ. A second obstruction (to get oordinates on M rather than on T^*B/Λ) is the existence of a global Lagrangian section.

Duistermaat shows that for a very standard integrable system, the spherical pendulum, the monodromy obstruction is non-zero. We will discuss this example in Section 6.4 below.

Exercise 6.19. Suppose (M, ω) is a symplectic manifold such that ω is exact: $\omega = \mathsf{d}\gamma$ for some 1-form γ. Let $\pi : M \to B$ be a Lagrangian fibration with compact connected fibers, with B simply connected. Given $b \in B$ let

$$A_1(b), \ldots, A_n(b) : \mathbb{R}/\mathbb{Z} \to \pi^{-1}(b)$$

be smooth loops in $\pi^{-1}(b)$ generating the fundamental group of the fiber. Suppose that the $A_i(b)$ define continuous functions $A_i : B \times \mathbb{R}/\mathbb{Z} \to M$. Show that the formula

$$I_j(m) := \int_{A_j(\pi(m))} \gamma$$

defines a set of action variables.

6.3. **Completely integrable systems.** After this lengthy general discussion let us finally make the connection with the theory of completely integrable systems. Let (M, ω) be a compact symplectic manifold, $H \in C^\infty(M, \mathbb{R})$ a Hamiltonian and X_H its vector field. In general the flow of X_H can be very complicated, unless there are many "integrals of motion". An integral of motion is a function $G \in C^\infty(M, \mathbb{R})$ such that $X_H(G) = 0$, or equivalently $\{H, G\} = 0$. An integral of motion defines itself a Hamiltonian flow X_G, which commutes with the flow of X_H since $[X_H, X_G] = X_{\{H,G\}} = 0$.

Definition 6.20. The mechanical system (M, ω, H) is called *completely integrable* if there are n integrals of motion $G_1, \ldots, G_n \in C^\infty(M, \mathbb{R})$, $\{G_j, H\} = 0$ such that

(a) The G_i are "in involution", i.e. they Poisson-commute: $\{G_i, G_j\} = 0$.
(b) The map $G = (G_1, \ldots, G_n)\colon M \to \mathbb{R}^n$ is a submersion almost everywhere, i.e. the open set of points where G has maximal rank is dense.

Suppose (M, ω, H) is completely integrable, and suppose also that the map G is *proper* (i.e., pre-images of compact sets are compact). Then all the fibers of M are compact, but possibly disconnected. Replacing M with the pre-image of the set of regular values, we may assume that G is a submersion everywhere. Let $B = M/\sim$ be the quotient under the equivalence relation, where $m' \sim m$ if m, m' are in the same connected component of a fiber of G. This is a finite cover of $G(M)$, and so $\pi\colon M \to B$ is a Lagrangian submersion with compact, connected fibers.

Hence, we may introduce local action-angle coordinates. Since $\{G_j, H\} = 0$ for all j the Hamiltonian H is constant along the fibers of G. In other words, H is a function of the action variables I_i only. The Hamiltonian vector field becomes

$$X_H = \sum_j \frac{\partial H}{\partial I_j} \frac{\partial}{\partial s_j}.$$

The flow is therefore straightforward to compute. The result is:

Theorem 6.21 (Liouville-Arnold)**.** [3] *Let (M, ω, H) be a completely integrable Hamiltonian dynamical system, with integrals of motion G_j. Suppose G is a proper map. Let $M' \subseteq M$ be the subset on which G is a submersion, let B be the set of connected components of fibers of $G|_{M'}$, and $\pi : M' \to B$ the induced map. Then $\pi : M \to B$ is a Lagrangian fibration with compact connected fibers, hence it is an affine torus bundle. The flow of X_H is vertical and preserves the affine structure; in local action-angle coordinates it is given by*

$$\begin{aligned} I_j(t) &= I_j(0), \\ s_j(t) &= s_j(0) + t\,\frac{\partial H}{\partial I_j}. \end{aligned}$$

The flow on these *Liouville-Arnold tori* is quasi-periodic, i.e., is linear in local affine coordinates on the tori.

6.4. **The spherical pendulum.** As one of the simplest non-trivial examples of an integrable system let us briefly discuss the spherical pendulum. We first give the general description of the motion of a particle on a Riemannian manifold Q in a potential $V \in C^\infty(Q)$.

One defines the kinetic energy $T \in C^\infty(TQ)$ by

$$T(v) = \frac{1}{2}||v||^2.$$

Using the identification $g^\flat : TQ \to T^*Q$ given by the Riemannian metric, view T as a function on T^*Q. The Hamiltonian is the total energy

$$H = T + V \in C^\infty(T^*Q).$$

In local coordinates q_i on Q,

$$T(v) = \frac{1}{2}\sum_{ij} g(q)_{ij}\dot{q}_i\dot{q}_j,$$

where g_{ij} is the metric tensor. The relation between velocities and momenta in local coordinates is $p_i = \sum_j g_{ij}\dot{q}_j$. Thus

$$T(q,p) = \frac{1}{2}\sum_{ij} h(q)_{ij}\; p_i p_j$$

where $h(q)_{ij}$ is the inverse matrix to $g(q)_{ij}$, and

$$H(q,p) = \frac{1}{2}\sum_{ij} h(q)_{ij}\; p_i p_j + V(q).$$

Consider now the spherical pendulum. Its configuration space Q is the 2-sphere, which by an appropriate normalization we can take to be the unit sphere

$$Q = S^2 \subseteq \mathbb{R}^3.$$

Let $\phi \in [0, 2\pi], \psi \in (0, \pi)$ be polar coordinates on S^2, that is

$$x_1 = \sin\psi\cos\phi, \quad x_2 = \sin\psi\sin\phi, \quad x_3 = \cos\psi.$$

The potential energy is

$$V = \cos\psi$$

and the kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{3}\dot{x}_i^2 = \frac{1}{2}(\dot{\psi}^2 + \sin^2\psi\dot{\phi}^2).$$

Thus

$$H = \frac{1}{2}\big(p_\psi^2 + \frac{1}{\sin^2\psi}\, p_\phi^2\big) + \cos\psi.$$

(The apparent singularity at $\psi = 0, \pi$ comes only from the choice of coordinates.) An integral of motion for this system is given by the angular momentum

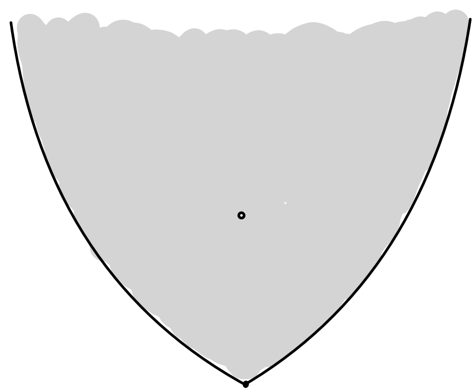
$$G = p_\phi.$$

Indeed,

$$\{H, G\} = 0$$

because H does not depend on ϕ (i.e. because the problem has rotational symmetry around the x_3-axis). Since $\dim T^*S^2 = 4$, it follows that the spherical pendulum is a completely integrable system.

The image of the map (G, H) has the form $H \geq f(G)$ where $u \mapsto f(u)$ is a symmetric function shaped roughly like a parabola, but non-smooth at $u = 0$.



The minimum of f is the point $(G, H) = (0, -1)$, corresponding to the stable equilibrium. The set of singular values of (G, H) consists of the boundary of the region, i.e. the range of the function f, together with the *unstable* equilibrium $(0, 1)$ (corresponding to the configuration where the pendulum is vertical).

Removing these singular points as well as the boundary from the image of (G, H), we obtain a non-simply connected region B, and one can raise the question about existence of global action-angle variables. Duistermaat shows that they do not exist in this system: The lattice bundle $\Lambda \to B$ is non-trivial, i.e. the monodromy obstruction does not vanish.

7. Symplectic group actions and moment maps

7.1. **Background on Lie groups.** Let us start with a rapid review of Lie groups. A Lie group is a group G with a manifold structure on G such that group multiplication is a smooth map. (This implies that inversion is a smooth map also.)

7.1.1. *Cartan's theorem.* A Lie subgroup $H \subseteq G$ is a subgroup which is also a submanifold. By a theorem of Cartan, every (topologically) closed subgroup of a Lie group is an (embedded) Lie subgroup (i.e, smoothness is automatic). In this case the homogeneous space G/H inherits a unique manifold structure such that the quotient map is smooth. A closed subgroup of the group $\mathrm{GL}(n, \mathbb{R})$ of invertible matrices (for some n) will be called a *matrix Lie group.*

7.1.2. *Lie algebra of a Lie group.* The Lie algebra $\mathfrak{g}$ of a Lie group G is defined as

$$\mathfrak{g} = T_e G,$$

with Lie bracket defined by the identification $T_e G \cong \mathfrak{X}^L(G)$ with the Lie algebra of left-invariant vector fields on G. Here $X \in \mathfrak{X}(G)$ is called left-invariant if it satisfies $(L_a)_* X = X$ under all left translations $L_a \colon G \to G,\ g \mapsto ag$; a left-invariant vector field is uniquely determined by its value $X(e) = \xi$ at the group unit. The construction is functorial: for a Lie group morphism $\phi\colon G_1 \to G_2$, the differential at the group unit gives a Lie algebra morphism $T_e\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. For matrix Lie groups, the Lie bracket coincides with the commutator of matrices.

7.1.3. *Exponential map.* Every $\xi \in \mathfrak{g}$ determines a unique Lie group morphism (*1-parameter group*)

$$\gamma_\xi \colon \mathbb{R} \to G, \quad t \mapsto \gamma_\xi(t)$$

with the property that $\frac{d}{dt}|_{t=0}\gamma_\xi(t) = \xi$. One defines the exponential map

$$\exp : \mathfrak{g} \to G$$

by $\exp(\xi) = \gamma_\xi(1)$. It restricts to a diffeomorphism between open neighborhoods of $0 \in \mathfrak{g}$ and $e \in G$. We have $\gamma_\xi(t) = \exp(t\xi)$.

The construction is functorial: For a Lie group morphism $\phi\colon G_1 \to G_2$, we have $\exp \circ T_e\phi = \phi \circ \exp$. For matrix Lie groups, exp is the usual exponential of matrices.

7.1.4. *Adjoint actions and coadjoint actions.* For $a \in G$ let

$$\mathrm{Ad}_a \colon G \to G,\ g \mapsto aga^{-1}.$$

This is a Lie group automorphisms of G, so it induces a Lie algebra automorphism $T_e \mathrm{Ad}_a$ of $\mathfrak{g} = T_e G$. For simplicity, this is again denoted by Ad_a:

$$\mathrm{Ad}_a \colon \mathfrak{g} \to \mathfrak{g},\ \xi \mapsto (T_e \mathrm{Ad}_a)(\xi).$$

This is a linear representation of G on $\mathfrak{g}$; its dual representation on $\mathfrak{g}^*$ is called the *coadjoint action* or *coadjoint representation*:

$$a \colon \mathfrak{g}^* \to \mathfrak{g}^*,\ \mu \mapsto a \cdot \mu = (\mathrm{Ad}_{a^{-1}})^*(\mu).$$

Some authors use the notation $\mathrm{Ad}^*_a = (\mathrm{Ad}_{a^{-1}})^*$, but this can get confusing and we shall avoid it.

Remark 7.1. If the Lie algebra has a nondegenerate Ad-invariant symmetric bilinear form B, used to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, then the adjoint and coadjoint actions are identified as well. For example, in the case of matrix Lie algebras $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R})$ with the property $X \in \mathfrak{g} \mapsto X^\top \in \mathfrak{g}$ (e.g., the Lie algebras of $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{SL}(n,\mathbb{R}),\ldots$) we can use $B(X,Y) = \mathrm{tr}(XY)$. This is nondgenerate due to the fact that $B(X,X^\top) = \mathrm{tr}(\mathrm{X}^\top\mathrm{X}) \geq 0$ with equality only if $X = 0$. For complex Lie algebras $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{C})$ with the property $X \in \mathfrak{g} \mapsto X^\dagger \in \mathfrak{g}$ (e.g., the Lie algebras of $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{SL}(n,\mathbb{C})$, $\ldots$) we may similarly use $B(X,Y) = \mathrm{Re}(\mathrm{tr}(XY))$. Observe however that the identification $\mathfrak{g} \cong \mathfrak{g}^*$ does depend on the choice of such B, in general. Furthermore, not all Lie algebras admit a nondegenerate Ad-invariant bilinear form.

From the Lie group representations one obtains Lie algebra representations,

$$\mathrm{ad}_\xi \colon \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}_\xi(\eta) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(t\xi)}(\eta).$$

It turns out that $\mathrm{ad}_\xi(\eta) = [\xi,\eta]$, which gives an alternative way of defining the Lie bracket on $\mathfrak{g}$. We also have the coadjoint representation on $\mathfrak{g}^*$, $\xi \cdot \mu = -(\mathrm{ad}_\xi)^*(\mu)$.

For matrix Lie groups, Ad_a is simply conjugation of matrices by a, while ad_ξ is commutator with ξ.

7.2. **Generating vector fields for group actions.**

Definition 7.2. Let G be a Lie group. An action of G on a manifold Q is a smooth map

$$\mathcal{A} : G \times Q \to Q,\ (g,q) \mapsto \mathcal{A}_g(q) = g \cdot q$$

such that the map $G \to \mathrm{Diff}(Q)$, $\; g \mapsto \mathcal{A}_g$ is a group morphism.

We refer to Q as a G-manifold. A map $F : Q_1 \to Q_2$ between two G-manifolds is called equivariant if it intertwines the G-actions, that is, $g.F(q_1) = F(g.q_1)$.

Example 7.3.
(a) There are three natural actions of any Lie group G on itself: The action by left multiplication, the action by right multiplication, and the adjoint action:

$$g.a = ga, \quad g.a = ag^{-1}, \quad g.a = gag^{-1}.$$

(b) A linear representation of G on a finite dimensional vector space V is a G-action on V viewed as a manifold.
(c) Given an action $g \mapsto \mathcal{A}_g$ of G on a manifold Q, one obtains actions on the tangent and cotangent bundles by *tangent lift* and *cotangent lift*,

$$g \mapsto T\mathcal{A}_g, \quad g \mapsto (T\mathcal{A}_{g^{-1}})^*.$$

(d) Given a closed subgroup $H \subseteq G$, any G-manifold Q becomes an H-manifold by restriction. Similarly, if a submanifold $P \subseteq Q$ is invariant under the G-action, then it becomes a G-manifold by restriction.

(e) Given a G-manifold Q, and a given point $q \in Q$, the stabilizer G_q is a closed subgroup of G, and hence is a Lie subgroup. Its action on TQ restricts to a linear action of G_q on T_qQ called the *isotropy representation*. For example, the adjoint action of G on G fixes e, and the corresponding isotropy representation is the adjoint representation on $\mathfrak{g}$. Dually, by restricting the action on T^*G we obtain the coadjoint action (representation) on $\mathfrak{g}^*$. The two actions are related by

$$\langle g \cdot \mu, \xi \rangle = \langle \mu, g^{-1} \cdot \xi \rangle, \quad \mu \in \mathfrak{g}^*, \ \xi \in \mathfrak{g}.$$

From now on, we will use the notation Ad_g for both the adjoint and the co-adjoint G-action: That is, $\langle \mathrm{Ad}_g \, \mu, \xi \rangle = \langle \mu, \mathrm{Ad}_{g^{-1}} \, \xi \rangle$ for $\mu \in \mathfrak{g}^*, \ \xi \in \mathfrak{g}$.

Definition 7.4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A *Lie algebra action* of $\mathfrak{g}$ on Q is a smooth vector bundle map

$$\mathfrak{g} \times Q \to TQ, \ (q, \xi) \mapsto \xi_Q(q)$$

such that the map $\mathfrak{g} \to \mathfrak{X}(Q), \ \xi \mapsto \xi_Q$ is a Lie algebra morphism.

Lie algebra action often arise by differentiating Lie group actions $G \times Q \to Q$. Informally, we think of $\mathfrak{X}(Q)$ as the Lie algebra of $\mathrm{Diff}(Q)$. If these were finite-dimensional Lie groups, we would just apply Lie differentiation to $G \to \mathrm{Diff}(Q)$ in to obtain a Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(Q)$. To make it rigorous, we think of vector fields as operators on functions. The action $\mathcal{A}_g$ gives a G-representation on $C^\infty(Q)$ by

$$(g \cdot f)(q) = f(g^{-1} \cdot q),$$

and the idea is to differentiate that:

Definition 7.5. Suppose Q is a G-manifold, with action map $g \mapsto \mathcal{A}_g$. The *generating vector fields*

$$\xi_Q \in \mathfrak{X}(Q), \quad \xi \in \mathfrak{g}$$

are defined in terms of the action on functions by

$$(\mathcal{L}_{\xi_Q} f)(q) = \frac{d}{dt}\Big|_{t=0} f(\exp(-t\xi) \cdot q).$$

In other words, ξ_Q is the vector field whose flow is given by $t \mapsto \mathcal{A}_{\exp(t\xi)}$.

Remark 7.6. Another formulation: The evaluation of ξ_Q at $q \in Q$ is the tangent vector represented by the curve $\exp(-t\xi).q$:

$$\xi_Q(q) = \frac{d}{dt}\Big|_{t=0} \exp(-t\xi).q. \tag{18}$$

Example 7.7. For $\xi \in \mathfrak{g}$ we denote by ξ^L the left-invariant vector field with $\xi^L|_e = \xi$. Similarly, define ξ^R to be the right-invariant vector field with $\xi^R|_e = \xi$. Note that

$$[\xi, \eta]^L := [\xi^L, \eta^L],$$

by definition of the Lie bracket.

The generating vector field for the left-action $g \cdot a = ga$ of G on itself is right-invariant (since the left-action commutes with the right-action), and its value at e is $-\xi$. Hence the left-action is generated by $-\xi^R$. Similarly the right-action $g \cdot a = ag^{-1}$ is generated by ξ^L, and the adjoint action $g \cdot a = gag^{-1}$ is generated by $\xi^L - \xi^R$.

If $F : Q_1 \to Q_2$ is a G equivariant map, then F intertwines the action of $\exp(t\xi)$ on the two manifolds, and hence relates the generating vector fields:

$$\xi_{Q_1} \sim_F \xi_{Q_2}.$$

Proposition 7.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any action of a Lie group G on a manifold Q the map*

$$\mathfrak{g} \to \mathfrak{X}(Q), \quad \xi \mapsto \xi_Q$$

is a Lie algebra action of $\mathfrak{g}$ on Q. In particular,

$$[\xi_Q, \eta_Q] = [\xi, \eta]_Q.$$

For $g \in G$ one has

$$(\mathcal{A}_g)_* \xi_Q = (\mathrm{Ad}_g\, \xi)_Q.$$

If G is simply connected, then a $\mathfrak{g}$-action on Q integrates to a G-action if and only if all the generating vector fields ξ_Q are complete. In particular, this is the case if Q is compact.

Example 7.9. Consider the action of $G = \mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ given by matrix multiplication. The Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ is the space of all real matrices, with exponential map the usual exponential map for matrices. For $A \in \mathrm{gl}(n, \mathbb{R})$, we calculate the generating vector field $A_{\mathbb{R}^n}$ of the action through the action on functions:

$$\begin{aligned}
(\mathcal{L}_{A_{\mathbb{R}^n}} f)(q) &= \frac{d}{dt}\Big|_{t=0} f(\exp(-tA)q) \\
&= -\sum_j (Aq)_j \frac{\partial f}{\partial q_j} \\
&= -\sum_{j,k} A_{jk} q_k \frac{\partial f}{\partial q_j},
\end{aligned}$$

that is,

$$A_{\mathbb{R}^n} = -\sum_{j,k} A_{jk} q_k \frac{\partial}{\partial q_j}. \tag{19}$$

You may verify directly that $[A_{\mathbb{R}^n}, B_{\mathbb{R}^n}] = [A, B]_{\mathbb{R}^n}$ (with the matrix commutator on the right hand side). Note that the generating vector fields in this example are exactly the *linear* vector fields on $\mathbb{R}^n$, i.e., the vector fields for which the coefficients are linear functions. Equivalently, this is the space of vector fields that are invariant under scalar multiplications $\gamma_t\colon \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto tx$ for $t \neq 0$.

Example 7.10. The additive group $G = \mathbb{R}^n$ has $\mathfrak{g} = \mathbb{R}^n$ (with zero bracket) as its Lie algebra, with exponential map the identity map $\exp(b) = b$. (This looks a bit funny, but remember that this is the exponential map for Lie groups, not for real numbers.) Let G act on $\mathbb{R}^n$ by translation:

$$b \cdot x = x + b$$

(the sign is just a convention). We calculate the generating vector fields $b_{\mathbb{R}^n}$ for $b \in \mathbb{R}^n$ as

$$(\mathcal{L}_{b_{\mathbb{R}^n}} f)(q) = \frac{d}{dt}\Big|_{t=0} f(\exp(-tb) \cdot q) == \frac{d}{dt}\Big|_{t=0} f(q - tb) = -\sum_j b_j \frac{\partial f}{\partial q_j}.$$

Thus

$$b_{\mathbb{R}^n} = -\sum_j b_j \frac{\partial}{\partial q_j}. \tag{20}$$

Example 7.11. For the basis-free version of these examples, consider a vector space V with the natural action of $\mathrm{GL}(V)$. For $v \in V$ we have $T_v V \cong V$; hence the generating vector fields may be regarded as V-valued functions on V. The calculation above shows that for $\xi \in \mathfrak{gl}(V) = \mathrm{End}(V)$, the generating vector field is

$$\xi_V|_v = -\xi \cdot v \tag{21}$$

(which is also immediate from (18)). This specializes to the generating vector fields for any G-representation on a vector space, for example the adjoint and coadjoint action of G on $\mathfrak{g}$ and $\mathfrak{g}^*$.

$$\xi_{\mathfrak{g}}|_\zeta = -\operatorname{ad}_\xi \zeta = -[\xi, \zeta], \quad \xi_{\mathfrak{g}^*}|_\mu = -\xi \cdot \mu = (\operatorname{ad}_\xi)^*(\mu).$$

The generating vector fields for the translation action $v \mapsto v - b$ of V on itself are given by $b_V|_v = b$.

7.3. **Hamiltonian group actions.** A G-action $\mathcal{A}\colon G \times M \to M$ on a symplectic manifold (M, ω) is called *symplectic* if $\mathcal{A}_g \in \mathrm{Diff}(M, \omega)$ for all g. It is thus described by a group morphism

$$G \to \mathrm{Diff}(M, \omega).$$

The generating vector fields of such an action give a Lie algebra action $\mathfrak{g} \to \mathfrak{X}(M, \omega)$. The G-action is called *weakly Hamiltonian* if its generating vector fields ξ_M are Hamiltonian vector fields. In other words, the infinitesimal action map takes values in the subalgebra $\mathfrak{X}_{\mathrm{Ham}}(M, \omega)$. For a weakly Hamiltonian action, one can choose a lift [13]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M) & \longrightarrow & \mathfrak{X}_{\mathrm{Ham}}(M,\omega) & \longrightarrow & 0 \\ & & & & & {\scriptstyle \Phi}\nwarrow & \uparrow & & \\ & & & & & & \mathfrak{g} & & \end{array}$$

of the infinitesimal action map to a linear map $\Phi\colon \mathfrak{g} \to C^\infty(M)$: Define on basis vectors and extend by linearity. Thus

$$\xi_M = X_{\Phi(\xi)}, \quad \xi \in \mathfrak{g}.$$

The linear map $\xi \mapsto \Phi(\xi)$ can also be viewed as a function $\Phi \in C^\infty(M) \otimes \mathfrak{g}^*$, i.e. as a map $\Phi\colon M \to \mathfrak{g}^*$. The latter version is called a *moment map*. The original version $\Phi\colon \mathfrak{g} \to C^\infty(M)$ is often called a *comoment map*. (Thus, $\Phi(\xi) = \langle \Phi, \xi \rangle$. It is essentially 'the same' map.)

Definition 7.12. [43] A symplectic G-action on a symplectic manifold (M, ω) is called *weakly Hamiltonian* if there exists a smooth *moment map*

$$\Phi\colon M \to \mathfrak{g}^*$$

satisfying

$$\mathsf{d}\langle \Phi, \xi \rangle = -\iota(\xi_M)\omega, \quad \xi \in \mathfrak{g}.$$

It is called *Hamiltonian* if the map Φ is G-equivariant.

Thus, for a Hamiltonian action we have the equivariance property

$$\Phi(g \cdot m) = g \cdot \Phi(m).$$

Equivalently,

$$\langle \Phi(g \cdot m), \mathrm{Ad}_g\, \xi \rangle = \langle \Phi(m), \xi \rangle$$

for all $\xi \in \mathfrak{g}$, $m \in M$.

Remarks 7.13. (a) Hamiltonian G-spaces where introduced around 1970 by Kostant [24], Smale [42], and Souriau [43] (independently). The terminology *moment* in the context of Hamiltonian actions was introduced in Souriau's book [43], and has since been called *application moment* in the French literature. The term 'moment map' is a mistranslation, which has become more or less standard. A better translation is *momentum map* (also widely used) since it relates to the physics concepts of angular and linear momentum (as opposed to 'moment of inertia' and such things). A similar mistranslation is common in the German literature (Momentenabbildung, as opposed to the correct Impulsabbildung).

[13] In this diagram we assume M is connected; otherwise $\mathbb{R}$ should be replaced with the space $H^0(M)$ of locally constant functions

(b) To repeat: The coadjoint action of G on $\mathfrak{g}^*$ is defined by

$$\langle g\cdot\mu,\ \zeta\rangle = \langle \mu, \mathrm{Ad}_{g^{-1}}\zeta\rangle$$

for $\mu \in \mathfrak{g}^*,\ \zeta \in \mathfrak{g}$. Infinitesimally, the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ is defined by

$$\langle \xi\cdot\mu, \zeta\rangle = -\langle \mu, [\xi,\zeta]\rangle.$$

(c) If a Lie group G acts on M in a Hamiltonian way, and if $H \to G$ is a Lie group morphism (e.g. inclusion of a subgroup) then the action of H is Hamiltonian; the moment map is the composition of the G-moment map with the dual map $\mathfrak{g}^* \to \mathfrak{h}^*$.

(d) Writing $g = \exp(-t\xi)$ and taking the derivative at $t = 0$ we find the infinitesimal version of the equivariance condition

$$\mathcal{L}_{\xi_M}\Phi = -\xi\cdot\Phi,\quad \xi\in\mathfrak{g}.$$

If G is connected, the G-equivariance of $\Phi\colon M \to \mathfrak{g}^*$ is equivalent to its infinitesimal counterpart.

(e) For an abelian group, the coadjoint action is trivial, so that equivariance simply means invariance.

The following fact uses the equivariance of the moment map.

Lemma 7.14. *Suppose $\mathcal{A}\colon G \to \mathrm{Diff}(M,\omega)$ is a weakly Hamiltonian action. If the action is Hamiltonian, then the comoment map*

$$\mathfrak{g} \to C^\infty(M),\ \xi \mapsto \langle \Phi, \xi\rangle$$

is a Lie algebra morphism with respect to the Poisson bracket on $C^\infty(M)$. If G is connected, the converse is true.

Proof. Write $\Phi^\xi = \langle \Phi, \xi\rangle$. If the action is Hamiltonian, we have:

$$\{\Phi^\xi, \Phi^\eta\} = \mathcal{L}_{X_{\Phi^\xi}}(\Phi^\eta) = \mathcal{L}_{\xi_M}\Phi^\eta = -\langle \xi\cdot\Phi, \eta\rangle = \Phi^{[\xi,\eta]}.$$

Conversely, if $\xi \mapsto \Phi^\xi$ is a Lie algebra morphism, a similar calculation gives the $\mathfrak{g}$-equivariance condition $\mathcal{L}_{\xi_M}\Phi = -\xi\cdot\Phi$. This implies, by integration, for all t,

$$\mathcal{A}^*_{\exp(-t\xi)}(\exp(-t\xi)\cdot\Phi) = \Phi$$

as one verifies by taking a t-derivative. Hence, the equivariance property $\mathcal{A}^*_{g^{-1}}(g^{-1}\cdot\Phi) = \Phi$ holds for 'small' g, and hence for all g. (We are using that a connected Lie group is generated by any open neighborhood of the group unit.) ∎

Proposition 7.15. *A weakly Hamiltonian action of a Lie group G on a connected symplectic manifold (M, ω) is necessarily Hamiltonian, in each of the following cases:*

(a) *G is compact,*
(b) *M is compact,*
(c) *the G-action on M has a fixed point.*

That is, in these cases one can always take the (weak) moment map to be G-equivariant.

Proof. Let $\mathcal{E}$ be the set of all weak moment maps $\Phi\colon M \to \mathfrak{g}^*$. Since any two weak moment maps differ by addition of a constant $\mathfrak{g}^*$-valued functions, the set $\mathcal{E}$ is an affine space under the action of $\mathfrak{g}^*$. We have an (affine) action of G on the affine space $\mathcal{E}$, by

$$(g \cdot \Phi)(m) = g \cdot \Phi(g^{-1} \cdot m).$$

To check that $g \cdot \Phi$ is again a weak moment map for the G-action, we calculate

$$\begin{aligned} \mathsf{d}\langle g \cdot \Phi, \xi\rangle &= (\mathcal{A}_{g^{-1}})^* \mathsf{d}\langle \Phi, \mathrm{Ad}_g(\xi)\rangle \\ &= -(\mathcal{A}_{g^{-1}})^* (\mathcal{A}_g)^* \iota(\xi_M)\omega \\ &= -\iota(\xi_M)\omega. \end{aligned}$$

Note that Φ is G-equivariant if and only if $g \cdot \Phi = \Phi$; that is, the (equivariant) moment maps are the G-fixed points of the action on $\mathcal{E}$.

Consider now the three settings. (a) If G is compact, then an affine G-action on a finite-dimensional affine space has a fixed point, obtained by averaging. Concretely, given any $\Phi \in \mathcal{E}$, an equivariant moment map is obtained as

$$\overline{\Phi}(m) = \int_G (g \cdot \Phi)(m) \; |\mathsf{d}g|$$

where $|\mathsf{d}g|$ is the normalized bi-invariant measure on G. Indeed, one checks that $h \cdot \overline{\Phi}(m) = \overline{\Phi}(m)$. (b) If If M is compact, we may normalize $\Phi \in \mathcal{E}$ by the condition

$$\int_M \Phi \omega^n = 0.$$

This property determines Φ uniquely. Since $g \cdot \Phi$ is aain normalized, we must have $g \cdot \Phi = \Phi$. (c) If the action has a G-fixed point $m_0 \in M$, we can normalize Φ by requiring $\Phi(m_0) = 0$; since $g \cdot \Phi$ also vanishes at m_0 it follows that $g \cdot \Phi = \Phi$. □

Remark 7.16. Suppose M is connected, and let $\mathfrak{g} \to \mathfrak{X}_{\mathrm{Ham}}(M, \omega)$ be a weakly Hamiltonian action. Let $\widehat{\mathfrak{g}} \subseteq \mathfrak{g} \times C^\infty(M)$ be the set of pairs (ξ, f) such that $\xi_Q = X_f$. This is a Lie subalgebra, and defines a central extension

$$0 \to \mathbb{R} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0 \to 0.$$

The map $\widehat{\mathfrak{g}} \to C^\infty(M)$ is, by construction, a Lie algebra morphism. Hence, if G is connected, then a G-equivariant moment map is equivalent to a Lie algebra splitting of the central extension, $\mathfrak{g} \to \widehat{\mathfrak{g}}$.

The central extensions of a Lie algebra $\mathfrak{g}$ by $\mathbb{R}$ are classified by the Lie algebra cohomology $H^2(\mathfrak{g}, \mathbb{R})$. Hence, if $H^2(\mathfrak{g}, \mathbb{R}) = 0$ we can always choose a splitting. This is the case, for example, for all semisimple Lie algebras. On the other hand, abelian Lie algebras $\mathfrak{g}$ of dimension at least 2 admit nontrivial central extensions. In fact, every skew-symmetric bilinear form $\phi\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defines a central extension.

From now on, we will always assume that the moment map is equivariant unless stated otherwise.

7.4. **Examples of Hamiltonian actions and moment maps.**

7.4.1. *Linear momentum and angular momentum.* Recall from Proposition 3.24 that for any vector field Y on a manifold Q, the cotangent lift $Y_{T^*} \in \mathfrak{X}(T^*Q)$ is a Hamiltonian vector field $Y_{T^*} = X_H$, for the Hamiltonian function $H = -\iota_{Y_{T^*}}\theta$. In local coordinates $q_1, \ldots, q_n$ on Q and corresponding coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ on T^*Q, the Hamiltonian for the cotangent lift of $Y = \sum_j Y_j(q) \frac{\partial}{\partial q_j}$ is

$$H(q, p) = -\sum_j Y_j(q) p_j.$$

Exercise 7.17. Show that the linear map

$$\mathfrak{X}(Q) \to C^\infty(T^*Q),\ Y \mapsto -\iota_{Y_{T^*}}\theta$$

is a Lie algebra morphism (using the Poisson bracket on T^*Q), which furthermore intertwines the action of $\mathrm{Diff}(Q)$ on both sides.[14]

Hence if Q is a G-manifold, we obtain a comoment map for the cotangent lift of the G-action to T^*Q by composing this map with the generating vector fields:

$$\mathfrak{g} \mapsto \mathfrak{X}(Q) \to C^\infty(T^*Q)$$

Let us specialize to $Q = \mathbb{R}^n$, thus $M = T^*(\mathbb{R}^n) = \mathbb{R}^{2n}$ with standard symplectic coordinates q_j, p_j. Let $G = \mathbb{R}^n$ act on itself by translation

$$b \cdot x = x + b.$$

In (19) we had calculated the generating vector fields as $b_{\mathbb{R}^n} = -\sum_j b_j \frac{\partial}{\partial q_j}$. Hence, the moment map for the cotangent lift is

$$\langle \Phi, b \rangle = \sum_j b_j p_j.$$

Using the standard inner product on $\mathbb{R}^n$ to identify $(\mathbb{R}^n) \cong (\mathbb{R}^n)^*$, this means

$$\Phi(q, p) = p.$$

[14]One might say that the latter claim is 'clear' since the construction is coordinate-free.

That is, the moment map is just *linear momentum.*

Consider similarly the cotangent lift of the action of $G = \mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$. We had found $A_{\mathbb{R}^n} = -\sum_{j,k} A_{jk} q_k \frac{\partial}{\partial q_j}$. Hence,

$$\langle \Phi, A\rangle = \sum_{j,k} A_{jk} p_j q_k.$$

Using the non-degenerate bilinear form $(A, A') = \mathrm{tr}(AA')$ on $\mathfrak{gl}(n,\mathbb{R})$ to identify $\mathfrak{gl}(n,\mathbb{R}) \cong \mathfrak{gl}(n,\mathbb{R})^*$, this means

$$\Phi(q,p)_{ij} = q_i p_j.$$

Let us restrict the action to the orthogonal group $\mathrm{O}(n)$. The Lie algebra of $\mathfrak{o}(n)$ consists of skew-symmetric matrices, $A = -A^\top$. Using again the trace form to identify the Lie algebra and its dual, the moment map $\Psi : T^*\mathbb{R}^n \to \mathfrak{o}(n)^*$ for the action of $\mathrm{O}(n)$ reads,

$$\Psi(q,p)_{ij} = \frac{1}{2}(q_i p_j - q_j p_i).$$

For $n = 3$, we can further identify $\mathrm{so}(3)^* \cong \mathbb{R}^3$ with the standard rotation action of $\mathrm{SO}(3)$, and Ψ just becomes just *angular momentum*

$$\vec{q} \times \vec{p} = \begin{pmatrix} q_2 p_3 - q_3 p_2 \\ q_3 p_1 - q_1 p_3 \\ q_1 p_2 - q_2 p_1 \end{pmatrix}$$

(up to an irrelevant factor, which again just depends on the chosen identification $\mathfrak{o}(3) \cong \mathfrak{o}(3)^*$).

7.4.2. *Exact symplectic manifolds.* The cotangent bundle example generalizes to *exact* symplectic manifolds. A symplectic manifold (M,ω) is called exact if $\omega = -\mathsf{d}\theta$ for a 1-form θ (which is sometimes called a symplectic potential). Note that a compact symplectic manifold is never exact (unless it is 0-dimensional): Indeed, if $\omega = -\mathsf{d}\theta$ is exact, then also the Liouville form $\omega^n = -\mathsf{d}\theta \wedge \omega^{n-1}$ is exact. Hence if M were compact, Stokes' theorem would show that M has zero volume, a contradiction.

Proposition 7.18. *Suppose (M,ω) is an exact symplectic manifold, $\omega = -\mathsf{d}\theta$. Then every G-action on M preserving θ is Hamiltonian, with moment map*

$$\langle \Phi, \xi\rangle = -\iota(\xi_M)\theta.$$

Proof. We calculate

$$-\mathsf{d}\iota(\xi_M)\theta = \iota(\xi_M)\mathsf{d}\theta - \underbrace{L(\xi_M)\theta}_{=0} = -\iota(\xi_M)\omega.$$

The resulting moment map is equivariant:

$$\mathcal{A}_g^*\langle \Phi, \xi\rangle = \mathcal{A}_g^*(\iota(\xi_M)\theta) = \iota(\mathcal{A}_g^*(\xi_M))\mathcal{A}_g^*\theta = \iota(\mathcal{A}_g^*(\xi_M))\theta = \iota((\mathrm{Ad}_g\,\xi)_M)\theta = \langle \Phi, \mathrm{Ad}_g\,\xi\rangle.$$

□

Remark 7.19. Note that if ω is exact, and G is compact, one can construct a G invariant θ by averaging. If $H^1(M,\mathbb{R})=0$ then Φ is independent of the choice of invariant θ.

The examples considered above were of the form $M=T^*Q$, with G acting by the cotangent lift of a G-action on Q. Another example is provided by the defining action of $\mathrm{U}(n)$ on $\mathbb{C}^n=\mathbb{R}^{2n}$, discussed below.

7.4.3. *Symplectic representations.* Generalizing the case of unitary representations, consider *symplectic representation* of G on a symplectic vector space (E,ω). That is, G acts by a Lie group morphism $G\to \mathrm{Sp}(E)$ into the symplectic group.

Proposition 7.20. *The action of $G=\mathrm{Sp}(E)$ on E is Hamiltonian, with moment map given by the formula,*

$$\langle \Phi(v),\xi\rangle = -\frac{1}{2}\omega(v,\xi.v).$$

Proof. if we identify $T_vE=E$ the generating vector field for $\xi\in\mathfrak{g}$ is just $\xi_E(v)=-\xi.v$. Thus for $w\in E$ we have

$$\omega(\xi_E(v),w)=-\omega(\xi.v,w)=\omega(v,\xi.w).$$

On the other hand, the map $\Phi^\xi=\langle\Phi,\xi\rangle$ defined above satisfies

$$\begin{aligned}
\mathsf{d}\Phi^\xi|_v(w) &= \frac{d}{dt}\Big|_{t=0}\Phi^\xi(v+tw)\\
&= -\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\omega(v+tw,\xi.(v+tw))\\
&= -\frac{1}{2}(\omega(w,\xi.v)+\omega(v,\xi.w))\\
&= \omega(\xi.v,w)\\
&= -\omega(\xi_E(v),w)
\end{aligned}$$

verifying that Φ is a moment map. Note that this is the unique moment map vanishing at 0. ☐

Remark 7.21. If desired, we may also express this in terms of the nondegenerate symmetric bilinear form $\mathrm{tr}(\xi\eta)$ on $\mathrm{sp}(E)$. Regard elements $v\in E$ as linear maps $v\colon\mathbb{R}\to E$, and let $v^*\colon E^*\to\mathbb{R}$ be the dual map. The map $\omega^\flat\colon E\to E^*$ is skew-adjoint. We have,

$$\omega(v,\xi v)=\langle\omega^\flat(v),\xi v\rangle=(\omega^\flat\circ v)^*\xi v=-v^*\omega^\flat\xi v=-\mathrm{Tr}(vv^*\omega^\flat\xi)=-\langle vv^*\omega^\flat,\xi\rangle.$$

This shows $\langle\Phi(v),\xi\rangle=\frac{1}{2}\mathrm{Tr}(vv^*\omega^\flat\xi)$. We read off that

$$\Phi(v)=\frac{1}{2}vv^*\omega^\flat.$$

7.4.4. *Unitary representations.* Consider now the case of a *complex* inner product space E, with corresponding unitary group $\mathrm{U}(E)$. Denote by $h\colon E\times E\to\mathbb{C}$ the corresponding Hermitian metric (linear in the second entry, conjugate linear in the first entry.)

In Section 2.5 we explained how this data is equivalent to symplectic structure ω and a compatible complex structure J, through the equation

$$h(v,w)=\omega(v,Jw)+\sqrt{-1}\omega(v,w).$$

Furthermore, $U(E)\subseteq \mathrm{Sp}(E,\omega)$. Hence, by the previous section we know that the $\mathrm{U}(E)$-action is Hamiltonian, with moment map $\langle\Phi(v),\xi\rangle=-\frac{1}{2}\omega(v,\xi.v)$. We may express this in terms of h, using that for $\xi\in\mathfrak{u}(E)$, the expression $h(v,\xi.v)$ is purely imaginary.

$$h(v,\xi.v)=-h(\xi.v,v)=-\overline{h(v,\xi.v)}$$

So, $h(v,\xi.v)=\sqrt{-1}\omega(v,\xi.v)$ and hence

$$\langle\Phi(v),\xi)=\frac{\sqrt{-1}}{2}h(v,\xi.v)$$

For any linear map $A\colon E\to F$ between complex inner product spaces, let $A^\dagger\colon F\to E$ be the adjoint map. Then $\mathrm{U}(E)$ consists of maps $A\colon E\to E$ such that $A^\dagger=A^{-1}$, and its Lie algebra $\mathfrak{u}(E)$ consists of maps ξ such that $\xi^\dagger=-\xi$. Use the $U(E)$-invariant positive define inner product on $\mathrm{u}(E)$,

$$\mathrm{Tr}(\xi^\dagger\eta)=-\mathrm{Tr}(\xi\eta)$$

to identify $\mathfrak{u}(E)^*\cong\mathfrak{u}(E)$. We may regard the 'column vector' $v\in E$ as a linear map $\mathbb{C}\to E$, hence the corresponding 'row vector' $v^\dagger$ is a linear map $E\to\mathbb{C}$. We may thus write [15]

$$\langle\Phi(v),\xi\rangle=\frac{\sqrt{-1}}{2}h(v,\xi.v)=\frac{\sqrt{-1}}{2}v^\dagger\xi v=\frac{\sqrt{-1}}{2}\mathrm{Tr}(vv^\dagger\xi).$$

The endomorphism $vv^\dagger$ of E is self-adjoint, hence $\frac{\sqrt{-1}}{2}vv^\dagger$ is skew-adjoint and so defines an element of $\mathfrak{u}(E)$. We arrive at the following formula:

Proposition 7.22. *The $\mathrm{U}(E)$-action on a complex inner product space E is Hamiltonian, with moment map given by*

$$\Phi(v)=\frac{1}{2\sqrt{-1}}vv^\dagger,$$

using the identification $\mathfrak{u}(E)^*\cong\mathfrak{u}(E)$ given by the inner product $\langle\xi,\eta\rangle=\mathrm{Tr}(\xi^\dagger\eta)$ on $\mathfrak{u}(E)$.

We may express these results also in terms of a complex orthonormal basis, identifying $E=\mathbb{C}^n=\mathbb{R}^{2n}$ with complex coordinates $z_j=q_j+\sqrt{-1}p_j$. These define complex

[15] We used that $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$ for linear maps $A\colon E\to F,\ B\colon F\to E$.

differentials

$$\mathsf{d}z_i = \mathsf{d}q_i + \sqrt{-1}\mathsf{d}p_i, \quad \mathsf{d}\overline{z_i} = \mathsf{d}q_i - \sqrt{-1}\mathsf{d}p_i$$

and dual complex-valued vector fields

$$\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial q_i} - \sqrt{-1}\frac{\partial}{\partial p_i}), \quad \frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}(\frac{\partial}{\partial q_i} + \sqrt{-1}\frac{\partial}{\partial p_i}).$$

The map $A \mapsto A^\dagger$ is the conjugate transpose. The Hermitian inner product is $h(v,w) = \sum_i \overline{v}_i w_i = v^\dagger w$. The symplectic form reads as

$$\omega = \sum_j \mathsf{d}q_j \wedge \mathsf{d}p_j = \frac{\sqrt{-1}}{2}\sum_j \mathsf{d}z_j \wedge \mathsf{d}\overline{z_j} = \frac{1}{2\sqrt{-1}} h(\mathsf{d}z, \mathsf{d}z),$$

The Lie algebra $\mathfrak{u}(n)$ of $\mathrm{U}(n)$ consists of skew-Hermitian matrices ξ, i..e, $\xi^\dagger = -\xi$ The generating vector fields are

$$\xi_{\mathbb{C}^n} = -\sum_{j,k}\big(\xi_{jk}\, z_k \frac{\partial}{\partial z_j} - \xi_{kj}\, \overline{z_k}\frac{\partial}{\partial \overline{z_j}}\big)$$

The moment map $\Phi(z) = \frac{1}{2\sqrt{-1}} z z^\dagger$ (using the positive define inner product on u(n), $-\operatorname{Tr}(\xi\eta)$ to identify the Lie algebra and its dual, we arrive at

$$\Phi(z)_{ij} = \frac{1}{2\sqrt{-1}} z_i \bar{z}_j$$

More generally, any finite dimensional unitary representation $G \to \mathrm{U}(n)$ defines a Hamiltonian action of G on $\mathbb{C}^n$; the moment map is the composition of the $\mathrm{U}(n)$ moment map with the projection $\mathfrak{u}(n)^* \to \mathfrak{g}^*$ dual to $\mathfrak{g} \to \mathfrak{u}(n)$. For example, the moment map for the diagonal $\mathrm{U}(1) \subseteq \mathrm{U}(n)$-action is given by $\frac{1}{2\sqrt{-1}}||z||^2$. (It looks complex valued since we think of $\mathfrak{u}(1)$ as $\sqrt{-1}\mathbb{R}$.)

Exercise 7.23. Show that $\omega = -\mathsf{d}\theta$, where

$$\theta = \frac{\sqrt{-1}}{4}\sum_j \big(\bar{z}_j \mathsf{d}z_j - z_j \mathsf{d}\bar{z}_j\big) = -\frac{1}{2}\operatorname{Im}(h(z, \mathsf{d}z)),$$

is preserved under the unitary group.

7.4.5. *Projective Representations.* The action of $\mathrm{U}(n+1)$ on $\mathbb{C}^{n+1}$ induces an action on $\mathbb{C}P(n)$ which is again Hamiltonian (as follows already from the fact that $\mathrm{U}(n+1)$ is compact). In homogeneous coordinates $[z_0 : \ldots : z_n]$, the moment map is

$$\Phi([z_0 : \ldots : z_n]) = \frac{1}{2\sqrt{-1}||z||^2} z z^\dagger \tag{22}$$

(note that this is well-defined). We will verify this fact later in the context of symplectic reduction.

7.4.6. *Constructions.* There are a number of simply constructions, obtaining new examples of Hamiltonian spaces from old.

(a) **Products.** Let $(M_i, \omega_i), i = 1, 2$ be Hamiltonian G-spaces, with moment maps $\Phi_i\colon M_i \to \mathfrak{g}^*$. Then the product $M_1 \times M_2$ with the diagonal G-action is a Hamiltonian G-space, with moment map the pointwise sum

$$\Phi = \Phi_1 \circ \mathrm{pr}_1 + \Phi_2 \circ \mathrm{pr}_2 .$$

(Here $\mathrm{pr}_i\colon M_1 \times M_2 \to M_i$ are the two projections.) Direct products are a classical analog to tensor products of G-representations.

(b) **Conjugates.** Let (M, ω) be a Hamiltonian G-space, with moment map Φ. Then $(M, -\omega)$ (with the same G-action) is a Hamiltonian G-space, with moment map $-\Phi$. This is known as the conjugate (sometimes denoted M^* or M^-) since it is a classical analog of dual representation of a Lie group.

(c) **Restriction to subgroups.** Let (M, ω) be a Hamiltonian G-space with moment map $\Phi\colon M \to \mathfrak{g}^*$, and $H \subseteq G$ a subgroup. Then the action of $H \subseteq G$ on (M, ω) has moment map

$$\Phi_H = p \circ \Phi\colon M \to \mathfrak{h}^*$$

where $p\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is the projection dual to the inclusion $j\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$. (We leave details as an exercise.) More generally, for every group homomorphism $H \to G$, the space (M, ω) becomes a Hamiltonian H-space with moment map Φ_H as above, where $p\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is the dual map to the infinitesimal map $j\colon \mathfrak{h} \to \mathfrak{g}$.

Exercise 7.24. Let $G \to G \times G$ be the diagonal embedding, and $j\colon \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ its differential. Show that the dual map $p\colon \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ is addition. Use this to explain how item (a) above may be seen as a special case of (c).

7.5. **Coadjoint Orbits.** Recall that a *homogeneous G-space* is a manifold M with a transitive G-action. In this case, any choice of a base point $m_0 \in M$ identifies $M = G/H$ where $H = G_{m_0}$ is the stabilizer. For a homogeneous space, the tangent spaces to M are spanned by generating vector fields.

Consider now a homogeneous Hamiltonian G-space (M, ω, Φ). (Example: $M = \mathbb{C}P(n)$ with the natural action of $U(n+1)$.) By the moment map condition, $\iota(\xi_M)\omega = -\mathsf{d}\langle \Phi, \xi\rangle$, the symplectic structure of a homogeneous Hamiltonian G-space is completely determined by the moment map $\Phi\colon M \to \mathfrak{g}^*$. In fact, we have

$$\omega(\xi_M, \eta_M) = \iota(\eta_M)\iota(\xi_M)\omega = -\iota(\eta_M)\mathsf{d}\Phi, \xi\rangle = -\langle \mathcal{L}(\eta_M)\Phi, \xi\rangle = -\langle \Phi,\ \eta.\xi\rangle = \langle \Phi, [\xi, \eta]\rangle.$$

We obtain the formula

$$\omega(\xi_M, \eta_M) = \langle \Phi, [\xi, \eta]\rangle.$$

The first examples of homogeneous Hamiltonian G-spaces are the orbits of the coadjoint action.

Theorem 7.25 (Kirillov-Kostant-Souriau)**.** [23, 24, 43] *Let*

$$\mathcal{O} \subseteq \mathfrak{g}^*$$

be an orbit for the coadjoint action of G on $\mathfrak{g}^*$. There exists a unique invariant symplectic structure on $\mathcal{O}$ such that the action is Hamiltonian, with moment map given by the inclusion $\Phi : \mathcal{O} \hookrightarrow \mathfrak{g}^*$.

Proof. For fixed $\mu \in \mathcal{O}$, the generating vector fields are given in terms of the coadjoint action by $\xi_{\mathcal{O}}(\mu) = -\xi \cdot \mu$. In particular, the stabilizer algebra $\mathfrak{g}_\mu$ consists of all ξ such that $\xi \cdot \mu = 0$. By the discussion above, the only candidate for ω is given by the formula

$$\omega(\xi_{\mathcal{O}}, \eta_{\mathcal{O}})|_\mu = \langle \mu, [\xi, \eta] \rangle. \tag{23}$$

The right hand side may also be written as

$$\langle \mu, [\xi, \eta] \rangle = -\langle \xi \cdot \mu, \eta \rangle = \langle \eta.\mu, \xi \rangle,$$

which shows that the right hand side only depends on $\xi_{\mathcal{O}}|_\mu$, $\eta_{\mathcal{O}}|_\mu$. Forthermore, $\xi_{\mathcal{O}}|_\mu$ lies in the kernel of $\omega_{\mathcal{O}}|_\mu$ if and only if (23) vanishes for all η, if and only if $\langle \xi \cdot \mu, \eta \rangle$ vanishes for all η, if and only if $\xi \cdot \mu = 0$, if and only if $\xi_{\mathcal{O}}|_\mu = 0$. This shows that ω is nondegenerate. The calculation

$$\mathcal{A}_g^*(\omega(\xi_{\mathcal{O}}, \eta_{\mathcal{O}})) = \langle \mathcal{A}_g^*\Phi, [\xi, \eta] \rangle = \langle \Phi, [\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta] \rangle = \omega(\mathcal{A}_g^*\xi_{\mathcal{O}}, \mathcal{A}_g^*\eta_{\mathcal{O}})$$

shows that the resulting 2-form ω on $\mathcal{O}$ is G-invariant, and the moment map condition holds by construction. To check $\mathsf{d}\omega = 0$, we compute:

$$\iota(\xi_{\mathcal{O}})\mathsf{d}\omega = L(\xi_{\mathcal{O}})\omega - \mathsf{d}\iota(\xi_{\mathcal{O}})\omega = 0.$$

As remarked above, the moment map uniquely determines the symplectic form. ☐

Example 7.26. Let $G = \mathrm{SO}(3)$. Identify the Lie algebra $\mathfrak{so}(3)$ with $\mathbb{R}^3$, by identifying the standard basis vectors of $\mathfrak{so}(3)$ as follows:

$$e_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} e_2 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} e_3 \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This identification takes the adjoint action of SO(3) to the standard rotation action on $\mathbb{R}^3$, and takes the invariant inner product $(A, B) \mapsto -\frac{1}{2}\operatorname{tr}(AB)$ on $\mathfrak{so}(3)$ to the standard inner product on $\mathbb{R}^3$. The inner product also identifies $\mathfrak{so}(3)^* \cong \mathbb{R}^3$. The coadjoint orbits for SO(3) are the 2-spheres around 0, together with the origin $\{0\}$.

Example 7.27. Let $G = \mathrm{U}(n)$, so that $\mathfrak{u}(n)$ are the skew-adjoint matrices. Use the inner product $\mathrm{Tr}(\xi^\dagger \eta)$ to identify $\mathfrak{u}(n)^* \cong \mathfrak{u}(n)$, and hence identify coadjoint orbits with adjoint orbits. Now, the skew-adjoint matrices are those of the form $\xi = \sqrt{-1}A$ where

A is self-adjoint, and the adjoint orbits correspond to matrices with a prescribed set of eigenvalues

$$(\sqrt{-1}\lambda_1, \ldots, \sqrt{-1}\lambda_n)$$

where $\lambda_1 \geq \cdots \geq \lambda_n$ are real numbers. (By linear algebra, two self-adjoint matrices are $\mathrm{U}(n)$-conjugate if and only they have the same set of eigenvalues.) Hence, every such n-tuple

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \cdots \geq \lambda_n$$

determines a unique coadjoint orbit. We have $\mathcal{O} = G/H$ where $H \subseteq \mathrm{U}(n)$ is the stabilizer of the diagonal matrix

$$\sqrt{-1}\begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & \cdots \\ 0 & \lambda_2 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & \lambda_n \end{pmatrix}.$$

This stabilizer is of the form $H = U(k_1) \times \cdots \times U(k_r)$ depending on coincidences among the eigenvalues. For example, if $n = 5$ and $\lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 = \lambda_5$ then $H = \mathrm{U}(2) \times \mathrm{U}(1) \times \mathrm{U}(2)$.

Note that for $1 \leq k \leq n$, if $\lambda_1 = \ldots = \lambda_k > \lambda_{k+1} = \ldots = \lambda_n$ the coadjoint orbit is the *Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$*

$$\mathrm{Gr}_{\mathbb{C}}(k, n) = \mathrm{U}(n)/(\mathrm{U}(k) \times \mathrm{U}(n-k)). \tag{24}$$

Indeed, the action of $\mathrm{U}(n)$ on $\mathbb{C}^n$ induces a transitive action on the set of k-dimensional subspaces, and the stabilizer of $\mathbb{C}^k \oplus 0 \subseteq \mathbb{C}^n$ is $H = \mathrm{U}(k) \times \mathrm{U}(n-k)$, thought of as matrices in block diagonal form.

Example 7.28. The discussion is similar for $\mathrm{SU}(n)$, except that now the set of eigenvalues has the additional condition $\lambda_1 + \ldots + \lambda_n = 0$. Using this to eleminate λ_n, we get the conditions

$$\lambda_1 \geq \cdots \lambda_{n-1} \geq -(\lambda_1 + \ldots + \lambda_{n-1}).$$

Observe that the Grassmannian may alse be regarded as a coadjoint orbit G/H of $G = \mathrm{SU}(n)$, with $H = \mathrm{SU}(n) \cap (\mathrm{U}(k) \times \mathrm{U}(n-k))$.

Theorem 7.29 (Kostant, Souriau)**.** [24, 43] *Let G be a Lie group, and (M, ω, Φ) is a homogeneous Hamiltonian G-space. Then M is a covering space of a coadjoint orbit, with 2-form obtained by pull-back of the KKS form on $\mathcal{O}$.*

Proof. Since the action on M is transitive, the same is true for the action on $\Phi(M)$. Hence $\mathcal{O} = \Phi(M)$ is a single coadjoint orbit. Let

$$\pi \colon M \to \mathcal{O}$$

be the map Φ, regarded as a map to its image. By by G-equivariance, the range of $T_m\pi$ contains all orbit directions, and so is all of $T_{\pi(m)}\mathcal{O}$. Thus π is a submersion. Let $\omega_{\mathcal{O}}$ be the symplectic structure on $\mathcal{O}$. We had already seen that the 2-form on M is determined by the moment map condition, and the formula shows that

$$\omega = \pi^* \omega_{\mathcal{O}}.$$

Hence $T\pi$ must be injective everywhere (any element of its kernel would lie in the kernel of $\pi^*\omega_{\mathcal{O}}$). This shows $\pi\colon M \to \mathcal{O}$ is a local diffeomorphism, and hence is a covering. □

Still asuming that M is a homogeneous Hamiltonian G-space, let $m \in M$ so that

$$M = G/G_m, \quad \mathcal{O} = \Phi(M) = G/G_{\Phi(m)}.$$

The covering map π is a fibration with *discrete* fiber $G_{\Phi(m)}/G_m$. Hence, non-trivial coverings can be obtained only if the stabilizer $G_{\Phi(m)}$ is disconnected (for compact connected Lie group do not have nontrivial subgroups of the same dimension). If G is a *compact, connected* Lie group then it is known that all stabilizer groups G_μ for the (co)adjoint action are connected. (See Remark 7.32 below.) Thus:

Theorem 7.30. *If G is compact and connected, and (M, ω, Φ) is a homogeneous Hamiltonian G-space, then the moment map induces a symplectomorphism of M with the coadjoint orbit $\mathcal{O} = \Phi(M)$.*

Example 7.31. The space $M = \mathrm{CP}(n)$ with the Fubini-Study symplectic form and the natural action of $G = \mathrm{U}(n+1)$ is a homogeneous Hamiltonian G-space. Hence it is a coadjoint orbit of G. The stabilizer of $[1 : 0 : \cdots : 0]$ is $H \cong \mathrm{U}(1) \times \mathrm{U}(n)$, so it must be the coadjoint orbit of type $\mathrm{U}(n+1)/(\mathrm{U}(1) \times \mathrm{U}(n))$. To decide exactly which coadjoint orbit it is, it suffices to find the image of $[1 : 0 : \cdots : 0]$ under the moment map.

Remark 7.32. The connectedness of stabilizers G_μ for compact G can be shown as follows: Using the fibration

$$G \to G/G_\mu,$$

and the fact that G is connected, it suffices to show that the base G/G_μ is *simply connected*. (Given $p_0, p_1 \in G_\mu$ choose a path $\gamma : [0,1] \to G$ connecting them. γ projects to a closed path in G/G_μ, and can be contracted to a constant path. By the homotopy lifting property this homotopy can be lifted to a homotopy with fixed end points of γ. This will then be a path in G_μ connecting p_0, p_1.) In turn, the simply-connectedness of G/G_μ follows from Morse theory; this will be explained later.

7.6. **Poisson manifolds.** Moment maps fit very nicely into the more general category of Poisson manifolds.

Definition 7.33. A Poisson manifold is a manifold M together with a bilinear map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that

(a) $\{\cdot,\cdot\}$ is a Lie algebra structure on $C^\infty(M)$, and
(b) for all $H \in C^\infty(M)$, the map $C^\infty(M) \to C^\infty(M), F \mapsto \{H,F\}$ is a derivation.

Since any derivation of $C^\infty(M)$ is given by a vector field, any H defines a so-called Hamiltonian vector field X_H by $X_H(F) = \{H,F\}$.

Exercise 7.34. Show that X_H is a Poisson vector field. Show that the flow of any complete Poisson vector field is Poisson.

Examples of Poisson manifolds are of course symplectic manifolds, with the Poisson bracket associated to the symplectic structure. Another important example, due to Kirillov, is the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. For any $\mu \in \mathfrak{g}^*$ and any function $F \in C^\infty(\mathfrak{g}^*)$ identify

$$\mathsf{d}F_\mu \in T^*_\mu \mathfrak{g}^* \cong (\mathfrak{g}^*)^* = \mathfrak{g}.$$

Then define

$$\{F,G\}(\mu) := \langle \mu, [\mathsf{d}F_\mu, \mathsf{d}G_\mu].\rangle$$

Exercise 7.35. Verify that this is a Poisson structure on $\mathfrak{g}^*$.

Remark 7.36. This standard Poisson structure on the dual of a Lie algebra was originally found by Lie himself. It was independently rediscovered by Kirillov, Kostant, and Souriau and is usually called the Kirillov-Kostant-Souriau Poisson structure.

Definition 7.37. A smooth map $\phi : M_1 \to M_2$ between Poisson manifolds is called a Poisson map if

$$\phi^*\{F_1,F_2\} = \{\phi^*F_1, \phi^*F_2\}.$$

A vector field $X \in \mathfrak{X}(M)$ is called Poisson if

$$X\{F_1,F_2\} = \{X(F_1),F_2\} + \{F_1, X(F_2)\}.$$

Exercise 7.38. Show that the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ of a coadjoint orbit is a Poisson map.

The definition of a moment map carries over to Poisson manifolds: A G-action is Poisson if it preserves Poisson brackets, and any such action is *Hamiltonian* if there exists an equivariant smooth map $\Phi : M \to \mathfrak{g}^*$ such that

$$\xi_M = X_{\langle \Phi, \xi \rangle}$$

for all $\xi \in \mathfrak{k}$.

Exercise 7.39. Show that the coadjoint action of G on $\mathfrak{g}^*$ is Hamiltonian, with moment map the identity map.

Exercise 7.40. Show that if M is a Poisson manifold with a Hamiltonian G-action, the moment map is a Poisson map.

Exercise 7.41. Suppose G is a connected Lie group, and M a Poisson G-manifold with a map $\Phi\colon M \to \mathfrak{g}^*$ satisfying the moment map condition $\xi_M = \{\langle \Phi, \xi\mathbb{R}, \cdot\}$. Show that Φ is equivariant if and only if Φ is a Poisson map.

Conversely, show that for every Poisson map $\Phi : M \to \mathfrak{g}^*$, the equation $\xi_M = \{\langle \Phi, \xi\rangle, \cdot\}$ defines a Poisson $\mathfrak{g}$-action on M, i.e. $\mathcal{L}_{\xi_M}\{f, g\} = \{\mathcal{L}_{\xi_M} f, g\} + \{f, \mathcal{L}_{\xi_M} g\}$.

7.7. **2d Gauge Theory.** In this section we will discuss, somewhat informally, an interesting ∞-dimensional example due to Atiyah-Bott [5, 6] (see also [35]). We start with some background on connections, curvature, and gauge transformations. Let Σ be a manifold (later this will be a surface), and G a Lie group with Lie algebra $\mathfrak{g}$. We shall consider the action of the infinite-dimensional *gauge group*

$$\mathcal{G}(\Sigma) = C^\infty(\Sigma, G)$$

(with pointwise group multiplication) on the infinite-dimensional manifold

$$\mathcal{A}(\Sigma) := \Omega^1(\Sigma, \mathfrak{g})$$

of *connections*, given by

$$g \cdot A = \mathrm{Ad}_g(A) - \mathsf{d}g\, g^{-1}.$$

Here the first term is the pointwise adjoint action. The second term is written for matrix-groups so that $\mathsf{d}gg^{-1}$ makes sense as a 1-form on Σ with values in $\mathfrak{g}$. One may verify that this is indeed an action: $g_1 \cdot g_2 \cdot A = (g_1 g_2) \cdot A$.

Remark 7.42. For general Lie groups, the term $\mathsf{d}gg^{-1}$ is to be interpreted as the pull-back under $g : \Sigma \to G$ of the right-invariant Maurer-Cartan form $\theta^R \in \Omega^1(G, \mathfrak{g})$; i.e. θ^R is the unique right-invariant form such that for any right-invariant vector field ξ^R, $\iota(\xi^R)\theta^R = \xi$.

The formula for the gauge action may be motivated as follows. Consider a G-representation $G \to \mathrm{GL}(V)$ on a vector space V. By differentiation, one obtains a Lie algebra representation of $\mathfrak{g}$. The connection A defines an operator on the space of V-valued differential forms, called the *covariant derivative*

$$\mathsf{d}_A : \Omega^k(\Sigma, V) \to \Omega^{k+1}(\Sigma, V), \quad \mathsf{d}_A\sigma = \mathsf{d}\sigma + A \cdot \sigma$$

(using both the $\mathfrak{g}$-action and wedge product in the second term). On the other hand, the G-action on V gives an action of the gauge group $\mathcal{G}(\Sigma) = C^\infty(\Sigma, G)$ acts on V-valued forms, simply using the pointwise action on V. The gauge action on connections is defined in such a way that the following is true:

Lemma 7.43. *For all $\sigma \in \Omega^k(\Sigma, V)$,*

$$\mathsf{d}_{g\cdot A}(g \cdot \sigma) = g \cdot (\mathsf{d}_A\sigma).$$

Proof. For all $\sigma \in \Omega^k(\Sigma, V)$,

$$\begin{aligned} \mathsf{d}(g\cdot\sigma) &= \mathsf{d}g\cdot\sigma + g\cdot\mathsf{d}\sigma \\ &= (\mathsf{d}g g^{-1})\cdot g\cdot\sigma + g\cdot\mathsf{d}\sigma. \end{aligned}$$

This gives

$$\begin{aligned} \mathsf{d}_{g\cdot A}(g\cdot\sigma) &= \mathsf{d}(g\cdot\sigma)) + (\mathrm{Ad}_g(A) - \mathsf{d}g g^{-1})\cdot g\cdot\sigma \\ &= g\cdot(\mathsf{d}\sigma) + \mathrm{Ad}_g(A)\cdot g\cdot\sigma \\ &= g\cdot(\mathsf{d}_A\sigma). \end{aligned}$$

□

Due to the presence of the gauge term the square of the covariant derivative is usually not zero:

Lemma 7.44. *We have*

$$(\mathsf{d}_A)^2\sigma = \mathrm{curv}(A)\cdot\sigma$$

with the curvature $\mathrm{curv}(A) = \mathsf{d}A + \frac{1}{2}[A, A] \in \Omega^2(\Sigma, \mathfrak{g})$. *The curvature transforms equivariantly:*

$$\mathrm{curv}(g\cdot A) = \mathrm{Ad}_g\,\mathrm{curv}(A).$$

Proof. We compute

$$\begin{aligned} \mathsf{d}_A\mathsf{d}_A\sigma &= \mathsf{d}(A\cdot\sigma) + A\cdot\mathsf{d}\sigma + A\cdot A\cdot\sigma \\ &= (\mathsf{d}A)\cdot\sigma + \frac{1}{2}[A, A]\cdot\sigma. \end{aligned}$$

Similarly, equivariance of the curvature is verified by direct calculation (or using the equivariance of the covariant derivative). □

By the lemma, $\mathsf{d}_A^2 = 0$ provided that the curvature is zero. Connections with this property are called *flat*. Consider the infinite-dimensional affine space $\mathcal{A}(\Sigma)$ as a $\mathcal{G}(\Sigma)$-space. What are the generating vector fields? The Lie algebra of the gauge group is identified with $\Omega^0(\Sigma, \mathfrak{g})$.

Lemma 7.45. *The generating vector fields for* $\xi \in \Omega^0(\Sigma, \mathfrak{g})$ *are*

$$\xi_{\mathcal{A}(\Sigma)}(A) = \mathsf{d}_A\xi.$$

Here $\mathsf{d}_A\xi = \mathsf{d}\xi + [A, \xi]$ is defined using the adjoint representation on $V = \mathfrak{g}$.

Proof. Recall that the evaluation $\xi_M(m) \in T_m M$ of a generating vector field for a G-action on a manifold M is represented by the curve $\exp(-t\xi).m$. Using this, we calculate

$$\begin{aligned}
\xi_{\mathcal{A}(\Sigma)}(A) &= \frac{d}{dt}\Big|_{t=0} \exp(-t\xi) \cdot A \\
&= \frac{d}{dt}\Big|_{t=0} \big(\mathrm{Ad}_{\exp(-t\xi)}(A) - \mathsf{d}(\exp(-t\xi)) \exp(t\xi) \big) \\
&= -[\xi, A] + \mathsf{d}\xi \\
&= \mathsf{d}_A \xi.
\end{aligned}$$

□

We now specialize to the case that Σ is a compact, oriented surface $\dim \Sigma = 2$. Let us fix an invariant inner product on $\mathfrak{g}$ (unique up to scalar if G is simple). We will denote the inner product simply by a fat dot $\bullet$ (to avoid confusion with actions). Then $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ is an ∞-dimensional symplectic manifold: The 2-form is

$$\omega_A(a, b) = \int_\Sigma a \bullet b$$

for all $a, b \in T_A \Omega^1(\Sigma, \mathfrak{g}) \cong \Omega^1(\Sigma, \mathfrak{g})$, using the inner product and wedge product. This 2-form is $\mathcal{G}(\Sigma)$-invariant. Indeed, since the gauge action is an affine action, with linear part the adjoint action, the action on tangent vectors $a \in T_A \mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ is simply

$$g_* a = \mathrm{Ad}_g(a) \in T_{g \cdot A} \mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$$

Hence,

$$\omega_{g \cdot A}(g_* a, g_* b) = \int_\Sigma \mathrm{Ad}_g(a) \bullet \mathrm{Ad}_g(b) = \int_\Sigma a \bullet b = \omega_A(a, b).$$

Theorem 7.46. [6] *For a compact oriented surface Σ, the gauge action of $\mathcal{G}(\Sigma)$ on $\mathcal{A}(\Sigma)$ is Hamiltonian, with moment map the curvature $A \mapsto \mathrm{curv}(A)$, i.e.*

$$\langle \Phi(A), \xi \rangle = \int_\Sigma \mathrm{curv}(A) \bullet \xi.$$

Proof. We calculate: For all $a \in \Omega^1(\Sigma, \mathfrak{g})$, viewed as a constant vector field,

$$\begin{aligned}
\langle \mathsf{d}\langle \Phi, \xi\rangle\Big|_A, a\rangle &= \frac{d}{dt}\Big|_{t=0}\langle \Phi(A+ta), \xi\rangle \\
&= \frac{d}{dt}\Big|_{t=0}\int_\Sigma \operatorname{curv}(A+ta)\bullet \xi \\
&= \frac{d}{dt}\Big|_{t=0}\int_\Sigma \Big(\mathsf{d}A + t\mathsf{d}a + t[A,a] + \frac{1}{2}([A,A]+t^2[a,a])\Big)\bullet \xi \\
&= \int_\Sigma \xi \bullet \mathsf{d}_A a \\
&= -\int_\Sigma \mathsf{d}_A \xi \bullet a \\
&= -\omega(\xi_{\mathcal{A}(\Sigma)}(A), a).
\end{aligned}$$

□

It is interesting to extend this calculation to 2-manifolds with boundary $\partial\Sigma$. Everything carries over, but the partial integration produces an extra boundary term so that the (weak) moment map is

$$\langle \Phi(A), \xi\rangle = \int_\Sigma \operatorname{curv}(A)\bullet\xi + \int_{\partial\Sigma} A\bullet\xi.$$

That is, the (informally) dual space to the Lie algebra $\Omega^0(\Sigma, \mathfrak{g})$ of the gauge group is identified with $\Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g})$ with the natural pairing, and the moment map is

$$\Omega^0(\Sigma, \mathfrak{g}) \to \Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g}), \quad A \mapsto (\operatorname{curv}(A), \iota^*_{\partial\Sigma}A).$$

Notice however that this moment map is no longer equivariant in the usual sense, for the action on the second summand is still the gauge action! This leads one to define a central extension of the gauge group. Define a *Polyakov-Wiegmann cocycle*

$$c:\ \mathcal{G}(\Sigma)\times\mathcal{G}(\Sigma) \to \mathrm{U}(1),\ c(g_1, g_2) = \exp\left(-i\pi\int_\Sigma g_1^{-1}\mathsf{d}g_1 \bullet \mathsf{d}g_2 g_2^{-1}\right),$$

and let $\widehat{\mathcal{G}(\Sigma)} = \mathcal{G}(\Sigma)\times \mathrm{U}(1)$ with product

$$(g_1, z_1)(g_2, z_2) = \big(g_1g_2, z_1z_2\, c(g_1, g_2)\big).$$

One can show that this does indeed define a group structure (i.e. c is a cocycle). The Lie algebra of this new group is $\Omega^0(\Sigma, \mathfrak{g}) \oplus \mathbb{R}$ with defining cocycle (up to scalar factor) $\int_\Sigma \mathsf{d}\xi_1 \mathsf{d}\xi_2 = \int_{\partial\Sigma}\xi_1\mathsf{d}\xi_2$, and its dual is $\Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g}) \oplus \mathbb{R}$, with action

$$(g, z)\cdot(\alpha, \beta, \lambda) = (\operatorname{Ad}_g \alpha, \operatorname{Ad}_g \beta - \lambda \mathsf{d}g g^{-1}, \lambda).$$

It follows that the moment map for the action of the extended gauge group (where the extra circle acts trivially) is equivariant, the image of the original moment map is identified with the hyperplane $\lambda = 1$.

8. Symplectic Reduction

8.1. **The Meyer-Marsden-Weinstein Theorem.** Let (M, ω, Φ) be a Hamiltonian G-space. One of the basic properties of the moment map is the following:

Proposition 8.1. *For all $m \in M$, the kernel and image of the tangent map $T_m\Phi\colon T_mM \to T_{\Phi(m)}\mathfrak{g}^* = \mathfrak{g}^*$ are given by*

$$\begin{aligned} \ker(T_m\Phi) &= T_m(G\cdot m)^\omega, \\ \operatorname{ran}(T_m\Phi) &= \operatorname{ann}(\mathfrak{g}_m). \end{aligned}$$

Proof. By the defining condition of the moment map we have, for $v \in T_mM$ and $\xi \in \mathfrak{g}$,

$$\omega_m(\xi_M(m), v) = -\iota(v)\mathsf{d}\langle \Phi, \xi\rangle\Big|_m = -\langle (T_m\Phi)(v), \xi\rangle. \tag{25}$$

It follows that $(T_m\Phi)(v) = 0$ if and only if $\omega_m(\xi_M(m), v) = 0$ for all $\xi \in \mathfrak{g}$. This shows $\ker(T_m\Phi) = T_m(G\cdot m)^\omega$.

Equation (25) also shows that if $\xi \in \mathfrak{g}_m$ then $\langle (T_m\Phi)(v), \xi\rangle = 0$ for all v. Hence $\operatorname{ran}(T_m\Phi) \subseteq \operatorname{ann}(\mathfrak{g}_m)$. Equality follows by dimension count:

$$\begin{aligned} \dim(\operatorname{ran}(T_m\Phi)) &= \dim M - \dim(\ker(T_m\Phi)) \\ &= \dim M - \dim(T_m(G\cdot m))^\omega \\ &= \dim T_m(G\cdot m) \\ &= \dim(G\cdot m) \\ &= \dim G - \dim G_m \\ &= \dim \operatorname{ann}(\mathfrak{g}_m). \end{aligned}$$

□

Theorem 8.2. *A point $\mu \in \mathfrak{g}^*$ is a regular value of Φ if and only if for all $m \in \Phi^{-1}(\mu)$, the stabilizer group G_m is discrete. In this case, $\Phi^{-1}(\mu)$ is a constant rank submanifold. The leaf of the null foliation through $m \in \Phi^{-1}(\mu)$ is the orbit*

$$G_\mu \cdot m \subseteq \Phi^{-1}(\mu).$$

Proof. We have

$$\begin{aligned} \mu \text{ is a regular value of } \Phi &\Leftrightarrow \forall m \in \Phi^{-1}(\mu)\colon \operatorname{ran}(T_m\Phi) = \mathfrak{g}^* \\ &\Leftrightarrow \forall m \in \Phi^{-1}(\mu)\colon \operatorname{ann}(\mathfrak{g}_m) = \mathfrak{g}^* \\ &\Leftrightarrow \forall m \in \Phi^{-1}(\mu)\colon \mathfrak{g}_m = \{0\} \\ &\Leftrightarrow \forall m \in \Phi^{-1}(\mu)\colon G_m \subseteq G_\mu \text{ is discrete.} \end{aligned}$$

Assuming μ is a regular value, let $\iota : \Phi^{-1}(\mu) \hookrightarrow M$ be the inclusion, and consider $m \in \Phi^{-1}(\mu)$. Using $T_m(\Phi^{-1}(\mu)) = \ker(T_m\Phi)$ the kernel of $\iota^*\omega|_m$ is

$$\begin{aligned}\ker \iota^*\omega\Big|_m &= T_m(\Phi^{-1}(\mu)) \cap T_m(\Phi^{-1}(\mu))^\omega \\ &= T_m(\Phi^{-1}(\mu)) \cap T_m(G \cdot m) \\ &= T_m(\Phi^{-1}(\mu) \cap G \cdot m) \\ &= T_m(G_\mu \cdot m).\end{aligned}$$

Since the stabilizers for the G_μ-action on $\Phi^{-1}(\mu)$ are discrete, we see that the dimension of $\ker \iota^*\omega\Big|_m$ is equal to $\dim G_\mu$, and in particular is constant. ☐

Theorem 8.3 (Marsden-Weinstein, Meyer)**.** [31] [36] *Let (M, ω, Φ) be a Hamiltonian G-space. Suppose that $\mu \in \mathfrak{g}^*$ is a regular value of Φ, and that the foliation of $\Phi^{-1}(\mu)$ by G_μ-orbits is a fibration. Let $M_\mu = \Phi^{-1}(\mu) \to \Phi^{-1}(\mu)/G_\mu$ be the quotient, and denote by ι, π the projection and inclusion*

$$\begin{array}{ccc} \Phi^{-1}(\mu) & \xrightarrow[\iota]{} & M \\ {\scriptstyle\pi}\big\downarrow & & \\ M_\mu & & \end{array}$$

There exists a unique symplectic form ω_μ on M_μ such that

$$\iota^*\omega = \pi^*\omega.$$

Proof. Since the null-foliation is given by the G_μ-orbits, this is a special case of the theorem on reduction of constant rank submanifolds. ☐

Remark 8.4. The assumption that the quotient map is a fibration is satisfies when the G_μ-action on $\Phi^{-1}(\mu)$ is free and proper. (A Lie group action of G on Q is called proper if the map $G \times Q \to Q \times Q$, $(g, q) \mapsto (g \cdot q, q)$ is proper. This is automatic if G is compact.)

Definition 8.5. The space M_μ is called the *reduced space* or *symplectic reduction* at level μ. The reduced space at 0 is denoted

$$M_0 = M /\!\!/ G$$

and called the *symplectic quotient*.

The notation $M /\!\!/_\mu G$ in place of M_μ is also common; it is especially useful when several groups are involved.

The symplectic reductions M_μ depend only on the coadjoint orbit $\mathcal{O} = G.\mu$. Let $\mathcal{O}^-$ be the same G-space but with the opposite symplectic structure and minus the inclusion as a moment map. The moment map for the diagonal action on $M \times \mathcal{O}^-$ is

$$\tilde{\Phi}: M \times \mathcal{O}^- \to \mathfrak{g}^*, \ (m, \mu) \mapsto \Phi(m) - \mu.$$

Proposition 8.6 (Shifting-trick)**.** [41] *μ is a regular value of Φ if and only if 0 is a regular value of*

$$\tilde{\Phi}: M \times \mathcal{O}^- \to \mathfrak{g}^*, \ (m, \mu) \mapsto \Phi(m) - \mu.$$

Moreover the G_μ-action on $\Phi^{-1}(\mu)$ is free if and only if the G-action on $\tilde{\Phi}^{-1}(0)$ is free. There is a canonical symplectomorphism,

$$M_\mu \cong (M \times \mathcal{O}^-)/\!\!/G.$$

Proof. The zero level set of $\tilde{\Phi}$ consists of pairs (m, μ) such that $\mu \in \mathcal{O}$ and $\Phi(m) = \mu$. This gives a G-equivariant bijection

$$\Phi^{-1}(\mathcal{O}) \to \tilde{\Phi}^{-1}(0), \quad m \mapsto (m, \Phi(m)).$$

0 is a regular value of $\tilde{\Phi}$ if and only if the G-action on $\tilde{\Phi}^{-1}(0)$ is locally free, if and only if the action on $\Phi^{-1}(\mathcal{O})$ is locally free. Since $\Phi^{-1}(\mathcal{O}) = G.\Phi^{-1}(\mu)$, the G-action on $\Phi^{-1}(\mathcal{O})$ is locally free if and only if the G_μ-action on $\Phi^{-1}(\mu)$ is locally free, if and only if μ is a regular value of Φ. The map

$$M \to M \times \mathcal{O}^-, \quad m \mapsto (m, \mu)$$

(inclusion as $M \times \{\mu\}$) preserves 2-forms, hence so does its restriction

$$\Phi^{-1}(\mu) \to \tilde{\Phi}^{-1}(0), \quad m \mapsto (m, \mu).$$

It follows that the maps

$$M_\mu = \Phi^{-1}(\mu)/G_\mu \to (\tilde{\Phi}^{-1}(0) \cap (M \times \{\mu\}))/G_\mu \to \tilde{\Phi}^{-1}(0)/G$$

are all symplectomorphisms. □

Example 8.7. Let $M = \mathbb{C}^{n+1}$, with the standard symplectic form

$$\omega = \frac{i}{2} \sum_{j=0}^{n} \mathsf{d}z_j \wedge \mathsf{d}\bar{z}_j,$$

and scalar $S^1 = \mathbb{R}/\mathbb{Z}$-action given as multiplication by $\exp(2\pi i t)$. Under the identifications $\mathrm{Lie}(S^1) = \mathbb{R}, \ \mathrm{Lie}(S^1)^* = \mathbb{R}$ the moment map for this action is given by

$$\Phi(z) = -\pi ||z||^2.$$

The reduced space at level $-\pi$ is $\mathbb{C}P(n) = S^{2n+1}/S^1$, with the Fubini-Study form as described in Proposition 3.36. Reducing at a different value $-\lambda\pi$ amounts to rescaling the symplectic form by λ.

Example 8.8. Let (M, ω) be an exact symplectic manifold, $\omega = -\mathsf{d}\theta$, and suppose θ is invariant under some G-action. Let Φ be the corresponding moment map $\langle \Phi, \xi \rangle = -\iota(\xi_M)\theta$. Suppose 0 is a regular value of Φ and the G-action on $\Phi^{-1}(0)$ is free and proper. Then the pull-back $\iota^*\theta$ is G-invariant and horizontal, i..e, G-basic. and hence descends to a 1-form θ_0 on M_0 such that $\pi^*\theta_0 = \iota^*\theta$, and one has

$$\omega_0 = -\mathsf{d}\theta_0.$$

It follows that the symplectic quotient of an exact Hamiltonian G-space at 0 an exact symplectic manifold.

Example 8.9. As a sub-example, consider the case $M = T^*Q$, where G acts by the cotangent lift of a G-action on Q. The moment map is given by

$$\langle \Phi(m), \xi \rangle = \langle m,\, \xi_Q(q) \rangle$$

where $q \in Q$ is the base point of $m \in M = T^*Q$. This shows that the zero level set is the union of covectors orthogonal to orbits:

$$\Phi^{-1}(0) = \coprod_{q \in Q} \mathrm{ann}(T_q(G \cdot q)).$$

Since $\Phi^{-1}(0) \supseteq Q$, it is clear that the G-action on $\Phi^{-1}(0)$ is locally free if and only if the action on Q is locally free. If the action is free, we have

$$(T^*Q)/\!\!/G = T^*(Q/G).$$

To see this (at least set-theoretically), note that

$$T(Q/G) = \Big(\coprod_q T_qQ/T_q(G \cdot q) \Big)/G$$

so that

$$T^*(Q/G) = \Big(\coprod_q \mathrm{ann}(T_q(G \cdot q)) \Big)/G.$$

To identify the symplectic forms one has to identify the reduced canonical 1-form θ_0 with the canonical 1-form on $T^*(Q/G)$, we leave this as an exercise.

For an arbitrary G-space Q the singular reduced space $(T^*Q)/\!\!/G$ may be viewed as a cotangent bundle for the singular space Q/G.

Example 8.10. Returning to the Atiyah-Bott gauge theory example, the reduction

$$\mathcal{M}(\Sigma) = \mathcal{A}(\Sigma)/\!\!/\mathcal{G}(\Sigma)$$

is the moduli space of flat connections on Σ. Indeed, we identified the moment map with the curvature, hence the zero level set consists of flat connections, and the quotient passes to the moduli space.

8.2. **Reduced Hamiltonians.** [16] Suppose (M, ω, Φ) is a Hamiltonian G-space, that $\mu \in \mathfrak{g}^*$ is a regular value of the moment map, and that the action of G_μ on the level set $\Phi^{-1}(\mu)$ is free and proper. Then every invariant Hamiltonian $H \in C^\infty(M)^G$ descends to a unique function $H_\mu \in C^\infty(M_\mu)$ with $\pi^* H_\mu = \iota^* H$. Passing to the reduced Hamiltonian H_μ is often a first step in solving the equations of motion for H. From H-invariance of X_H it follows that the restriction $(X_H)\big|_{\Phi^{-1}(\mu)} \in \mathfrak{X}(\Phi^{-1}(\mu))$ is π-related to $X_{H_\mu} \in \mathfrak{X}(M_\mu)$, that is its flow projects down to the flow on M_μ. After one has solved the reduced system (i.e. determined its flow $F_\mu(t)$) it is a second step to lift $F_\mu(t)$ up to the level set $\Phi^{-1}(\mu)$.

Example 8.11. Consider the motion of a particle on $\mathbb{R}^2$ in a potential $V(q)$. It is described by the Hamiltonian on $T^*\mathbb{R}^2$,

$$H(q,p) = \frac{||p||^2}{2} + V(q).$$

Suppose the potential has rotational symmetry, i.e. that it depends only on $r = ||q||$. Then H is invariant under the cotangent lift of the rotation action of $G = S^1$. We had seen that the moment map for this action is angular momentum, $\Phi(q,p) = p_2 q_1 - q_2 p_1$. In polar coordinates, (r, θ) on $\mathbb{R}^2$ and corresponding cotangent coordinates on $T^* R^2$,

$$H(r, \theta, p_r, p_\theta) = \frac{1}{2}(p_r^2 + \frac{1}{r^2} p_\theta^2) + V(r)$$

and $\Phi = p_\theta$. The symplectic form on $T^*\mathbb{R}^2$ is $\omega = \mathsf{d}r \wedge p_r + \mathsf{d}\theta \wedge p_\theta$. Every value $\mu \neq 0$. is a regular value of Φ (since S^1 acts freely on the set where $p_\theta = r^2 \dot{\theta} \neq 0$). On $\iota : \Phi^{-1}(\mu) \hookrightarrow T^*\mathbb{R}^2$ the second term disappears, i.e. $\iota^* \omega = \mathsf{d}r \wedge p_r$. It follows that $M_\mu \cong T^*\mathbb{R}_{>0}$ symplectically, and the reduced Hamiltonian is

$$H_\mu(r, p_r) = \frac{1}{2} p_r^2 + V_{\text{eff}}(r)$$

with the effective potential,

$$V_{\text{eff}}(r) = V(r) + \frac{\mu^2}{2r^2}.$$

Using conservation of energy

$$\frac{p_r^2}{2} + V_{\text{eff}}(r) = \frac{\dot{r}^2}{2} + V_{\text{eff}}(r) = E,$$

i.e. $\dot{r}^2 = 2(E - V_{\text{eff}}(r))$, one obtains the solution in implicit form,

$$t - t_0 = \int_{r_0}^{r} \frac{\mathsf{d}r}{\sqrt{2(E - V_{\text{eff}}(r))}}.$$

Using $r^2 \dot{\theta} = p_\theta = \mu$, one also obtains a differential equation for the trajectories,

$$\frac{\partial r}{\partial \theta} = \frac{r^2}{\mu} \sqrt{2(E - V_{\text{eff}}(r))},$$

[16] We will omit this discussion in class

with solutions,

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\mu \mathsf{d} r}{r^2\sqrt{2(E - V_{\text{eff}}(r))}}.$$

In the special case $V(r) = -\frac{1}{r}$ (Kepler problem) this integral can be solved and leads to conic sections – see any textbook on classical mechanics.

8.3. **Reduction in stages.** As a special case of "reduced Hamiltonian" one sometimes has a reduced moment map. For the simplest situation, suppose G, H are Lie groups, and (M, ω) is a Hamiltonian $G \times H$-space, with moment map

$$(\Phi, \Psi)\colon M \to \mathfrak{g}^* \times \mathfrak{h}^*.$$

The equivariance of the moment map means, in particular, that Φ is H-invariant and Ψ is G-invariant.

Let μ be a regular value of Φ, and that the G_μ-action on $\Phi^{-1}(\mu)$ is free and proper, so that the reduced space M_μ is defined.

Proposition 8.12. *The action of H on the G-reduced space M_μ is Hamiltonian, with moment map*

$$\Psi_\mu : M_\mu \to \mathfrak{h}^*$$

given by $\pi^*\Psi_\mu = \iota^*\Psi$.

We leave the proof as an exercise.

Proposition 8.13 (Reduction in Stages)**.** *Suppose μ is a regular value of Φ and (μ, ν) a regular value for (Φ, Ψ). Then ν is a regular value for Ψ_μ. If G_μ acts freely and properly on $\Phi^{-1}(\mu)$ and $G_\mu \times H_\nu$ acts freely and properly on $\Phi^{-1}(\mu) \cap \Psi^{-1}(\nu)$, then H_ν acts freely and properly on $\Psi_\mu^{-1}(\nu)$, and there is a natural symplectomorphism*

$$(M_\mu)_\nu \cong M_{(\mu,\nu)}.$$

Proof. If G_μ acts with finite (resp. trivial) stabilizers on $\Phi^{-1}(\mu)$ and $G_\mu \times H_\nu$ acts with finite (resp. trivial) stabilizers on $\Phi^{-1}(\mu) \cap \Psi^{-1}(\nu)$, the same is true for the H_ν-action on $\Psi_\mu^{-1}(\nu)$. This proves the first part since a level set having finite stabilizers is equivalent to the level being a regular value. The second part follows because the natural identifications

$$(M_\mu)_\nu = \Psi_\mu^{-1}(\nu)/H_\nu = (\Phi^{-1}(\mu) \cap \Psi^{-1}(\nu))/(G_\mu \times H_\nu) = M_{(\mu,\nu)}$$

all preserve 2-forms. $\square$

8.4. **The cotangent bundle of a Lie group.** Recall that a Lie group G acts on itself by left multiplication,

$$g \cdot a = ga,$$

with generating vector fields $-\xi^R$. It also acts on G by right multiplication

$$h \cdot a = ah^{-1}$$

with generating vector fields η^L. These two actions commute, and define a $G \times G$-action on G (where the first factor corresponds to left multiplication, the second to right multiplication), with generating vector fields

$$(\xi, \eta) \mapsto \eta^L - \xi^R.$$

By cotangent lift, we obtain a Hamiltonian $G \times G$-action on T^*G. The moment map

$$(\Phi, \Psi) \colon T^*G \to \mathfrak{g}^* \times \mathfrak{g}^*$$

is described using contractions of the canonical 1-form $\theta \in \Omega^1(T^*G)$ with the cotangent lifts $\eta^L_{T^*} - \xi^R_{T^*}$. but we would like a somewhat more concrete description.

To this end shall use *left-trivialization* of the tangent and cotangent bundles. For the tangent bundle, the left trivialization

$$TG \xrightarrow{\cong} G \times \mathfrak{g}$$

is given by the inverse of the map $(g, \xi) \mapsto \xi^L(g)$. Recall that the left-invariant Maurer-Cartan form $\theta^L \in \Omega^1(G, \mathfrak{g})$ is defined as $\iota(\xi^L)\theta^L = \xi$. by left-invariant 1-forms. We hence see that left trivialization is described in terms of the Maurer-Cartan form as $v \mapsto (g, \theta^L_g(v))$, for $v \in T_gG$.

Dual to the left-trivialization of TG there is the left trivialization of T^*G

$$T^*G \xrightarrow{\cong} G \times \mathfrak{g}^*.$$

The cotangent lift of the left-and right actions are given in this triviaization by

$$g \cdot (a, \mu) = (ga, \mu), \quad h \cdot (a, \mu) = (ah^{-1},\, h \cdot \mu)$$

(with the coadjoint action $\mu \mapsto h \cdot \mu$). We read off the generating vector fields

$$-\xi^R_{T^*} = (-\xi^R, 0), \qquad \eta^L_{T^*} = (\eta^L, \eta_{\mathfrak{g}^*}).$$

Lemma 8.14. *The canonical 1-form $\theta \in \Omega^1(T^*G)$ is described in left trivialization by the 1-form* [17]

$$\langle \mu, \theta^L \rangle \in \Omega^1(G \times \mathfrak{g}^*).$$

[17] In the following expression, θ^L lives on the first factor of $G \times \mathfrak{g}^*$, and μ is understood as the variable on the second factor. More pedantically, letting $\mathrm{pr}_1 \colon G \times \mathfrak{g}^* \to G,\ \mathrm{pr}_2 \colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ the two projections, it is given by $\langle \mathrm{pr}_2, \mathrm{pr}_1^*\,\theta^L \rangle$.

Proof. Recall that the canonical 1-form is characterized by the property that it vanishes on vertical vector fields, and its contractions with cotangent lifts of $Y \in \mathfrak{X}(Q)$ satisfy

$$\iota(Y_{T^*})\theta|_\mu = \langle \mu, Y|_q \rangle$$

for all $\mu \in T_q^*Q$.

The 1-form $\tilde{\theta} = \langle \mu, \theta^L \rangle$ vanishes on vertical vectors, and its contractions with cotangent lifts of left-invariant vector fields are, for $\mu \in \mathfrak{g}^* = T_g^*G$

$$\iota(\eta_{T^*}^L)\tilde{\theta} = \langle \mu, \eta \rangle.$$

This is the pairing of $\mu \in \mathfrak{g}^* = T_g^*G$ with $\eta^L|_g \in T_gG$ under left trivialization. □

Using this description, we can compute the contractions of our generating vector fields with the canonical 1-form, and hence reas off the moment map:

Proposition 8.15. *The moment map (Φ, Ψ) for the $G \times G$-action on T^*G is given in left trivialization by*

$$\Phi(g, \mu) = g \cdot \mu, \quad \Psi(g, \mu) = -\mu.$$

Proof. We have $\langle \Phi, \xi \rangle = \iota(\xi_{T^*}^R)\theta$, $\langle \Psi, \eta \rangle = -\iota(\eta_{T^*}^L)\theta$. In left trivialization, this becomes

$$\langle \Phi, \xi \rangle|_{(g,\mu)} = \langle \mu, \mathrm{Ad}_{g^{-1}} \xi \rangle = \langle g \cdot \mu, \xi \rangle, \qquad \langle \Psi, \eta \rangle|_{(g,\mu)} = -\langle \mu, \eta \rangle.$$

□

Remark 8.16. To remember which of these moment map components correspond to the left action, and which to the right action, recall that the moment map for the right-action must be left-invariant (since left and right actions commute). That is, in left trivialization the moment map for the right-action must be constant.

Note that each of the two factors in $G \times G$ acts freely on T^*G. (The action of the full group $G \times G$ is not free.) In particular, every $\nu \in \mathfrak{g}^*$ is a regular value for both moment maps.

Theorem 8.17. *The symplectic reduction $(T^*G)_{-\nu}$ by the right action, with G-action inherited from the left-action, is the coadjoint orbit $\mathcal{O} = G \cdot \nu$ with its KKS symplectic form.*

Proof. The left action of G on the level set $\Psi^{-1}(-\nu)$ is free and transitive, and the action on the quotient $\Psi^{-1}(-\nu)/G_\nu$ has stabilizer conjugate to G_ν. The moment map induced by Φ_L identifies gives a symplectomorphism onto $\mathcal{O} = G \cdot \nu$. □

Of course, the reduced spaces with respect to the left action are coadjoint orbits as well: the cotangent lift of the inversion map $G \to G$, $g \mapsto g^{-1}$ exchanges the roles of the left- and right action.

Theorem 8.18. *Let (M, ω, Φ) be a Hamiltonian G-space. Let G act diagonally on $T^*G \times M$, where the action on T^*G is the right action. Consider the reduced space at 0 as a Hamiltonian G-space, with G-action induced from the left-G-action on T^*G. Then there is a canonical isomorphism of Hamiltonian G-spaces,*

$$(T^*G \times M)/\!\!/G \cong M$$

Proof. Use left trivialization $T^*G = G \times \mathfrak{g}^*$. The moment map for the left G-action on $T^*G \times M$ is

$$\Phi_1 \colon (a, \mu, m) \mapsto g \cdot \mu$$

while that for the diagonal G-action is

$$\Phi_2 \colon (a, \mu, m) \mapsto \Phi(m) - \mu.$$

Let $i \colon Z \to T^*G \times M$ be the zero level set $Z = \Phi_2^{-1}(0)$ for the diagonal action. It consists of elements of the form $(a, \Phi(m), m)$, with the G-action $h \cdot (a, \Phi(m), m) = (ah^{-1}, h \cdot \Phi(m), h \cdot m)$. We hence see that $Z/G = M$, with the quotient map

$$\pi \colon Z \to M, \ (a, \Phi(m), m) \mapsto a \cdot m.$$

The inclusion

$$j \colon M \to T^*G \times M, \quad m \mapsto (e, \Phi(m), m)$$

is an embedding as a symplectic submanifold, contained in Z. Since $\pi \circ j = id_M$, it follows that $(T^*G \times M)/\!\!/G = M$ as a symplectic manifold.

This quotient map intertwines the left G-action with the given action on M:

$$\pi(g \cdot a, \Phi(m), m) = g \cdot \pi(a, \Phi(m), m).$$

Furthermore, the moment map Φ_1 descends to Φ:

$$\Phi_1 \circ i = \Phi \circ \pi.$$

This completes the proof. ∎

A modification of this construction allows us to construct Hamiltonian G-spaces from Hamiltoian spaces for any subgroup. Let H be a closed subgroup of G, and let (N, ω_N, Φ_N) be a Hamitlonian H-space. Then $T^*G \times N$ is a Hamiltonian $G \times H$-space, and

$$(T^*G \times N)/\!\!/H$$

is a Hamiltonian G-space. Note that the H-action on $T^*G \times N$ is free (and proper) since the action on the first factor is. Hence, the symplectic quotient is a well-defined symplectic manifold. The map $T^*G \times N \to G$ given by bundle projection for the first factor descends to a map $(T^*G \times N)/\!\!/H \to G/H$, making the reduction into a fiber bundle over $G \times H$.

Exercise 8.19. Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ admits an H-invariant complementary subspace $\mathfrak{p}$, so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ (as vector spaces). Show that $\mathrm{Ind}_H^G(N)$ contains N as an H-invariant symplectic submanifold, and

$$(T^*G \times N)/\!\!/H = (G \times (\mathrm{ann}(\mathfrak{h}) \times N))/H$$

as a fiber bundle over G/H.

Note that $(T^*G \times N)/\!\!/H$ is non-compact, in general, even when N is compact.

8.5. **Normal forms near the zero level set.** Let (M, ω, Φ) be a Hamiltonian G-space, where G is a compact Lie group. If 0 is a regular value of the moment map then so are nearby values $\mu \in \mathfrak{g}^*$. What is the relation between M_0 and reduced spaces M_μ at nearby values?

To investigate this question we describe the reduction process in terms of a normal form. Let

$$i\colon Z = \Phi^{-1}(0) \to M$$

be the inclusion of the zero level set. Since 0 is a regular value, the level set is a closed submanifold, and the G-action on Z is locally free.

Lemma 8.20. *There exists a Lie algebra valued 1-form*

$$\alpha \in \Omega^1(Z, \mathfrak{g}),$$

which is G-equivariant

$$\mathcal{A}_g^* \alpha = \mathrm{Ad}_{g^{-1}} \alpha, \quad g \in G$$

and satisfies

$$\iota(\xi_Z)\alpha = \xi.$$

for all $\xi \in \mathfrak{g}$.

In the terminology of principal bundles, α is a connection 1-form.

Proof. Since the G-action is locally free, the map

$$Z \times \mathfrak{g} \to TZ, \quad (m, \xi) \mapsto \xi_Z(m)$$

is an inclusion as a G-invariant subbundle. Choose a G-invariant Riemannian metric on Z (such a metric may be constructed by averaging), and let

$$\tilde{\alpha}\colon TZ \to Z \times \mathfrak{g}$$

be the corresponding orthogonal projection onto this subbundle. We may regard $\tilde{\alpha}$ as a $\mathfrak{g}$-valued 1-form on Z, $\tilde{\alpha}(v) = (m, \iota(v)\alpha|_m)$. The G-equivariance of $\tilde{\alpha}$ corresponds to G-equivariance of α, and the property $\iota(\xi_Z)\alpha = \xi$ holds since $\tilde{\alpha}$ restricts to the identity on $Z \times \mathfrak{g}$. ∎

Remark 8.21. The lemma, and the discussion below, holds more generally if G is not necessarily compact but the G-action on Z is proper, since it is still possible to find a

G-invariant Riemannian metric in this case. (However, the construction is more tricky since averaging over G is not defined.)

Using α, we construct the local normal form near the zero level set. The normal form is given by the product $Z \times \mathfrak{g}^*$, with the diagonal G-action, and with the G-equivariant map

$$\Psi\colon Z \times \mathfrak{g}^* \to \mathfrak{g}^*,\ (m, \mu) \mapsto \mu$$

given by projection to the second factor, and with the closed G-invariant 2-form

$$\sigma = i^*\omega + \mathsf{d}\langle \Psi, \alpha\rangle.$$

(Here $i^*\omega, \alpha$ are regarded as forms on the product $Z \times \mathfrak{g}^*$; more accurately one should write $\mathrm{pr}_1^* i^*\omega,\ \mathrm{pr}_1^* \alpha$.)

Theorem 8.22 (Local normal form near the zero level set)**.** (a) *The 2-form $\sigma \in \Omega^2(Z \times \mathfrak{g}^*)$ satisfies*

$$\iota(\xi_{Z\times\mathfrak{g}^*})\sigma = -\mathsf{d}\langle \Psi, \xi\rangle$$

for all $\xi \in \mathfrak{g}$. It is nondegenerate (i.e. symplectic) on some neighborhood of $Z \times \{0\}$.

(b) *There exists an equivariant symplectomorphism between neighbourhoods of Z in M and in $Z \times \mathfrak{g}^*$, intertwining the two moment maps.*

Proof. We have

$$\begin{aligned}\iota(\xi_{Z\times\mathfrak{g}^*})\sigma &= i^*\iota(\xi_Z)\omega + \iota(\xi_{Z\times\mathfrak{g}^*})\mathsf{d}\langle \Psi, \alpha\rangle \\ &= -i^*\mathsf{d}\Phi, \xi\rangle + \mathcal{L}(\xi_{Z\times\mathfrak{g}^*})\langle \Psi, \alpha\rangle) - \mathsf{d}\iota(\xi_{Z\times\mathfrak{g}^*})\langle \Psi, \alpha\rangle\end{aligned}$$

The first term vanishes since $i^*\mathsf{d}\langle \Phi, \xi\rangle = \mathsf{d}\langle i^*\Phi, \xi\rangle = 0$. The second term vanishes since the 1-form $\langle \Psi, \alpha\rangle$ is G-invariant. Finally, $\iota(\xi_{Z\times\mathfrak{g}^*})\langle \Psi, \alpha\rangle = \langle \Psi, \xi\rangle$ since $\xi_{Z\times\mathfrak{g}^*} = \xi_Z + \xi_{\mathfrak{g}^*}$ and $\iota(\xi_Z)\alpha = \xi$. We conclude $\iota(\xi_{Z\times\mathfrak{g}^*})\sigma = -\mathsf{d}\langle \Psi, \xi\rangle$.

To show that σ is nondegenerate near $Z \times \{0\}$, it suffices to show that it is nondegenerate at all points $(m, 0) \in Z \times \{0\}$. But

$$T_{(m,0)}(Z \times \mathfrak{g}^*) = T_m Z \times \mathfrak{g}^* = \ker(\alpha_m) \oplus (\mathfrak{g} \oplus \mathfrak{g}^*),$$

where $\mathfrak{g}$ is included via the generating vector fields. Note also that since Ψ vanishes on Z,

$$\sigma_m = (i^*\omega)_m + \langle \mathsf{d}\Psi|_m,\ \alpha_m\rangle.$$

The first summand $(i^*\omega)_m$ restricts to a nondegenerate symplectic form on $\ker(\alpha_m)$ and vanishes on all vectors of $(\mathfrak{g} \oplus \mathfrak{g}^*)$. The second summand vanishes on vectors of $\ker(\alpha_m)$ and restricts to a nondegenerate symplectic form on $\mathfrak{g} \oplus \mathfrak{g}^*$ (in fact, it restricts to the standard symplectic form there). This shows that $(T_{(m,0)}(Z \times \mathfrak{g}^*), \sigma_{(m,0)})$ is the a direct sum of symplectic vector spaces; in particular $\sigma_{(m,0)}$ is symplectic.

b) By the G-equivariant version of the co-isotropic embedding theorem (Theorem 5.18) it follows that neighborhoods of Z in M and of Z in X are equivariantly symplectomorphic. Since both moment maps vanish on Z, it is automatic that the symplectomorphism intertwines the moment maps. □

Remark 8.23. We can view $Z\times\mathfrak{g}^*$ also as a quotient $(Z\times T^*G)/G$, using left trivialization to identify $T^*G \cong G\times\mathfrak{g}^*$; the G-action on $Z\times\mathfrak{g}^*$ is induced from the cotangent lift of the right G-action.

Suppose the G-action on $Z=\Phi^{-1}(0)$ is free, so that M_0 is a symplectic manifold. The model shows that the G-action is free on some open neighborhood of Z. (This also follows since for any G-manifold, the subset where G acts freely is open.) We may hence compare M_0 with the reduced spaces M_μ for 'small' μ. In the model,

$$\Psi^{-1}(\mu)=Z\times\{\mu\},$$

and so

$$\Psi^{-1}(\mu)=(Z\times\{\mu\})/G_\mu\cong Z/G_\mu.$$

We can write this quotient also as

$$Z/G_\mu=(Z\times G/G_\mu))/G$$

which is a fiber bundle over $Z/G=M_0$ with fiber a coadjoint orbit $\mathcal{O}=G\cdot\mu=G/G_\mu$.

8.6. The symplectic slice theorem.

8.6.1. *The slice theorem for G-manifolds.* Let G be a compact Lie group, and H a closed subgroup.

Every G-equivariant vector bundle over the homogeneous space G/H is of the form

$$E=G\times_H W\equiv(G\times W)/H$$

where W is an H-representation, the quotient is taken by the H-action $h.(g,w)=(gh^{-1},h.w)$ and the G-action is given by $g_1.[g,w]=[g_1g,w]$. Indeed, given a G-equivariant vector bundle $E\to G/H$ one defines $W=E_m$ to be the fiber over the identity coset $m=eH$. The map $G\times_H W\mapsto E,\ [g,w]\mapsto g.w$ is a well-defined, equivariant vector bundle isomorphism.

For example, suppose M is a G-manifold, and $\mathcal{O}=G\cdot m\subseteq M$ an orbit for the G-action. Then the normal bundle $\nu(M,\mathcal{O})$ is a G-equivariant vector bundle. Let $H=G_m$ so that $\mathcal{O}=G/H$. Then

$$\nu(M,\mathcal{O})=G\times_H W$$

with the *slice representation* of H on

$$W=T_mM/T_m(G.m)$$

Using a G-equivariant tubular neighborhood embedding $\nu(M,\mathcal{O})\supset U\to M$, we obtain:

Theorem 8.24 (Slice theorem)**.** *Let G be a compact Lie group, and let M be a G-manifold. There exists a G-equivariant diffeomorphism from an invariant open neighborhood of any orbit $\mathcal{O} = G.m$ to a neighborhood of the zero section of $E = G \times_H W$, where $H = G.m$ with slice representation on $W = T_m M / T_m(G.m)$.*

Corollary 8.25. *Every G-orbit $\mathcal{O} = G/H \subseteq M$ has an invariant open neighbourhood U with the property that all stabilizer groups G_x, $x \in U$ are G-conjugate to subgroups of $H = G.m$. In particular, if M is compact there are only finitely many conjugacy classes of stabilizer groups.*

Proof. Identify some neighborhood of the orbit with the model $E = G \times_H W$. Let $x = g.y$ with $y \in W$. Then $G_x = \mathrm{Ad}_g(G_y)$. But G_y is a subgroup of H, since it preserves the fiber $W = E_m$. □

In the special case that G is abelian (e.g., a torus), the conjugation is trivial. We hence conclude that for an action of an abelian Lie group on a compact manifold, the set of stabilizer subgroups $H \subseteq G$ is finite.

Definition 8.26 (Orbit types). For any subgroup H of G one denotes its conjugacy class by (H), and calls the G-invariant subset

$$M_{(H)} = \{m \in M |\, G_m \text{ is } G\text{-conjugate to } H\},$$

the points *of orbit type* (H). One also defines

$$M^H = \{m \in M |\, G_m \supset H\}, \quad M_H = \{m \in M |\, G_m = H\}.$$

Proposition 8.27. *The connected components of $M_{(H)}, M_H$ and M^H are smooth submanifolds of M.*

Proof. Any orbit $\mathcal{O} \subseteq M_{(H)}$ contains a point $m \in M$ with $G_m = H$. In the model $E = G \times_H W$ near $\mathcal{O}$, we have

$$E_{(H)} = G \times_H W^H = G/H \times W^H$$

since W^H is a vector subspace of W, this is clearly a smooth subbundle of E. The connected components of M^H are smooth submanifolds, since for all $m \in M^H$, a neighborhood is H-equivariantly modeled by the H-action on $T_m M$ and $(T_m M)^H$ is a vector subspace. The closure $\overline{M}^H$ is a union of connected components of M^H. Since M_H is open in its closure, it is in particular a submanifold. □

The decomposition

$$M = \bigcup_{(H)} M_{(H)}$$

is called the *orbit type stratification of M*. Using the slice theorem, one can show that it is indeed a stratification in the technical sense. Note that since each $M_{(H)}/G$ is a (union of) smooth manifolds, it induces a decomposition (in fact, a stratification) of the (usually singular) orbit space M/G into smooth manifolds.

8.6.2. *The slice theorem for Hamiltonian G-manifolds.* In symplectic geometry one can go one step further and try to equip the total space to the normal bundle with a symplectic structure. Thus let (M, ω, Φ) be a Hamiltonian G-space. Assume that $m \in \Phi^{-1}(0)$ is in the zero level set. This implies that the orbit is an isotropic submanifold:

$$T_m\mathcal{O} \subseteq \ker(T_m\Phi) = T_m\mathcal{O}^\omega$$

where the first inclusion holds since Φ vanishes on $\mathcal{O}$. The symplectic vector space

$$V = (T_m\mathcal{O})^\omega / T_m\mathcal{O}$$

with the action of $H = G_m$ is called the *symplectic slice representation at m*.

To describe a model around the orbit $\mathcal{O}$, consider the T^*G as a Hamiltonian $G \times H$-space, with the cotangent lift of the action $(g, h) \cdot a = gah^{-1}$. It allows us to elevate any Hamiltonian H-space (N, ω_N) to a Hamiltonian G-space $(T^*G \times N)/\!\!/H$. We shall apply this construction to linear symplectic representations.

Definition 8.28. Let H act on a symplectic vector space (V, ω_V) by linear symplectic transformations, and let

$$\Phi_V : V \to \mathfrak{h}^*, \quad \langle \Phi(v), \xi \rangle = -\frac{1}{2}\omega(v, \xi.v)$$

be its moment map (cf. 7.4.3). The symplectic quotient

$$E = (T^*G \times V)/\!\!/H$$

is called the *model* defined by V. We let $\Phi_E :\ E \to \mathfrak{g}^*$ be the moment map for the G-action on E inherited from the cotangent lift of the left-G-action on T^*G.

The orbit $\mathcal{O} = G/H$ is naturally embedded as an isotropic submanifold of E, namely as the zero section of $T^*G/\!\!/H = T^*(G/H)$. Its symplectic normal bundle in E is an associated bundle, $G \times_H V$.

Remark 8.29. Suppose H is compact. Then we may choose an H-invariant complement $\mathfrak{p}$ to $\mathfrak{h} \subseteq \mathfrak{g}$, for example the orthogonal complement for an H-invariant inner product. Dually we obtain

$$\mathfrak{g}^* = \mathrm{ann}(\mathfrak{h}) \oplus \mathrm{ann}(\mathfrak{p}) = \mathrm{ann}(\mathfrak{h}) \oplus \mathfrak{h}^*.$$

Identify $T^*G = G \times \mathfrak{g}^*$ using left trivialization. The zero level set for the H-action consists of points (g, μ, v) such that $-\operatorname{pr}_{\mathfrak{h}^*} \mu + \Phi_V(v) = 0$. In terms of the decomposition of $\mathfrak{g}^*$, it consists of points $(g, \Phi_V(v) + \nu, v)$ with $\nu \in \mathrm{ann}(\mathfrak{h})$, and is therefore isomorphic to $G \times \mathrm{ann}(\mathfrak{h}) \times V$. Thus

$$E \cong G \times_H (\mathrm{ann}(\mathfrak{h}) \times V).$$

In this description the moment map Φ_E is given by

$$\Phi_E([g, \nu, v]) = g.(\nu + \Phi_V(v)).$$

We stress that this identification depends on the choice of splitting.

Theorem 8.30 (Symplectic slice theorem)**.** *Let (M, ω, Φ) be a Hamiltonian G-manifold, and*

$$\mathcal{O} = G.m \subseteq \Phi^{-1}(0)$$

an orbit in the zero level set. There exists a G-equivariant symplectomorphism between neighbourhoods of $\mathcal{O}$ in M and in the model E defined by the symplectic slice representation $V = T_m\mathcal{O}^\omega / T_m\mathcal{O}$ of $H = G_m$, intertwining the two moment maps.

Proof. This follows from (equivariant version of) the constant rank embedding theorem: The symplectic normal bundles of $\mathcal{O}$ in both spaces are $G \times_H V$. □

Remark 8.31. The symplectic slice theorem is extremely useful: For example we obtain a model for the singularities of $M/\!\!/G$ in case 0 is a singular value. Indeed, by reduction in stages we have

$$(T^*G \times V/\!\!/H)/\!\!/G = (T^*G \times V/\!\!/G)/\!\!/H = V/\!\!/H$$

which shows that the singularities are modeled by symplectic reductions of unitary representations. Since the moment map for a unitary representation is homogeneous, the zero level set $\Phi_V^{-1}(0)$ is a cone and hence the singularities are *conic singularities*. This discussion can be carried much further, see the paper Sjamaar-Lerman [41].

Proposition 8.32. *Let (M, ω, Φ) be a Hamiltonian G-space, $H \subseteq G$ a closed subgroup. The connected components of M^H and M_H are symplectic submanifolds of M. For every connected open subset $U \subseteq M_H$, the image $\Phi(U)$ is an open subset of an affine subspace $\mu + \mathrm{ann}(\mathfrak{h})^H \subseteq \mathfrak{g}^*$ for some $\mu \in (\mathfrak{g}^*)^H$.*

Proof. For all $m \in M^H$, the tangent space $T_m(M^H)$ is equal to $(T_mM)^H$. But for any symplectic representation V of a compact Lie group H, the subspace V^H is symplectic. (Proof: Choose an H-invariant compatible complex structure. Then V^H is a complex, hence symplectic, subspace.) This shows that M^H and the open subset $M_H \subseteq M^H$ are symplectic.

The second part follows from the local model, or alternatively follows: Let $Z = Z_G(H)$ be the centralizer and $K = N_G(H)$ the normalizer of H in G, respectively. Thus $Z \subseteq$

$K \subseteq G$ and $\mathfrak{z} = \mathfrak{k}^H = \mathfrak{g}^H$. Dually, identify $\mathfrak{z}^* = (\mathfrak{k}^*)^H = (\mathfrak{g}^*)^H$. By equivariance of the moment map, $\Phi(M_H) \subseteq \Phi(M^H) \subseteq (\mathfrak{g}^*)^H = \mathfrak{z}^*$. The action of $K \subseteq G$ preserves M_H. Its moment map $\Psi : M_H \to \mathfrak{k}^*$ is the restriction of Φ followed by projection $\mathfrak{g}^* \to \mathfrak{k}^*$, but since it takes values in $(\mathfrak{k}^*)^H = \mathfrak{z}^*$ it is actually just the restriction of Φ. Since $\operatorname{ran} T_m\Psi = \operatorname{ann}_{\mathfrak{k}^*}(\mathfrak{h}) = \operatorname{ann}(\mathfrak{h})^H$ is independent of $m \in U$, we conclude that $\Phi(U)$ is an open neighborhood of $\Phi(m)$ in $\Phi(m) + \operatorname{ann}(\mathfrak{h})^H$. $\square$

9. Hamiltonian torus actions

Throughout this section, we will consider the case that the group G is compact, connected and abelian, i.e. a *torus* T.

9.1. **Duistermaat-Heckman theorem.** Let (M, ω, Φ) be a Hamiltonian T-space, and suppose 0 is a regular value of the moment map. As discussed in Section **??**, we have the local model

$$Z \times \mathfrak{t}^*, \quad \sigma = i^*\omega + \mathsf{d}\langle \Psi, \alpha \rangle$$

near the zero level set. Suppose for simplicity that the T-action on $Z = \Phi^{-1}(0)$ is *free*, so that $M_0 = M/\!\!/T$ is a symplectic manifold. Since the coadjoint action on $\mathfrak{t}^*$ is trivial, the reduced spaces of $Z \times \mathfrak{t}^*$ at nearby values μ are diffeomorphic:

$$M_\mu \cong Z/T$$

Writing

$$\sigma = \pi^*\omega_0 + \langle \mathsf{d}\Psi,\ \alpha \rangle + \langle \Psi, \mathsf{d}\alpha \rangle$$

we see that the pullback under $j_\mu \colon Z \to Z \times \{\mu\} \subseteq Z \times \mathfrak{t}^*$ is given by

$$j_\mu^*\sigma = \pi^*\omega_0 + \langle \mu, \mathsf{d}\alpha \rangle.$$

The $\mathfrak{t}$-valued 2-form $\langle \mu, \mathsf{d}\alpha \rangle$ is basic for the projection $Z \to Z/T$, since it is T-invariant and

$$\iota(\xi_Z)\mathsf{d}\alpha = \mathcal{L}(\xi_Z)\alpha - \mathsf{d}\xi = 0.$$

Hence it descends to a $\mathfrak{t}$-valued 2-form $M_0 = Z/T$ denoted

$$F^\alpha \in \Omega^2(M_0, \mathfrak{t}).$$

This is the *curvature form* of the connection α. As a consequence we find that the symplectic form on $M_\mu = Z/T$ changes according to

$$\omega_\mu = \omega_0 + \langle \mu, F^\alpha \rangle$$

Observe that this change is *linear in* μ. This result depends on our identification of $M_\mu \cong M_0$. This identification is not natural, since it depends on the symplectomorphism of neighbourhoods of Z in M and in the model $Z \times \mathfrak{t}^*$. But any two such identifications are related by an isotopy of M_0. Since cohomology classes are stable under isotopies, it makes sense to compare cohomology classes, and the above discussion proves the following result.

Theorem 9.1 (Duistermaat-Heckman)**.** *The cohomology class of the symplectic form changes according to*

$$[\omega_\mu] = [\omega_0] + \langle \mu, c \rangle$$

where $c \in H^*(M_0) \otimes \mathfrak{t}$ *is the first Chern class of the torus bundle* $\Phi^{-1}(0) \to M_0$.

In particular this change is *linear in* μ. As a consequence one has:

Corollary 9.2. *Let (M, ω, Φ) be a Hamiltonian T-space, and let U be a connected component of the set of regular values of Φ. Suppose the T-action on $\Phi^{-1}(U)$ is free. Then the volume function $U \to \mathbb{R}$, $\mu \mapsto \mathrm{Vol}(M_\mu)$ is given by a polynomial of degree at most $k = \frac{1}{2}\dim M - \dim T$.*

Proof. The reduced spaces have dimension

$$\dim M_\mu = \dim Z - \dim T = \dim M - 2\dim T = 2k.$$

Let us prove polynomiality near a given point $\mu_0 \in U$. Replacing Φ with $\Phi - \mu_0$, we may assume $\mu_0 = 0$. The volume of reduced spaces near $\mu_0 = 0$ is obtained by integrating

$$\frac{1}{k!}[\omega_\mu]^k = \frac{1}{k!}([\omega_0] + \langle \mu, c\rangle)^k$$

over Z/T. As a function of $\mu \in \mathfrak{t}^*$, this expression is a polynomial on $\mathfrak{t}^*$ of degree k, hence so is its integration. □

The assumption that the T-action on $\Phi^{-1}(U)$ is free is somewhat strong. One obtains a more general version of the theorem by considering the Liouville volume form $\frac{1}{n!}\omega^n$ on M and the associated measure on M.

Definition 9.3. Let (M, ω, Φ) be a compact Hamiltonian T-space. The push-forward measure

$$\varrho = \Phi_* \left| \frac{1}{n!}\omega^n \right|$$

on $\mathfrak{t}^*$ is called the *Duistermaat-Heckman measure.*

Remark 9.4. We recall that a measure ϱ on a topological space X may be defined as a continuous linear functional on the space of compactly supported continuous functions, usually written as an integral

$$\langle \varrho, f\rangle = \int_X f\,\varrho.$$

For $\mathbb{R}^n$ one has the translation invariant measure $|\mathsf{d}x|$ as in the definition of Riemann's integral. Similarly, on vector spaces one can consider translation invariant measures; these are unique up to a constant. At another extreme, given $x_0 \in X$ one can consider the delta-measure δ_{x_0} given by $f \mapsto f(x_0)$.

For a continuous proper map $\Phi\colon X \to Y$ to another topological space, one defines a push-forward measure $\Phi_*\varrho$ by $\int_Y g(\Phi_*\varrho) = \int_X (\Phi^* g)\varrho$. If X, Y are manifolds, Φ is smooth, and ϱ is a smooth measure, then $\Phi_*\varrho$ is a smooth measure over the set of regular values of Φ. By contrast, if Φ is a constant map onto a point y_0, then $\Phi_*\varrho$ will be a mukltiple of the delta-measure at y_0.

Hence, the Duistermaat-Heckman measure ϱ on t^* is smooth on the set of regular values of Φ. In other words, on each such component it is given by a smooth function times a 'constant' measure m on $\mathfrak{t}^*$. If the T-action on $\Phi^{-1}(U)$ is free, then this function is just the volume function $\mu \mapsto M_\mu$, up to a constant depending on the choice of m.

Exercise 9.5. Verify this claim, using the normal form.

For other components, the DH measure is 'essentially' the volume function, but M_μ must now be interpreted as an orbifold.

Theorem 9.6 (Duistermaat-Heckman). *Let (M, ω, Φ) be a compact Hamiltonian T-space. On each component of the set of regular values of Φ, the measure ϱ is a polynomial function of degree $\leq k = \frac{1}{2}\dim M - \dim T$ times the constant measure on $\mathfrak{t}^*$.*

Exercise 9.7. Use the local model to prove this theorem.

Example 9.8 (Archimedes ?). Consider the 2-sphere $S^2 \subseteq \mathbb{R}^3$ with its standard rotation invariant volume form ω of total integral 4π. Letting $\Phi\colon S^2 \to \mathbb{R}$ be projection to the z-axis, $\pi(x, y, z) = z$, we obtain the measure

$$\varrho = \Phi_*|\omega|$$

supported on $[-1, 1]$, given by the constant measure $2\pi|\mathsf{d}z|$ on $(-1, 1)$. This is easily checked in cylindrical coordinates,where

$$\omega = \mathsf{d}\phi \wedge \mathsf{d}z.$$

It may be seen as a special case of Duistermaat-Heckman since Φ can be regarded as the moment map for a Hamiltonian circle action on S^2.

9.2. **The Atiyah-Guillemin-Sternberg convexity theorem.** The convexity theorem for Hamiltonian T-spaces was proved independently by Atiyah [4] and Guillemin-Sternberg [18]. The argument presented below is similar to that in [18], see also [19].

9.2.1. *Motivation.* As a motivating example, which on first sight seems quite unrelated to symplectic geometry, consider the following problem about complex self-adjoint (i.e., Hermitian) matrices. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be an n-tuple of real numbers, and let $\mathcal{O}(\lambda)$ be the set of all self-adjoint complex $n \times n$-matrices having eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\pi : \mathcal{O}(\lambda) \to \mathbb{R}^n$ be the projection to the diagonal.

Theorem 9.9. *The image $\pi(\mathcal{O}(\lambda))$ is the convex hull*

$$\Delta = \mathrm{hull}\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}),\ \sigma \in \mathfrak{S}_n\}$$

where $\mathfrak{S}_n$ is the permutation group.

This and related results were proved by Schur and Horn, later greatly generalized by Kostant and Heckman.

The relation to symplectic geometry is as follows. First, instead of self-adjoint matrices we can equivalently consider skew-adjoint matrices, i.e. the Lie algebra $\mathfrak{g}$ of $G = \mathrm{U}(n)$. Since all matrices with given eigenvalues are conjugate, $\mathcal{O}(\lambda)$ is an orbit for the action of $\mathrm{U}(n)$. Using the inner product

$$(A, B) = \mathrm{tr}(A^\dagger B) = -\,\mathrm{tr}(AB)$$

we can also view it as a *coadjoint* orbit.

The projection π is just orthogonal projection onto the diagonal matrices, which are a maximal commutative subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$. Using the inner product to identify $\mathfrak{t} \cong \mathfrak{t}^*$ it becomes the moment map

$$\Phi \colon \mathcal{O} \to \mathfrak{t}^*$$

for the induced $T \subseteq G$ action. For this reason the Schur-Horn theorem can be viewed as a convexity theorem for Hamiltonian torus actions on coadjoint orbits of $\mathrm{U}(n)$. Nothing is special about $\mathrm{U}(n)$, analogous results hold for arbitrary compact groups. That is, if G is a compact Lie group and $T \subseteq G$ its maximal torus, then the image of $\mathcal{O} \subseteq \mathfrak{g}^*$ under projection $\mathfrak{g}^* \to \mathfrak{t}^*$ is a convex polytope given as the convex hull of $\mathcal{O} \cap \mathfrak{t}^*$. In fact, as it turns out, this type of result generalizes to moment map images for arbitrary Hamiltonian T-spaces.

9.2.2. *Moment map images of symplectic torus representations.* We need the notion of weights for a symplectic torus representation. Let T be a torus, and $\Lambda \subseteq \mathfrak{t}$ its integral lattice:

$$\Lambda = \{\xi \in \mathfrak{t} |\ \exp(\xi) = e\}.$$

A unitary representation of T on $\mathbb{C}$ is the same as a group morphism

$$T \to \mathrm{U}(1) \cong S^1 = \mathbb{R}/\mathbb{Z}$$

Its differential defines a map $\mathfrak{t} \to \mathbb{R}$, which restricts to a morphism of lattices, $\alpha : \Lambda \to \mathbb{Z}$ called the *weight* of the representation. One calls

$$\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subseteq \mathfrak{t}^*$$

the *weight lattice*. Conversely, given $\alpha \in \Lambda^*$ one defines a 1-dimensional representation $\mathbb{C}_\alpha$ by

$$\exp(\xi) \cdot z = e^{2\pi i \langle \alpha, \xi \rangle}\, z.$$

By Schur's Lemma, any unitary representation of T on a Hermitian vector space V splits into a direct sum of 1-dimensional representations[18], Thus

$$V \cong \oplus_{j=1}^{n} \mathbb{C}_{\alpha_j}$$

[18]Proof: The elements of T act as commuting operators on V. Hence they have a joint eigenvector v. The span of v is T-invariant, hence so is its orthogonal complement V'. Now proceed by induction: Pick a joint eigenvector in V', etc.

where α_j are called the weights of V. Given a *symplectic* vector space V with a symplectic T-representation, one chooses a G-invariant compatible complex structure J, which makes V into a unitary T-representation. The weights α_j for this representation are independent of the choice of J, since any two J's are deformation equivalent.[19] They are called the weights of the symplectic T-representation.

Lemma 9.10. *Let (V, ω_V, Φ_V) be a symplectic T-representation with moment map (see Proposition 7.20)*

$$\langle \Phi_V(v), \xi \rangle = -\frac{1}{2}\omega(v, \xi\, v).$$

The image of the moment map Φ_V is a convex, rational polyhedral cone spanned by minus the weights $\alpha_j \in \Lambda^*$ of the representation:

$$\Phi_V(V) = -\operatorname{cone}\{\alpha_1, \ldots, \alpha_n\}$$

The map Φ_V is open as a map onto its image, and has path connected fibers. Every open neighborhood of 0 contains a T-invariant open neighborhood U such that $\Phi_V|_U$ has path connected fibers.

Here we use the following notation, for any subset $S = \{v_1, \ldots, v_k\}$ of vectors in a real vector space:

$$\operatorname{cone}(S) = \{t_1 v_1 + \ldots t_k v_k |\ v_i \in S,\ t_i \geq 0\}$$

(the convex cone generated by S).

Proof. By the discussion above, we may assume that $V = \bigoplus \mathbb{C}_{\alpha_j}$ as a T-representation, where each summand carries the standard symplectic structure on $\mathbb{C} \cong \mathbb{R}^2$. The moment map for the $S^1 = \mathbb{R}/\mathbb{Z}$-action on $\mathbb{C} = \mathbb{R}^2$ is $-\pi|z|^2$. Hence, the moment map for the T-action on $\mathbb{C}_\alpha$ for $\alpha \in \Lambda^*$ is $\phi_\alpha \colon C_\alpha \to \mathfrak{t}^*$,

$$\langle \Phi_\alpha, \xi \rangle = -\pi \langle \alpha, \xi \rangle |z|^2.$$

The moment map for the T-action on $V = \bigoplus \mathbb{C}_{\alpha_j}$ becomes

$$\langle \Phi_V, \xi \rangle = -\pi \sum_{j-1}^{n} |z_j|^2 \alpha_j.$$

from this, we readily read off the moment map image. The second claim follows by writing Φ_V as a composition of the map

$$\Phi_0 \colon (z_1, \ldots, z_n) \mapsto (|z_1|^2, \ldots, |z_n|^2)$$

(which is easily seen to be open to its image, since the map $z \mapsto |z|^2$ is) with the linear map $(t_1, \ldots, t_n) \mapsto -\pi \sum_j t_j \alpha_j$.

[19] Proof: Recall that an inner product on V determines an ω-compatible complex structure in a canonical way. Taking the metric to be invariant (as one may, by averaging), the resulting J will be invariant. The interpolation between any two J's is obtained by interpolating between the associated metrics.

To see that the fibers are connected, we want to show the set of solutions z to $\sum |z_j|^2\alpha_j = \mu$ is connected, for any given μ. For this, it suffices to show that the solutions set of $\sum x_j\alpha_j = \mu,\ x_j \geq 0$ is connected. But this solution set is convex, and hence is connected. The same argument works if we require $\sum |z_j|^2 < \epsilon$, leading to the extra condition $\sum x_j < \epsilon$. □

9.2.3. *Local convexity.* In order to understand how images of moment maps for Hamiltonian T-spaces look like, we first have to understand how they look like "locally". We will work with the symplectic slice theorem, Theorem 8.30, giving a local model for the T-action near any orbit $\mathcal{O} = T \cdot m_0$. This theorem uses the assumption $\mathcal{O} \subseteq \Phi^{-1}(0)$, but this can be arranged by adding a constant to the moment map. Letting $H = T_{m_0}$ be the stabilizer, we arrive at the model

$$E = (T \times \mathfrak{t}^* \times V)/\!\!/H$$

where $V = (T_{m_0}\mathcal{O})^\omega / T_{m_0}\mathcal{O}$ is a symplectic H-representation, and $T \times \mathfrak{t}^*$ may be thought of as the cotangent bundle of T, or as a quotient of $\mathfrak{t} \times \mathfrak{t}^*$. The moment map for the H-action on $T^* \times \mathfrak{t}^* \times V$ is

$$(t, \tau, v) \mapsto -\operatorname{pr}_{\mathfrak{h}^*}(\tau) + \Phi_V(v);$$

its zero level set is given by the condition $\operatorname{pr}_{\mathfrak{h}^*}(\tau) = \Phi_V(v)$. On the other hand, the moment map

$$\Phi_E \colon E \to \mathfrak{t}^*$$

for the T-action on E is induced from the map $(t, \tau, v) \mapsto \Phi(m_0) + \tau$. This shows that the image of the moment map Φ_E is given by

$$\Phi_E(E) = \Phi(m_0) + (\operatorname{pr}_{\mathfrak{h}^*})^{-1}(\Phi_V(V)).$$

Lemma 9.11. *The image $\Phi_E(E) \subseteq \mathfrak{t}^*$ of the moment map for the local model is the affine polyhedral cone*

$$C_{m_0} = \Phi(m_0) - (\operatorname{pr}_{\mathfrak{h}^*})^{-1} \operatorname{cone}\{\beta_1, \ldots, \beta_k\} \tag{26}$$

Here $\beta_i \in \mathfrak{h}^*$ are the weights for the H^o-action on V. The map Φ_E is open as a map onto its image, and has connected fibers. Every neighborhood of $\mathcal{O} = G/H$ in E contains a T-invariant neighborhood U such that $\Phi_E|_U$ has connected fibers.

Proof. The map Φ_V is also the moment map for the identity component $H^o \subseteq H$, itself a torus. Hence, by Lemma 9.10 its image is of the form $-\operatorname{cone}\{\beta_1, \ldots, \beta_k\}$ where $\beta_1, \ldots, \beta_k \in \mathfrak{h}^*$ are the weights for the H-action on V. The description of $\Phi_E(E)$ follows. The connectivity properties of fibers are similarly obtained from those of Φ_V. □

We find it useful to make the following definition, for any Hamiltonian T-space (M, ω, Φ) and any $m_0 \in M$:

Definition 9.12. [40] The affine cone
$$C_{m_0} = \Phi(m_0) - (\mathrm{pr}_{\mathfrak{h}^*})^{-1}(\mathrm{cone}\{\beta_1, \dots, \beta_k\})$$
where $\beta_i \in \mathfrak{h}^*$ are the weights of the H^o-action on the symplectic vector space $T_m(T \cdot m_0)^\omega / T_m(T \cdot m_0)$, is called the *local moment cone* at $m_0 \in M$.

Note that the local moment cone always contains the affine subspace
$$\Phi(m_0) + \mathrm{ann}(\mathfrak{t}_{m_0}) \subseteq \mathfrak{t}^*.$$

To summarize the discussion, we obtain:

Theorem 9.13 (Local convexity theorem)**.** *Let (M, ω, Φ) be a Hamiltonian T-space, $m_0 \in M$. Then there exists a T-invariant open neighborhood U of $T \cdot m_0$ such that*

- *$\Phi|_U$ is an open map to C_{m_0},*
- *the fibers of $\Phi|_U$ are connected.*

In the first item, the openness refers to the relative topology of C_{m_0}. In particular, $\Phi(U)$ is an open neighborhood of the vertex $m_0 \in C_{m_0}$.

9.2.4. *The global convexity theorem.* The global version of the convexity theorem is the following result, obtained independently by Atiyah [4] and Guillemin-Sternberg [18]

Theorem 9.14 (Atiyah, Guillemin-Sternberg)**.** *Let (M, ω, Φ) be a compact connected Hamiltonian T-space. Then all fibers of Φ are connected, and*
$$\Delta = \Phi(M)$$
is a convex polytope. In fact, it is the convex hull of the image of the fixed point set:
$$\Phi(M) = \mathrm{hull}(\Phi(M^T)).$$

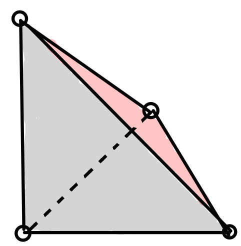
Example 9.15. Consider the complex projective space $M = \mathbb{C}P(n)$ with its standard Fubini-Study symplectic structure (as a symplectic reduction of $\mathbb{C}^n$). The moment map for the $\mathrm{U}(n+1)$-action is $\Psi \colon \mathbb{C}P(n) \to \mathfrak{u}(n+1)^* \cong \mathfrak{u}(n+1)$ where
$$\Psi([z])_{ij} = c\frac{1}{||z||^2} z_i \overline{z}_j,$$
where c is a non-zero scalar depending on the choice of identification $\mathfrak{u}(n+1)^* \cong \mathfrak{u}(n+1)$. Restrict to the action of $U(1)^n \subseteq \mathrm{U}(n+1)$, corresponding to diagonal matrices with entries $(1, u_1, \dots, u_n)$ down the diagonal where $|u_i| = 1$. The corresponding moment

map is

$$[z] \mapsto c\frac{1}{||z||^2}(|z_1|^2, \ldots, |z_n|^2).$$

Ignoring c, its image is the *standard n-simplex* in $\mathbb{R}^n$, spanned by the origin together with the standard basis vectors:

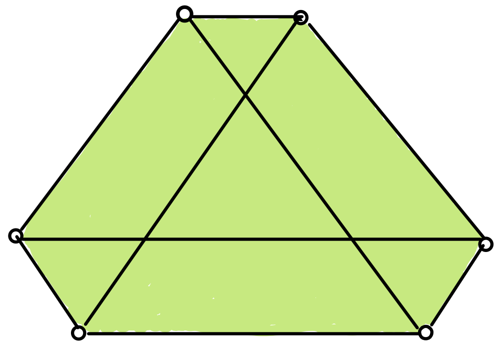
$$(0, \ldots, 0), (1, 0, \ldots, 0), (0, 1, \ldots, 0), (0, 0, \ldots, 1).$$



Example 9.16. Let G be a compact connected Lie group, and $T \subseteq G$ a torus (e.g., a maximal torus). Recall that coadjoint orbits $\mathcal{O} \subseteq \mathfrak{g}^*$ are Hamiltonian G-spaces, with moment map the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$. Restricting the action, it becomes a Hamiltonian T-space, with moment map the inclusion followed by projection $\mathfrak{g}^* \to \mathfrak{t}^*$ (dual to $\mathfrak{t} \hookrightarrow \mathfrak{g}$). The convexity theorem says that $\mathrm{pr}_{\mathfrak{t}^*}(\mathcal{O}) \subseteq \mathfrak{t}^*$ is a convex polytope, and is the convex hull of $\mathrm{pr}_{\mathfrak{t}^*}(\mathcal{O}^T)$. If T is a *maximal* torus, it is known that $\mathfrak{g}^T$ is exactly $\mathfrak{t}$. Dually, $(\mathfrak{g}^*)^T$ is identified with $\mathfrak{t}^*$. In terms of this inclusiion $\mathfrak{t} \to \mathfrak{g}^*$, we have that $\mathcal{O}^T = \mathcal{O} \cap \mathfrak{t}^*$, and the polytope is

$$\Delta = \mathrm{hull}(\mathcal{O} \cap \mathfrak{t}^*).$$

For example, if $G = \mathrm{U}(n)$, with $T \subseteq \mathrm{U}(n)$ the diagonal matrices, the coadjoint orbits are identified with self-adjoint matrices with a given set of eigenvalues, and $\mathcal{O} \cap \mathfrak{t}^*$ corresponds to diagonal matrices with entries these eigenvalues (in any order). As a standard example , for $n = 3$, consider the (co)-adjoint orbit of the matrix $A = i\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{u}(3)$ for some $\lambda_1 > \lambda_2 > \lambda_3$. This is of the form $\mathcal{O} = \mathrm{U}(3)/T$. The T-fixed points are given by diagonal matrices with entries any permutation of $\lambda_1, \lambda_2, \lambda_3$; there are six of them. The image of the moment map is the convex hull of these points; note that it is contained in the 2-dimensional affine subspace of $\mathbb{R}^3$ defined by $x_1 + x_2 + x_3 = \lambda_1 + \lambda_2 + \lambda_3$. The resulting polytope looks something like this:



Here the vertices correspond to the fixed points. The edges correspond to matrices with spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$, having one of the standard basis vectors e_1, e_2, e_3 as an eigenvector, with eigenvalue from this set. (There are 9 possibilities, but 3 of those edges lie in the interior of the polytope.

Example 9.17 (Sub-example). Recall that the Grassmannian

$$\mathrm{Gr}(2,4)$$

of 2-dimensional subspaces in C^4 is a coadjoint orbit $\mathcal{O}$ under the action of SU(4). Under the identification of $\mathfrak{su}(4)^* \cong \mathfrak{su}(4)$ with the space of trace-free self-adjoint 4×4-matrices, we may take $\mathcal{O}$ to be the set of matrices A with eigenvalues ± 1, each with multiplicity 2. The correspondence with $\mathrm{Gr}(2,4)$ assigns to any such matrix the space $E = \ker(A + I)$.

Let $T \subseteq G$ be the maximal torus, consisting of diagonal matrices $\mathrm{diag}(z_1, z_2, z_3, z_4)$ with $|z_i| = 1$ and $\prod z_i = 1$. A choice of lattice basis identifies $\mathfrak{t} = \mathbb{R}^3$, hence Δ will be a convex polytope in $\mathfrak{t}^* = \mathbb{R}^3$.

For $I \subseteq \{1,2,3,4\}$ let $\mathbb{C}^I$ be the corresponding coordinate subspace. The fixed point set for the action on the Grassmannian are the coordinate subspaces such that I has cardinality 2. There are six of them:

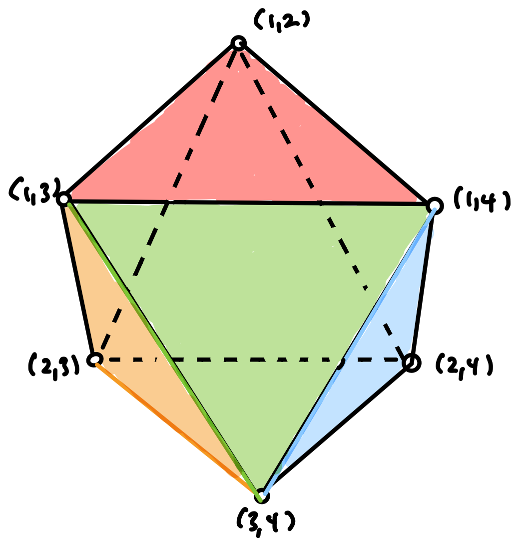
$$\mathbb{C}^{(1,2)},\ \mathbb{C}^{(1,3)},\ \mathbb{C}^{(1,4)},\ \mathbb{C}^{(2,3)},\ \mathbb{C}^{(2,4)},\ \mathbb{C}^{(3,4)}.$$

If I, J have cardinality 2, $I \cap J$ has cardinality 1, then $I \cup J$ has cardinality 3, and the 2-dimensional subspaces E satisfying

$$\mathbb{C}^{I\cap J} \subsetneq E \subsetneq \mathbb{C}^{I+J}$$

are preserved under a 2-dimensional subgroup of T. Thus T_E is 2-dimensional, for E is viewed as a point of $\mathrm{Gr}(2,4)$. This corresponds to edges in the polytope, connecting the vertices I, J. (For example, $\mathbb{C}^{(1,2)}, \mathbb{C}^{(1,3)}$ are connected by a 1=parameter family of 2-dimensional subspaces of $\mathbb{C}^{(1,2,3)}$, all containing the axis $\mathbb{C}^{(1)}$.

Finally, a 2-dimensional subspace E containing a coordinate line $\mathbb{C}^{(i)}$ but not contained in any 3-dimensional coordinate subspace has a 1-dimensional stabilizer, and corresponds to a face of the polytope. Likewise, 2-dimensional subspaces that are contained in a 3-dimensional coordinate subspace (but not containing a lower-dimensional coordinate subspace) have 1-dimensional stabilizer as well. The resulting polytope is an octahedron (6 vertices, 12 edges, and 8 faces).



We shall outline two proofs of global convexity. The first is due to Condevaux-Dazord-Molino [8], the second due to Guillemin-Sternberg. (Atiyah's argument is very instructive as well, but we won't explain it here.)

Remark 9.18. A general *local-to-global-principle*, deducing the global convexity result from local convexity (and generalizing the argument in [8]), was formulated by Hilgert-Neeb-Planck [20]. See also [14, 22] for further developments.

9.2.5. *Condevaux-Dazord-Molino argument.* The first step is the following stability property of local moment cones.

Proposition 9.19 (Condevaux-Dazord-Molino [8], Sjamaar [40])**.** *Let* (M, ω, Φ) *be a Hamiltonian* T*-space. Then all points in a path component of a given fiber* $\Phi^{-1}(\mu)$ *have the same local moment cones:*

$$\Phi(m_1) = \Phi(m_0) \quad \Rightarrow C_{m_0} = C_{m_1}.$$

Proof. Consider first the linear version: Given a representation of T on $V = \mathbb{C}^n$ with weights $\alpha_1, \ldots, \alpha_n$, and given any $v \in \Phi^{-1}(0)$, we will show that

$$C_v = C_0 = -\operatorname{cone}\{\alpha_1, \ldots, \alpha_n\}.$$

Recall that the moment map is given by

$$\Phi(z_1, \ldots, z_n) = -\pi \sum_i |z_i|^2 \alpha_i,$$

Let $H = T_v$, with Lie algebra $\mathfrak{h}$ obtained from the weights as

$$\mathrm{ann}(\mathfrak{h}) = \mathrm{span}\{\alpha_i |\ |z_i|^2 \neq 0\}.$$

Renumbering the coordinates, we may assume that $\alpha_i \in \mathrm{ann}(\mathfrak{h})$ for $i \leq k$ and $\alpha_i \notin \mathrm{ann}(\mathfrak{h})$ for $i > k$. Let $V' \cong \mathbb{C}^k,\ V'' \cong \mathcal{V}^{n-k}$ be the corresponding coordinate subspaces.

Since $\exp(\xi)$ acts by scalars $e^{2\pi i\langle\alpha_i,\xi\rangle}$ on the i-th summand of $\mathbb{C}^n$, we see that V' is exactly the subspace fixed by H^o (the identity component of H). In particular,

$$T \cdot v \subseteq V' \oplus \{0\}.$$

The symplectic normal space

$$W = T_v(T \cdot v)^\omega / T_v(T \cdot v)$$

is the direct sum $W = W' \oplus W''$, where W' is the symplectic normal space of $T \cdot v$ as a submanifold of V' while $W'' = V''$ regarded as an H-representation. Since H^o acts *trivially* on V', the corresponding weights for the action of H (if any) are all zero. On the other hand, the normal weights for the H^o-action on V'' are the projections $\beta_i = \mathrm{pr}_{\mathfrak{h}^*}(\alpha_i),\ i > k$. This shows that

$$C_v = -\,\mathrm{pr}_{\mathfrak{h}*}^{-1}\,\mathrm{cone}\{\beta_{k+1}, \ldots, \beta_n\}$$

which we may also write as

$$C_v = \mathrm{ann}(\mathfrak{h}) - \mathrm{cone}\{\alpha_{k+1}, \ldots, \alpha_n\}.$$

But $\mathrm{ann}(\mathfrak{h}) = \mathrm{span}\{\alpha_1, \ldots, \alpha_k\}$. Using again the condition $\Phi(v) = 0$, we have $\sum_{i=1}^k |z|_i^2 \alpha_i = 0$. Hence, each α_i with $i \leq k$ is a positive linear combination of $-\alpha_1, \ldots, -\alpha_k$. This shows

$$\mathrm{ann}(\mathfrak{h}) = \mathrm{span}\{\alpha_1, \ldots, \alpha_k\} = -\,\mathrm{cone}\{\alpha_1, \ldots, \alpha_k\}$$

and completes the proof of $C_v = C_0$. The proof for linear T-actions implies the analogous result for the local models $E = (T \times \mathfrak{t}^* \times V)/\!\!/H$; the general case follows by passing to the local models. ☐

Proof of global convexity theorem, Theorem 9.14 (after [8]*).* We may assume, with no loss of generality, that the generic stabilizer for the T-action on M is trivial. This then means that the local moment cones have dimension equal to $\dim T$.

Introduce an equivalence relation on M, by declaring $m_0 \sim m_1$ whenever m_0, m_1 are in the same path component of a fiber of Φ. Let

$$\check{M} = M/\sim$$

the set of equivalence classes, with quotient map denoted $\pi\colon M \to \check{M}$. The moment map descends to a map on the quotient:

$$\check{\Phi}\colon \check{M} \to \mathfrak{t}^*.$$

By Proposition 9.19 the local moment cone C_m of $m \in M$ depends only on its image $x = \check{m} \in \check{M}$, and will be denoted C_x.

Claim. The quotient $\check{M}$ is Hausdorff space, and admits a finite covering by 'charts' $\check{U}$ around points $x \in \check{M}$ such that $\check{\Phi}$ restricts to a bijection from $\check{U}$ onto an open neighborhood of the vertex $\check{\Phi}(x)$ inside C_x. (Using the relative topology).

Proof of claim: By compactness, each fiber $\Phi^{-1}(\mu)$ has a finite number of components, corresponding to the elements $x \in \check{\Phi}^{-1}(\mu)$. Choose disjoint open neighbourhoods U_x^1 of these components. By compactness, and using the local convexity theorem, each component $\pi^{-1}(x) \subseteq \Phi^{-1}(\mu)$ may be covered by finitely many open subsets

$$U_i \subseteq U_x^1$$

with the property that $\Phi|_{U_i}$ has connected fibers and $\Phi(U_i)$ is an open neighborhood of the vertex of C_x. Let $O_x \subseteq C_x$ be an open neighborhood of the vertex, with the property that $\mathcal{O}_x \subseteq \Phi(U_i)$ for all i, and also $O_x \subseteq \Phi(U_i \cap U_j)$ for every pair i, j such that $U_i \cap U_j$ meets $\pi^{-1}(x)$. Let

$$U_x = \bigcup_i U_i \cap \pi^{-1}(O).$$

Then U_x is an open neighborhood of $\pi^{-1}(x)$, with $\Phi(U_x) = O$, and $\Phi|_{U_x}$ has connected fibers. We let $\check{U}_x = \pi(U_x)$. By construction, $\check{\Phi}$ restricts to a bijection $\check{U}_x \to O$. The U_x corresponding to distinct elements of $\pi^{-1}(\mu)$ are disjoint; as a consequence the corresponding sets $\check{U}_x$ for distinct elements of $\pi^{-1}(\mu)$ are disjoint. [20] This implies the Hausdorff property, and finishes the proof of the claim.

We might call $\check{M}$ a *polyhedral manifold* of dimension $\dim T$. It is not technically a 'manifold with corners', since the cones are not spanned by linearly independent vectors in general, but we may still speak of smooth functions etc on $\check{M}$. The map $\check{\Phi}$ is a morphism of affine polyhedral manifolds of the same dimension, and has maximal rank everywhere. Hence, it is a local diffeomorphism onto its image.

To prove the convexity theorem, it remains to show that $\check{\Phi}$ is actually a bijection onto its image. (This will then show that the image is a compact and locally convex polyhedral subset of $\mathfrak{t}^*$, and hence is convex.) This fact is certainly unsurprising since we are dealing with an affine map, and since $\check{M}$ is compact. The detailed arguments in [8] are somewhat technical, though, and we will skip this part. □

[21]

9.2.6. *Guillemin-Sternberg argument.* The original argument of Guillemin-Sternberg [18] proceeded by first proving connectivity of the fibers of Φ. In turn, this was shown by looking at Morse

[20] In more detail, if x_n is a sequence with $x_n \to x$, and $m_n \in \pi^{-1}(x_n)$, we claim that $m_n \in U_x$ for n sufficiently large. If not, we could construct a subsequence with $m_n \notin U_x$ for all n. A subsequence of this subsequence converges. But then the limit m_∞ most be in $M - U_x$, a contradiction to $\pi(m_\infty) = x$.

[21] This is roughly how far we got in Fall 2024. The rest of these notes are from 2000, and have not been revised.

We now come to the key observation of . Given $\xi \in \mathfrak{t}$ consider the corresponding component $\Phi^\xi = \langle \Phi, \xi \rangle$ of the moment map. A value $s \in \mathbb{R}$ is called a local minimum for Φ^ξ if there exists $m \in M$ with $\Phi^\xi(m) = s$ and $\Phi^\xi \geq s$ on some neighborhood.

Lemma 9.20 (Guillemin-Sternberg)**.** *Let (M, ω, Φ) be a compact connected Hamiltonian T-space. Then all fibers of Φ are connected. Moreover, the function Φ^ξ has a unique local minimum/maximum.*

We will prove this Lemma in the next section. For any subset $S \subseteq \mathfrak{t}^*$ and $\mu \in \mathfrak{t}^*$ let

$$\mathrm{cone}_\mu(S) = \{\mu + t(\nu - \mu) | \, \nu \in S\}$$

be the cone over S at μ.

Proof of global convexity, after Guillemin-Sternberg. Since local convexity of a compact set implies global convexity it suffices to prove

$$C_m = \mathrm{cone}_{\Phi(m)}(\Delta). \tag{27}$$

The inclusion $\supset$ follows from local convexity. To see the opposite inclusion, we define, for all $\xi \in \mathfrak{t}$, the affine linear functional $f_\xi = \langle \cdot, \xi \rangle - \langle \mu, \xi \rangle$ on $\mathfrak{t}^*$. We have to show that for all ξ,

$$f_\xi|_{C_m} \geq 0 \Rightarrow f_\xi|_\Delta \geq 0.$$

But $f_\xi \geq 0$ on C_m means, by the model, that $\langle \mu, \xi \rangle$ is a local minimum for Φ^ξ. By the lemma, this has to be a global minimum, or equivalently $f_\xi \geq 0$ on Δ. $\square$

We obtain the following description of the faces and the "fine structure" of Δ. Let $H \subseteq T$ be in the (finite) list of stabilizer groups, and M_H the points with stabilizer H. Recall again that M_H is an open subset of the symplectic submanifold M^H. Each connected component of M^H is a Hamiltonian T-space in its own right, with H acting trivially. Thus its moment map image is a convex polytope of dimension $\dim(T/H)$ inside an affine subspace $\mu + \mathrm{ann}(\mathfrak{h})$, with the corresponding component of M_H mapping to its interior. That is, the (open) faces of Δ correspond to orbit type strata, and in particular the vertices of Δ correspond to fixed points M^T. That is,

$$\Delta = \mathrm{hull}(\Phi(M^T))$$

is the convex hull of the fixed point set. Note however that some of the polytopes $\Phi(M^H)$ get mapped to the *interior* of Δ. Thus Δ gets subdivided into polyhedral subregions, consisting of regular values of Φ.

Theorem 9.21. *Let (M, ω, Φ) be a Hamiltonian T-space, with T acting effectively, and $\Delta \subseteq \mathfrak{t}^*$ its moment polytope. For any closed face Δ_i of Δ of codimension d_i, the pre-image $\Phi^{-1}(\Delta_i)$ is symplectic, and is a connected component of the fixed point set for some d_i-dimensional stabilizer group $H_i \subseteq T$ where $\mathfrak{h}_i$ is the subspace orthogonal to Δ_i. In particular, the vertices of Δ correspond to fixed point manifolds.*

Proof. We note that each $\Phi^{-1}(\Delta_i) \subseteq M$ is closed and connected, by connectedness of the fibers of Φ. Hence it is a connected component of some M^{H_i}, where $\mathrm{ann}(\mathfrak{h}_i)$ is parallel to Δ_i. □

In particular, Hamiltonian torus actions on compact symplectic manifolds are *never fixed point free*. (This shows immediately that the standard 2k-torus action on itself cannot be Hamiltonian.)

Exercise 9.22. Let (M, ω, Φ) be a compact, connected Hamiltonian T-space where T acts effectively. Let $M_* = M_{\{e\}}$ be the subset on which the action is free. Show that M_* is connected, and that its image $\Phi(M_*)$ is precisely the interior of the moment polytope $\Delta = \Phi(M)$.

Let us assume that the image of Φ contains regular values. The images of the fixed point manifolds for non-trivial stabilizer algebras define a subdivison of the polytope Δ into chambers, given as the connected components of the set of regular values of Φ. By the Duistermaat-Heckman theorem, the Duistermaat-Heckman measure

$$\varrho = \Phi_* \left| \frac{\omega^n}{n!} \right|$$

is polynomial on each of these chambers.

Remark 9.23. Duistermaat-Heckman [13] used this fact to derive a remarkable "exact integration formula", which we will in Section 9.4.

9.3. **Some basic Morse-Bott theory.** The proof of the fact that every component $f = \Phi^\xi$ of the moment map has a unique local minimum relies on the idea of viewing f as a Morse-Bott function. For any function $f \in C^\infty(M, \mathbb{R})$ on a manifold M, the set of critical points is the closed subset

$$\mathcal{C} = \{m |\, \mathsf{d}f(m) = 0\}.$$

For all $m \in \mathcal{C}$ there is a well-defined symmetric bilinear form on T_mM, called the Hessian

$$\mathsf{d}^2 f(m)(X_m, Y_m) = (L_X\, L_Y\, f)(m)$$

for all $X, Y \in \mathrm{Vect}(M)$. In local coordinates, the Hessian is simply given by the matrix of second derivatives of f.

The function f is called a Morse function if $\mathcal{C}$ is discrete and for all $m \in \mathcal{C}$ the Hessian is non-degenerate. More generally, f is called Morse-Bott if the connected components $\mathcal{C}^j$ of $\mathcal{C} = \{m |\, \mathsf{d}f(m) = 0\}$ are smooth manifolds, and for all $m \in \mathcal{C}^j$ we have

$$\ker(\mathsf{d}^2 f(m)) = T_m\mathcal{C}^j.$$

Given a Riemannian metric on M, consider the negative gradient flow of f, i.e. the flow F^t of the vector field $-\nabla(f) \in \mathrm{Vect}(M)$. For all connected components $\mathcal{C}^j$ we can consider the sets

$$W_+^j = \{m \in M,\ \lim_{t\to\infty} F^t(m) \in \mathcal{C}^j\}$$

and

$$W_-^j = \{m \in M,\ \lim_{t\to\infty} F^t(m) \in \mathcal{C}^j\}.$$

If f is Morse-Bott then all $W_\pm^j$ are smooth manifolds, and one has natural finite decompositions

$$M = \cup_j W_-^j = \cup_j W_+^j$$

into unstable/stable manifolds. The dimension of W_-^j (resp. W_+^j) is equal to the dimension of $\mathcal{C}^j$ plus the dimension of the negative eigenspace of $\mathrm{Hess}(f)$, denoted $n_\pm^j$. Thus

$$n_\mp^j = \mathrm{codim}(W_\pm^j).$$

The number n_-^j is called the index of $\mathcal{C}^j$.

Proposition 9.24. *If none of the indices n_-^j is equal to 1, there exists a unique critical manifold of index 0, i.e. a unique local minimum of f. If moreover all $n_+^j \neq 1$ then all level sets $f^{-1}(c)$ are connected.*

Proof. The condition $n_-^j \neq 1$ means that all W_j^+ of positive index have codimension at least 2, so that their complement is connected. Hence there is a unique stable manifold W_j^+ with $n_j^- = 0$. If in addition $n_+^j \neq 1$, the set M_* obtained from M by removing all M_+^j with $n_-^j > 0$ and all M_-^j with $n_+^j > 0$ is open, dense and connected in M. Notice that M_* consists of all points which flow to the (unique) minimum of f for $t \to \infty$ and to the (unique) maximum of f for $t \to -\infty$. If $\min(f) < c < \max(f)$ then every trajectory of the gradient flow of a point in M_* intersects $f^{-1}(c)$ in a unique point. Therefore the map

$$(f^{-1}(c) \cap M_*) \times \mathbb{R} \to M_*,\ (m,t) \to F^t(m)$$

is a diffeomorphism, and in particular $f^{-1}(c)\cap M_*$ is connected. To prove the proposition it suffices to show that $f^{-1}(c)\cap M_*$ is dense in $f^{-1}(c)$. Let $m \in f^{-1}(c)$ and U a connected open neighborhood of m. Since c is neither maximum or minimum, $U \cap M^*$ meets both the sets where $f < c$ and $f > c$, and since it is connected it meets $f^{-1}(c)$. ☐

Returning to the symplectic geometry context, we need to show:

Theorem 9.25. *Let (M, ω, Φ) be a Hamiltonian G-space, $\xi \in \mathfrak{g}$. Then $f = \Phi^\xi$ is a Morse-Bott function. Moreover all critical manifolds $\mathbb{C}^j$ are symplectic submanifolds of M, and the indices n_-^j are all even.*

Proof. Let $H \subseteq G$ be the closure of the 1-parameter subgroup generated by ξ. Then H is a torus. The critical set of f is given by the condition

$$0 = \mathsf{d}\langle \Phi, \xi\rangle(m) = \iota(\xi_M(m))\omega_m.$$

Since ω is non-degenerate, it is precisely the set of zeroes of the vector field ξ_M, or equivalently the fixed point set for the 1-parameter subgroup $\{\exp(t\xi)|\, t \in \mathbb{R}\} \subseteq G$. Let

$$H = \overline{\{\exp(t\xi)|\, t \in \mathbb{R}\}}.$$

then H is abelian and connected, hence is a torus, and $\mathbb{C}$ is just the set of fixed points for this torus action. Let $m \in \mathbb{C}$, and equip $T_m M = V$ with an H-invariant compatible complex structure. As a unitary representation, V is equivalent to $V = \oplus \mathbb{C}_{\alpha_j}$ where α_j are the weights for the action. By the equivariant Darboux-theorem, V serves as a model for the H-action near m. In particular the fixed point manifold $\mathbb{C} = M^H$ gets modeled by the space of fixed vectors V^H, which is a complex, hence also symplectic subspace. This shows that all $\mathbb{C}_j$ are symplectic manifolds. Moreover the moment map in this model is (a constant plus)

$$z \mapsto \pi \sum_j |z_j|^2 \alpha_j = \pi \sum_j (q_j^2 + p_j^2)\ \alpha_j,$$

in particular

$$f = \pi \sum_j |z_j|^2 \alpha_j = \pi \sum_j (q_j^2 + p_j^2)\ \langle \alpha_j, \xi \rangle.$$

From this it is evident that f is Morse-Bott and that all indices are even. □

The fact that all indices are even has very strong implications in Morse theory: It implies that the so-called lacunary principle applies, and the Morse-Bott polynomial is equal to the Poincare polynomial. (I.e. the Morse inequalities are equalities – Morse functions for which this is the case are called *perfect*.) This gives a powerful tool to calculate the cohomology of Hamiltonian G-spaces: in particular for isolated fixed points, this gives

$$\dim H^k(M, \mathbb{Q}) = \#\{\text{ critical points of index } k\ \ \};$$

in particular all cohomology sits in even degree if all indices are even.

Corollary 9.26. *Suppose M admits a Morse-Bott function f such that the minimum of f is an isolated point and all $n_-^j \neq 1$. Then M is simply connected.*

Proof. Given any $m \in X$ and a loop $\gamma \in X$ based at m, one can always perturb γ so that it does not meet the stable manifolds of index > 0. Applying the gradient flow to γ contracts γ to the minimum. □

Examples are coadjoint orbits of a compact Lie group (the fact that coadjoint orbits are compact submanifolds of a vector space allows one to show that for generic components of the moment map the minimum is isolated.) Thus coadjoint orbits are simply connected. (We remark that this is not true in general for conjugacy classes.). Let G/G_μ be a coadjoint orbit where G is compact, connected. View G_μ as the fiber over the identity coset. Given any two points in G_μ they can be joined by a path in G. The projection to G/G_μ is a closed path, hence can be contracted. Lifting the contraction to G produces a path in G_μ connecting the two points. Thus all stabilizer groups for the (co)-adjoint action are connected.

9.4. **Localization formulas.** Let (M,ω,Φ) be a compact Hamiltonian T-space. For simplicity we assume that the set M^T of fixed points is finite. (This is for example the case for the action of a maximal torus $T\subseteq G$ on a coadjoint orbit $\mathcal{O}=G\cdot\mu$.) Given $p\in M^T$ let $a_1(p),\ldots,a_n(p)\in\Lambda^*\subseteq\mathfrak{t}^*$ be the weights for the action on T_pM.

Theorem 9.27 (Duistermaat-Heckman)**.** *Let $\xi\in\mathfrak{t}^{\mathbb{C}}$ be such that $\langle a_j(p),\xi\rangle\neq 0$ for all p,j. Then one has the exact integration formula*

$$\int_M e^{\langle\Phi,\xi\rangle}\ \frac{\omega^n}{n!}=\sum_{p\in M^T}\frac{e^{\langle\Phi(p),\xi\rangle}}{\prod_j\langle a_j(p),\xi\rangle}.$$

One way of looking at this result is to say that the stationary phase approximation for the integral $\int_M e^{it\langle\Phi,\xi\rangle}\ \frac{\omega^n}{n!}$ is *exact*!

Our proof of the DH-formula will follow an argument of Berline-Vergne [7]. Notice first that the integrand is just the top form degree part of

$$e^{\omega+\langle\Phi,\xi\rangle}=e^{\langle\Phi,\xi\rangle}\sum_{j=0}^{n}\frac{\omega^j}{j!}\in\Omega^*(M).$$

Consider the derivation

$$\mathsf{d}_\xi:\ \Omega^*(M)\to\Omega^*(M),\mathsf{d}_\xi:=\mathsf{d}-\iota(\xi_M).$$

The differential form $\omega+\langle\Phi,\xi\rangle$ is d_ξ-closed, i.e. killed by d_ξ:

$$\mathsf{d}_\xi(\omega+\langle\Phi,\xi\rangle)=-\iota(\xi_M)\omega+\mathsf{d}\langle\Phi,\xi\rangle=0.$$

Moreover $\alpha:=e^{\omega+\langle\Phi,\xi\rangle}$ is d_ξ-closed as well. Berline-Vergne prove the following generalization of the DH-formula:

Theorem 9.28. *Let M be a compact, oriented T-manifold with isolated fixed point set. Given $p\in M^T$ let $a_j(p)$ be the weights for the action on T_pM, defined with respect to some choice of T-invariant complex structure on T_pM. Suppose $\xi_M\neq 0$ on $M\backslash M^T$. Then for all forms $\alpha\in\Omega^*(M)$ such that $\mathsf{d}_\xi\alpha=0$, one has the integration formula*

$$\int_M\alpha_{[\dim M]}=\sum_{p\in M^T}\frac{\alpha_{[0]}(p)}{\prod_j\langle a_j(p),\xi\rangle}.$$

In the proof we will use the useful notion of *real blow-ups*. Consider first the case of a real vector space V. Let

$$S(V)=V\backslash\{0\}/\mathbb{R}_{>0}$$

be its sphere, thought of as the space of rays based at 0. Define $\hat{V}$ as the subset of $V \times S(V)$,

$$\hat{V} := \{(v, x) \in V \times S(V) | \, v \text{ lies on the ray parametrized by } x\}.$$

Then $\hat{V}$ is a manifold with boundary. (In fact, if one introduces an inner product on V then $\hat{V} = S(V) \times \mathbb{R}_{\geq 0}$). There is a natural smooth map $\pi : \hat{V} \to V$ which is a diffeomorphism away from $S(V)$. If M is a manifold and $m \in M$, one can define its blow-up $\pi : \hat{M} \to M$ by using a coordinate chart based at m. Just as in the complex category, one shows that this is independent of the choice of chart (although this is actually not important for our purposes).

Suppose now that M is a T-space as above. Let $\pi : \hat{M} \to M$ be the manifold with boundary obtained by real blow-up at all the fixed points M^T. The T-action on M lifts to a T-action on $\hat{M}$ *with no fixed points*. In particular $\xi_{\hat{M}}$ has no zeroes. Choose an invariant Riemannian metric g on $\hat{M}$, and define

$$\theta := \frac{g(\xi_{\hat{M}}, \cdot)}{g(\xi_{\hat{M}}, \xi_{\hat{M}})} \in \Omega^1(\hat{M}).$$

Then θ satisfies $\iota(\xi_{\hat{M}})\theta = 1$ and $\mathsf{d}_\xi^2 \theta = L_{\xi_M} \theta = 0$. Therefore

$$\gamma := \frac{\theta}{\mathsf{d}_\xi \theta} = \frac{\theta}{\mathsf{d}\theta - 1} = -\theta \wedge \sum_j (\mathsf{d}\theta)^j$$

is a well-defined form satisfying $\mathsf{d}_\xi \gamma = 1$. The key idea of Berline-Vergne is to use this form for partial integration:

$$\begin{aligned}
\int_M \alpha &= \int_{\hat{M}} \pi^* \alpha \\
&= \int_{\hat{M}} \pi^* \alpha \wedge \mathsf{d}_\xi \gamma \\
&= \int_{\hat{M}} \mathsf{d}_\xi (\pi^* \alpha \wedge \gamma) \\
&= \int_{\hat{M}} \mathsf{d} (\pi^* \alpha \wedge \gamma) \\
&= \sum_{p \in M^T} \int_{S(T_p M)} \pi^* \alpha \wedge \gamma \\
&= \sum_{p \in M^T} \alpha_{[0]}(p) \int_{S(T_p M)} \gamma
\end{aligned}$$

Thus, to complete the proof we have to carry out the remaining integral over the sphere. We will do this by a trick, defining a d_ξ-closed form α where we can actually compute the integral by hand.

Consider the T-action on $T_pM = \sum_{j=1}^n \mathbb{C}_{a_j(p)}$ for a given $p \in M^T$. Introduce coordinates $r_j \geq 0, t_j \in [0,1]$ by $z_j = r_j\, e^{2\pi i t_j}$. Given $\epsilon > 0$ let $\chi \in C^\infty(\mathbb{R}_{\geq 0})$ be a cut-off function, with $\chi(r) = 1$ for $r \leq \epsilon$ and $\sigma = 0$ for $\epsilon \geq 2$. Define a form

$$\alpha = \prod_{j=1}^n (-\mathsf{d}_\xi(\chi(r_j)\,\mathsf{d}t_j)) = \prod_{j=1}^n \big(\langle a_j(p), \xi\rangle - \chi'(r_j)\mathsf{d}r_j \wedge \mathsf{d}t_j\big).$$

Note that this form is well-defined (even though the coordinates are not globally well-defined), compactly supported and d_ξ-closed. Its integral is equal to

$$\int_{T_pM} \alpha = \prod_{j=1}^n (-\chi'(r_j)\mathsf{d}r_j) = 1.$$

On the other hand $\alpha_{[0]} = \prod_{j=1}^n (\langle a_j(p), \xi\rangle)$.

Choosing ϵ sufficiently small, we can consider α as a form on M, vanishing at all the other fixed points. Applying the localization formula we find

$$1 = \int_M \alpha = \prod_{j=1}^n (\langle a_j(p), \xi\rangle) \int_{S(T_pM)} \gamma,$$

thus

$$\int_{S(T_pM)} \gamma = \frac{1}{\prod_{j=1}^n \langle a_j(p), \xi\rangle}.$$

Q.E.D.

The above discussion extends to non-isolated fixed points, in this case the product $\prod_{j=1}^n \langle a_j(p), \xi\rangle$ is replaced by the equivariant Euler class of the normal bundle of the fixed point manifold.

One often applies the Duistermaat-Heckman theorem in order to compute Liouville volumes of symplectic manifolds with Hamiltonian group action. Consider for example a Hamiltonian $S^1 = \mathbb{R}/\mathbb{Z}$-action with isolated fixed points. Identify Lie(S^1), so that the integral lattice and its dual are just $\Lambda = \mathbb{Z}$, $\Lambda^* = \mathbb{Z}$. Let $H = \langle \Phi, \xi\rangle$ where ξ corresponds to $1 \in \mathbb{R}$. By Duistermaat-Heckman,

$$\int_M e^{tH}\frac{\omega^n}{n!} = \frac{1}{t^n} \sum_{p \in M^{S^1}} \frac{e^{tH(p)}}{\prod_j a_j(p)}.$$

Notice by the way that the individual terms on the right hand side are singular for $t = 0$. This implies very subtle relationships between the weight, for example one must have

$$\sum_{p \in M^{S^1}} \frac{H(p)^k}{\prod_j a_j(p)} = 0$$

for all $k < n$. For the volume one reads off,

$$\mathrm{Vol}(M) = \frac{1}{n!} \sum_{p \in M^{S^1}} \frac{H(p)^n}{\prod_j a_j(p)}.$$

9.5. **Frankel's theorem.** As we have seen, Hamiltonian torus actions are very special in many respects: In particular they always have fixed points. It is a classical result of Frankel [15] (long before moment maps were invented) that on Kähler manifolds the converse is true:

Theorem 9.29. *Let M be a compact Kähler manifold, with Kähler form ω. Consider a symplectic S^1-action on M with at least one fixed point. Then the action is Hamiltonian.*

Proof. Let $\dim M = 2n$. We need one non-trivial result from complex geometry, which is a particular case of the hard Lefschetz theorem: Wedge product with ω^{n-1} induces an isomorphism in cohomology,

$$\wedge\, [\omega]^{n-1}: \; H^1(M) \cong H^{2n-1}(M).$$

Let $X \in \mathrm{Vect}(M)$ be the vector field corresponding to $1 \in \mathbb{R} = \mathrm{Lie}(S^1)$. We need to show that $\iota_X \omega$ is exact. By hard Lefschetz, this is equivalent to showing that $\iota_X \omega^n$ is exact. Let $m \in M^{S^1}$ be a fixed point. In a neighborhood of m we can identify M as a T-space with $T_m M$. Let $\sigma \in \Omega^{2n}(T_m M)$ be an invariant form supported in an ϵ-ball around $T_m M$, normalized so that $\int_{T_m M} \sigma = \int_M \omega^n$. Choosing ϵ sufficiently small we can view σ as a form on M. Since σ and ω^n have the same integral, it follows that $\omega^n - \sigma = \mathsf{d}\beta$ for some invariant form $\beta \in \omega^{2n-1} M$.Then

$$\iota(X)(\sigma - \omega^n) = \iota(X)\mathsf{d}\beta = L_X \beta - \mathsf{d}\iota(X)\beta = -\mathsf{d}\iota(X)\beta,$$

showing that $\iota(X)(\sigma - \omega^n)$ is exact. We thus need to show that $\iota(X)\sigma$ is exact. This, however, follows from the Poincare lemma since it is supported in a ball around m, where one can just apply the homotopy operator. ☐

9.6. **Delzant spaces.**

Definition 9.30. A Hamiltonian T-space (M, ω, Φ) with proper moment map Φ is called multiplicity-free if all reduced space M_μ are either empty or 0-dimensional. We call (M, ω, Φ) a *Delzant*-space if in addition M is connected, the moment map is proper, and the number of orbit type strata is finite.[22]

Thus, if T acts effectively, (M, ω, Φ) is Delzant if and only if $\dim M = 2 \dim T$.

[22] The finiteness assumption is not very important, and is of course automatic if M is compact.

Examples 9.31. (a) $M = \mathbb{C}^n$ with the standard action of $T = (S^1)^n$. The moment map image is the positive orthant $\mathbb{R}^n_+ \subseteq \mathbb{R}^n \cong \mathfrak{t}^*$. More abstractly, if V is a Hermitian vector space, the action of the maximal torus $T \subseteq U(V)$ on V is Delzant.

(b) $M = \mathbb{C}P(n)$ with the action of $T = (S^1)^{n+1}/S^1$ (quotient by diagonal subgroup) coming from the action of $(S^1)^{n+1}$ on $\mathbb{C}^{n+1}$. The moment map image is a simplex, given as the intersection of the positive orthant $\mathbb{R}^{n+1}_+$ with the hyperplane $\sum_{i=0}^n t_i = \pi$. More generally, if V is a Hermitian vector space, the action of the maximal torus $T \subseteq \mathrm{U}(V)$ on the projectivization $P(V)$ is Delzant.

(c) $M = T^*(T)$ with the cotangent lift of the left-action of T on itself. The moment map image is all of $\mathfrak{t}^*$. We will call this, from now on, the standard T-action on $T^*(T)$.

(d) Suppose (M, ω, Φ) is a Delzant T-space, and $H \subseteq T$ is a subgroup acting freely on the level set of $\mu \in \mathfrak{h}^*$. Then the H-reduced space $(M_\mu, \omega_\mu, \Phi_\mu)$ is Delzant. The moment map image $\Phi(M_\mu) \in \mathfrak{t}^*$ is the intersection of $\Phi(M)$ with the affine subspace $\mathrm{pr}_{\mathfrak{h}^*}^{-1}(\mu)$. We can view M_μ as a Delzant T/H-space, after choosing a moment map for the T/H-action; such a choice amounts to choosing a point in $\mathrm{pr}_{\mathfrak{h}^*}^{-1}(\mu)$.

The moment map images for Delzant spaces can be characterized as follows. Let $\Lambda \subseteq \mathfrak{t}$ be the integral lattice, i.e. the kernel of $\exp : \mathfrak{t} \to T$. Let $\Delta \subseteq \mathfrak{t}^*$ be a rational convex polyhedral set of dimension $d = \dim T$, with k boundary hyperplanes. That is, Δ is of the form

$$\Delta = \bigcap_{i=1}^{k} \mathcal{H}_{v_i, \lambda_i} \tag{28}$$

where $v_i \in \Lambda$ are primitive lattice vectors and $\lambda_i \in \mathbb{R}$, and

$$\mathcal{H}_{v_i, \lambda_i} = \{\mu \in \mathfrak{t}^* | \langle \mu, v_i \rangle \leq \lambda_i\}.$$

For any subset $I \subseteq \{1, \ldots, k\}$ let Δ_I be the set of all μ with $\langle \mu, v_i \rangle = \lambda_i$ for $i \in I$. We set $\Delta_\emptyset = \mathrm{int}(\Delta)$.

Definition 9.32. The polyhedral set $\Delta \subseteq \mathfrak{t}^*$ is called *Delzant* if for all I with $\Delta_I \neq \emptyset$, the vectors v_i, $i \in I$ are linearly independent, and

$$\mathrm{span}_{\mathbb{Z}}\{v_i | \, i \in I\} = \Lambda \cap \mathrm{span}_{\mathbb{R}}\{v_i | \, i \in I\}.$$

Remark 9.33. For compact polyhedral sets, (that is, polytopes) it is enough to check the Delzant condition at the vertices. The Delzant condition means in particular that each v_i has to be a *primitive* normal vector, i.e. is not of the form $v_i = a\, v_i'$ where $v_i' \in \Lambda$ and $a \in \mathbb{Z}_{>0}$.

Example 9.34. Let $T = (S^1)^2$ and identify $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}^2$ and $\Lambda \cong \Lambda^* = \mathbb{Z}^2$. The polytope with vertices at $(0,0), (0,1), (1,0)$ is Delzant. However, the polytope with vertices at

$(0,0),(0,2),(1,0)$ is not Delzant. Indeed, for the vertex at $(1,0)$ the two primitive normal vectors are $v_1=(0,-1)$ and $v_2=(2,1)$, and they do not span the lattice $\mathbb{Z}^2$.

The Delzant condition for $\Delta_I\neq\emptyset$ says that $\sum_{j\in I}s_jv_j\in\Lambda\Leftrightarrow s_j\in\mathbb{Z}$ for all $j\in I$, or equivalently,

$$\exp(\sum_{j\in I}s_jv_j)=1\Leftrightarrow s_j=0\,\mathrm{mod}\,\mathbb{Z}\text{ for all }j\in I.$$

Thus if we define a Lie group morphism

$$\phi_\Delta:\ (S^1)^k\to T,\ [(s_1,\ldots,s_k)]\mapsto\exp(\sum_{i=1}^k s_iv_i)$$

and let

$$(S^1)^I=\{[(s_1,\ldots,s_k)]\in(S^1)^k|\ s_j=0\,\mathrm{mod}\,\mathbb{Z}\ \text{ for }\ j\notin I\}$$

be the product of S^1-factors corresponding to indices $j\in I$, the Delzant condition is equivalent to saying that ϕ_Δ restricts to an inclusion $\phi_\Delta:\ (S^1)^I\hookrightarrow T$. The image $H_I=\phi_\Delta((S^1)^I)\subseteq T$ is obtained by exponentiating $\mathfrak{h}_I=\mathrm{span}_\mathbb{R}\{v_j|\ j\in I\}$; by definition it is the subspace perpendicular to $\Delta_I\subseteq\mathfrak{t}^*$.

Theorem 9.35. *Let (M,ω,Φ) be a Delzant T-space with effective T-action. Then $\Delta=\Phi(M)$ is a Delzant polyhedron. For all open faces $F\subseteq\Delta$, the pre-image $\Phi^{-1}(F)$ is a connected component of the orbit type stratum $M_H\subseteq M$ for $H=\exp(\mathfrak{h}_F)$, where $\mathfrak{h}_F\subseteq\mathfrak{t}$ is the subspace perpendicular to F. In particular, all stabilizer groups are connected.*

Proof. Let $\mu\in F$, $\mathcal{O}=T.m\in\Phi^{-1}(\mu)$ an orbit, and $H=T_m$ the stabilizer group. We had seen that the $\mathrm{cone}_\mu(\Delta)$ is equal to the local moment cone

$$C_m=\mu+(\mathrm{pr}_\mathfrak{h}^*)^{-1}(C),$$

where $C\subseteq\mathfrak{h}^*$ is the cone spanned by the weights $\alpha_1,\ldots,\alpha_k\in\mathfrak{h}^*$ for the H-action on the symplectic vector space $V=T_m(\mathcal{O})^\omega/T_m(\mathcal{O})$. By dimension count, $k=\dim_\mathbb{C}V=\frac{1}{2}\dim M-\dim(T/H)=\dim H$. It follows that α_i are a basis of $\mathfrak{h}^*$. Since $\mathrm{ann}(\mathfrak{h})\subseteq\mathfrak{t}^*$ is the maximal linear subspace inside the cone $(\mathrm{pr}_{\mathfrak{h}^*})^{-1}(C)$, it must coincide with the space parallel to F. That is, $\mathfrak{h}=\mathfrak{h}_F$.

The action of H on V must be effective since the T-action on E is effective. Thus H acts as a compact abelian subgroup of $\mathrm{U}(V)$ of dimension $\dim H=\dim_\mathbb{C}V$. So its identity component H_0 is a maximal torus. But it is a well-known fact from Lie group theory that maximal tori are maximal abelian, so $H=H_0$. In particular, we have shown that all points in $\Phi^{-1}(F)$ have the same stabilizer group.

It follows that the map $H\to(S^1)^k$ defined by the roots is an isomorphism. This means that $\alpha_1,\ldots,\alpha_k$ are a basis for the weight lattice weight lattice $(\Lambda\cap\mathfrak{h})^*$ in $\mathfrak{h}^*$.

Equivalently, the dual basis $w_1, \ldots, w_k \in \mathfrak{h}$ are a basis for $\Lambda \cap \mathfrak{h}$. We have

$$C = \text{cone}\{\alpha_1, \ldots, \alpha_n\} = \{\nu \in \mathfrak{h}^* | \, \langle \nu, w_i \rangle \geq 0\},$$

which identifies the $\{w_1, \ldots, w_k\}$ with $\{v_i| \, i \in I\}$. ☐

Delzant gave an explicit recipe for constructing a Delzant space with moment polytope a given Delzant polyhedron. The following version of Delzant's construction is due to Eugene Lerman.

Let $(S^1)^k$ act on the cotangent bundle $T^*(T)$ via the composition of ϕ_Δ with the standard T-action on $T^*(T)$. In the left trivialization $T^*(T) = T \times \mathfrak{t}^*$, a moment map for the T-action is projection to $\mathfrak{t}^*$. Hence

$$\Psi_\Delta(t, \mu) = \sum_{j=1}^{k} \langle \mu, v_j \rangle e_j - \sum_j \lambda_j \, e_j$$

is a moment map for the action of $(S^1)^k$. Let $(S^1)^k$ act on $\mathbb{C}^k$ in the standard way, with moment map $\pi \sum_j |z_j|^2 \, e_j$.

Definition 9.36. For any polyhedron Δ let D_Δ be the symplectic quotient

$$D_\Delta = (T^*(T) \times \mathbb{C}^k) /\!\!/ (S^1)^k,$$

by the diagonal action, with T-action induced from the standard T-action on $T^*(T)$.

Theorem 9.37. *Suppose Δ is a Delzant polyhedron. Then the action of $(S^1)^k$ on the zero level set of $(T^*(T) \times \mathbb{C}^k)$ is free, and the quotient D_Δ is a Delzant-T-space. The moment map image of D_Δ is exactly Δ.*

Proof. Let $((t, \mu), z)$ in the zero level set. Thus

$$\langle \mu, v_i \rangle = \lambda_i - \pi |z_i|^2.$$

If $z_i \neq 0$ then the ith factor of $(S^1)^k$ acts freely at $((t, \mu), z)$. Thus we need only worry about the set I of indices i with $z_i = 0$. For these indices $\langle \mu, v_i \rangle = \lambda_i$. Let $(S^1)^I$ be the product of copies of S^1 corresponding to these indices. By the Delzant condition, ϕ_Δ restricts to an *embedding* $(S^1)^I \to T$. Since T acts freely on $T^*(T)$, so does $(S^1)^I$. This shows that the action is free, and D_Δ is a smooth symplectic manifold. To identify the image of the T-moment map note that, given $\mu \in \mathfrak{t}^*$, one can find t, z with $((t, \mu), z)$ is in the zero level set if and only if $\langle \mu, v_i \rangle \leq \lambda_i$. ☐

Definition 9.38 (Lerman [25]). Let Δ be a Delzant polyhedron, and (M, ω, Φ) a Hamiltonian T-space. The *cut space* defined by Δ is the symplectic quotient

$$M_\Delta = (M \times D_\Delta^-) /\!\!/ T$$

with T-action induced from the action on the first factor.

It is immediate that $T^*(T)_\Delta = D_\Delta$: In particular, $T^*(S^1)_{[0,\infty)} \cong \mathbb{C}$. We will now use these two facts to prove:

Theorem 9.39 (Delzant [11]). *Every Delzant space (M, ω, Φ) is determined by its moment polyhedron $\Delta = \Phi(M)$, up to equivariant symplectomorphism intertwining the moment maps.*

Proof. Usually this is proved using a Čech theoretic argument. Below we sketch a more elementary (?) approach. The idea is to present M as a symplectic cut $\tilde{M}_\Delta$ of a connected, multiplicity free Hamiltonian T-space $\tilde{M}$ with *free* T-action. Since the action of T on $\tilde{M}$ is free, the map $\tilde{\Phi}$ is a Lagrangian fibration over its image. Thus we can introduce action-angle variables which identifies $\tilde{M}$ as an open subset of $T^*(T)$. Therefore, $M = \tilde{M}_\Delta = T^*(T)_\Delta = D_\Delta$.

We now indicate how to construct such a space $\tilde{M}$. Let $i_1 \in \{1, \ldots, k\}$ be an index such that $\Delta_{i_1} \neq 0$, and $S = \Phi^{-1}(\Delta_{i_1})$ the symplectic submanifold obtained as its preimage. It is a connected component of the fixed point set of H_{i_1}, and has codimension 2. Let $\nu_S = TS^\omega$ be its symplectic normal bundle. After choosing a compatible complex structure it can be viewed as a Hermitian line bundle. Let $Q \subseteq \nu_S$ be the unit circle bundle inside Q. It is a T-equivariant principal S^1-bundle, and $\nu_S = Q \times_{S^1} \mathbb{C}$. Let $\pi_Q : Q \to S$ be the projection map. Let $\alpha \in \Omega^1(Q)^T$ be a T-invariant connection 1-form, and consider the closed 2-form

$$\omega_{Q\times\mathbb{C}} := \pi_Q^*\omega_S + \omega_\mathbb{C} + \pi \mathsf{d}(|z|^2\alpha).$$

It is easy to check that this 2-form is basic for the S^1-action, so it descends to a closed 2-form

$$\omega_{\nu_S} \in \Omega^2(\nu_S).$$

Furthermore, ω_{ν_S} is non-degenerate near $S = Q/S^1$. It follows that there exists an equivariant symplectomorphism between open neighborhoods of S in M and in ν_S. Now $\nu_S = (Q \times \mathbb{R})_{[0,\infty)}$ (cut with respect to the S^1-action), where $Q \times \mathbb{R}$ is equipped with the 2-form

$$\omega_{Q\times\mathbb{R}} = \pi_Q^*\omega_S + \omega_\mathbb{C} + \mathsf{d}(s\alpha).$$

We have a natural diffeomorphism between $Q \times \mathbb{R}_{>0}$ and $\nu_S \backslash S$, preserving 2-forms. We can thus glue $M\backslash S$ with a small neighborhood of Q in $Q \times \mathbb{R}_-$, to obtain a new connected multiplicity free Hamiltonian T-space (M_1, ω_1, Φ_1) with one orbit type stratum less. The original space is obtained from M_1 by cutting,

$$M = (M_1)_{\mathcal{H}_1}$$

where $\mathcal{H}_1$ is the affine half-space $\langle \mu, v_i \rangle \geq \lambda_i$. Continuing in this fashion, construct spaces $M_1, M_2, \ldots, M_n = \tilde{M}$ where n is the number of faces of Δ. We have

$$M = (M_1)_{\mathcal{H}_1} = (M_2)_{\mathcal{H}_1\cap\mathcal{H}_2} = \ldots = (M_n)_\Delta,$$

The final space $M_n = \tilde{M}$ no longer has 1-dimensional stabilizer groups, so the T-action is free as required. $\square$

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