Contemporary Mathematics

A least action principle for steepest descent in a non-convex landscape

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1. Introduction

Physical dynamics interpolate naturally between the dissipative and conservative extremes, in which friction either dominates or can be neglected. *Gradient flows* and *Hamiltonian systems* represent the archetypal examples of these two extremes. The orbits of a Hamiltonian system correspond to the critical paths of an *action* functional, but variational characterizations for the trajectories of a gradient flow are less familiar. For steepest descent into a *convex valley* such a characterization was formulated by Brezis-Ekeland [**4**], but their principle was not amenable to deducing existence of solutions. Later, Auchmuty [**2**] used min-max methods to establish the existence of solutions variationally, but under certain growth conditions on the convex potential. Recently, Ghoussoub and Tzou [**8**] deduced the existence of semigroup flows in full generality (for convex lower semicontinuous potentials) using a modified Brezis-Ekeland principle which is invariant under Bolza duality [**12**]. For convex gradient flows, their method now provides a direct alternative to proving existence by maximal monotone accretive operator theoretic semigroup methods or by time-step approximation; recent references for the latter techniques include [**3**] [**7**] [**10**] [**1**]. In a forthcoming paper [**9**], the first-named author develops further the scope of the duality method and proposes a general framework for a variational formulation of many equations which do not normally fit into standard Euler-Lagrange theory. This approach is based on a concept of *anti-self dual Lagrangians* which seems to be inherent in many important differential equations.

In the present article we streamline and broaden the scope of the Ghoussoub-Tzou approach to gradient flows, showing how it can be adapted to characterize the path of steepest descent of a *non-convex* potential as the global minimum of a *convex* action. This is achieved by *dynamically rescaling space*, to convert the energy landscape on which our descent takes place from a static but non-convex profile to a contracting convex one. This approach is inspired by the time-dependent change of coordinates used to find similarity variables in nonlinear partial differential equations, especially as employed by Otto to quantify long-time behaviour of porous medium flows [**11**]. Although that descent takes place on the 'Riemannian' manifold of probability measures metrized by Wasserstein distance, our present considerations will be restricted to the more traditional setting of a Hilbert space H with norm $|u| = \langle u, u\rangle^{1/2}$.

Let $[0,T]$ be a fixed real interval ($0 < T < +\infty$). Consider the Hilbert space $L^2_H := L^2([0,T];H)$ of Bochner integrable functions from $[0,T]$ into H with norm denoted by $\|\cdot\|_{L^2_H}$,

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and the Sobolev space

$$A_H^2 := \{u : [0,T] \to H;\ \dot{u} \in L_H^2\} =: AC^2([0,T];H)$$

consisting of all absolutely continuous vector-valued arcs $u : [0,T] \to H$, equipped with the norm

$$\|u\|_{A_H^2} = \left(|u(0)|^2 + \int_0^T |\dot{u}|^2 dt \right)^{\frac{1}{2}}.$$

Our main result is formulated under a *semiconvexity* assumption, which means that the energy landscape differs from a convex valley by a smooth function. In particular, this assumption is satisfied whenever the landscape is sufficiently smooth.

THEOREM 1.1 (Least action descent in a non-convex landscape). *Let $W : H \longrightarrow \mathbf{R} \cup \{+\infty\}$ be* semiconvex, *meaning for some $k \geq 0$ the function $\overline{W}(u) := W(u) + k|u|^2/2$ is strictly convex, lower semicontinuous on H, and not identically infinity. Then $V(t,v) := e^{-2kt}\overline{W}(e^{kt}v)$ is convex at each instant in time so let*

$$V^*(t,u) := \sup_{v \in H} \langle u, v\rangle - V(t,v)$$

be its Legendre-Fenchel transform. For any $u_0 \in \operatorname{dom} \partial W$, consider the functional

$$\Phi[u] = \frac{1}{2}(|u(0)|^2 + |u(T)|^2) - 2\langle u(0), u_0\rangle + |u_0|^2 + \int_0^T [V(t,u(t)) + V^*(t, -\dot{u}(t))]\, dt \tag{1}$$

on the path space A_H^2. Then there exists a unique v in A_H^2 such that

$$\Phi[v] = \inf_{u \in A_H^2} \Phi[u] = 0. \tag{2}$$

Moreover, the path $w(t) := e^{kt}v(t)$ is the unique solution in A_H^2 to

$$\begin{cases} -\dot{w}(t) & \in & \partial W(w(t)) \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ w(0) & = & u_0. \end{cases} \tag{3}$$

Here the set $\partial W(u)$ is related to the subdifferential of $\overline{W}$ by $\partial \overline{W}(u) = \partial W(u) + ku$, and $\operatorname{dom} \partial W := \{u \in H \mid W(u) < +\infty \text{ and } \partial W(u) \neq \emptyset\}$. An alternate representation (9) of the action $\Phi[\,\cdot\,]$ shows why it is minimized by the evolution (3), but conceals the convexity manifest in (1).

This theorem can of course be deduced from well-known existence results concerning semi-group flows. Our purpose here is to give a simple variational proof which implies these existence results. At the same time, we highlight the convex-analytic properties of our parabolic rescaling of space: it commutes with both Legendre transformation and Yosida regularization.

2. Duality and gradient flow in an evolving landscape

The path space A_H^2 is also a Hilbert space that can be identified with the product space $H \times L_H^2$, while its dual $(A_H^2)^*$ can be identified with $H \times L_H^2$. The duality is given by the formula:

$$\langle u, (a,p)\rangle_{A_H^2, H \times L_H^2} = \langle u(0), a\rangle_H + \int_0^T \langle \dot{u}(t), p(t)\rangle dt.$$

THEOREM 2.1 (Least action descent into an evolving valley). *Let $V : [0,T] \times H \to \mathbf{R} \cup \{+\infty\}$ be a measurable function with respect to the σ-field in $[0,T] \times H$ generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in H. Assume V satisfies the following conditions:*

- *For every $t \in [0,T]$, the function $V(t,\cdot)$ is convex and lower semicontinuous on H.*
- $\int_0^T V^*(t,0)dt < \infty$.
- *There is $\alpha \in L^\infty[0,T]$ such that:*

$$V(t,x) \leq \alpha(t)(1 + |x|^2) \quad \textit{for } (t,x) \in [0,T] \times H. \tag{4}$$

For any $u_0 \in H$, the functional

$$\Phi[u] = \frac{1}{2}(|u(0)|^2 + |u(T)|^2) - 2\langle u(0), u_0\rangle + |u_0|^2 + \int_0^T [V(t, u(t)) + V^*(t, -\dot{u}(t))]\, dt \tag{5}$$

admits a unique minimizer v in A^2_H and

$$\Phi[v] = \inf_{u \in A^2_H} \Phi[u] = 0. \tag{6}$$

Among paths in A^2_H, v is the unique solution to

$$\begin{cases} -\dot{v}(t) & \in & \partial V(t, v(t)) \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ v(0) & = & u_0. \end{cases} \tag{7}$$

Proof. First, we notice that $\Phi[\,\cdot\,]$ is convex and that:

$$\Phi[u] \geq 0 \quad \text{for all } u \text{ in } A^2_H. \tag{8}$$

Indeed, it is clear that

$$\Phi[u] = |u(0) - u_0|^2 + \int_0^T [V(t, u(t)) + V^*(t, -\dot{u}(t)) + \langle u(t), \dot{u}(t)\rangle]\, dt. \tag{9}$$

By the Fenchel-Young inequality, we have

$$V(t, u(t)) + V^*(t, -\dot{u}(t)) \geq \langle u(t), -\dot{u}(t)\rangle = -\frac{1}{2}\frac{d}{dt}|u(t)|^2 \quad \text{a.e. on } [0,T] \tag{10}$$

with equality holding if and only if $-\dot{u}(t) \in \partial V(t, u(t))$.

This also yields that $\Phi[u] \geq |u(0) - u_0|^2 \geq 0$, which means that (6) would automatically imply (7).

The rest of the section deals with the reverse inequality and the existence of a minimum. We follow the method of Ghoussoub-Tzou [**8**] by showing that $\Phi[\,\cdot\,]$ "behaves lower semicontinuously" with respect to certain perturbations. For that, we associate the following functional Ψ defined on $(A^2_H)^* = H \times L^2_H$ as:

$$\Psi[a, y] = \inf_{u \in A^2_H} \Big\{ \tfrac{1}{2}(|u(0) + a|^2 + |u(T)|^2) - 2\langle u(0) + a, u_0\rangle + |u_0|^2 + \int_0^T V(t, u(t) + y(t)) + V^*(t, -\dot{u}(t))dt \Big\} \tag{11}$$

so that

$$\Psi[0, 0] = \inf_{u \in A^2_H} \Phi[u]. \tag{12}$$

The following lemma establishes a key duality-symmetry between the two functionals.

Lemma 2.2 (Self-duality). *Defining $\Psi[\cdot, \cdot]$ by (11), the hypotheses of Theorem 2.1 imply*

$$\Psi^*[v] = \Phi[-v] \quad \text{for all } v \in A^2_H,$$

where Ψ^* is the Legendre transform of Ψ in the duality $(H \times L^2_H, A^2_H)$.

Proof of Lemma 2.2. For $v \in A^2_H$, write:

$$\Psi^*[v] = \sup_{a \in H} \sup_{y \in L^2_H} \sup_{u \in A^2_H} \Big\{ \langle a, v(0)\rangle - \tfrac{1}{2}(|u(0) + a|^2 + |u(T)|^2) + 2\langle u(0) + a, u_0\rangle - |u_0|^2 + \int_0^T [\langle y(t), \dot{v}(t)\rangle - V(t, u(t) + y(t)) - V^*(t, -\dot{u}(t)]dt \Big\}.$$

Making a substitution

$$u(0) + a = \tilde{a} \in H \quad \text{and } u + y = \tilde{y} \in L^2_H,$$

we obtain

$$\Psi^*[v] \geq \sup_{\tilde{a} \in H} \sup_{\tilde{y} \in L^2_H} \sup_{u \in A^2_H} \Big\{ \langle \tilde{a} - u(0), v(0)\rangle - \tfrac{1}{2}(|\tilde{a}|^2 + |u(T)|^2) + 2\langle \tilde{a}, u_0\rangle - |u_0|^2 + \int_0^T [\langle \tilde{y}(t) - u(t), \dot{v}(t)\rangle - V(t, \tilde{y}(t)) - V^*(t, -\dot{u}(t))]dt \Big\}.$$

Since $\dot{u} \in L^2_H$ and $u \in L^2_H$, we have:

$$\int_0^T \langle u, \dot{v}\rangle \, dt = \langle v(T), u(T)\rangle - \langle v(0), u(0)\rangle - \int_0^T \langle \dot{u}, v\rangle \, dt,$$

which implies

$$\Psi^*[v] \geq \sup_{\tilde{a}\in H} \sup_{\tilde{y}\in L^2_H} \sup_{u\in A^2_H} \Big\{ \quad \langle \tilde{a}, v(0)\rangle - \tfrac{1}{2}(|\tilde{a}|^2 + |u(T)|^2) + 2\langle \tilde{a}, u_0\rangle - |u_0|^2 - \langle u(T), v(T)\rangle$$
$$+ \int_0^T [\langle \tilde{y}, \dot{v}\rangle + \langle \dot{u}, v\rangle - V(t, \tilde{y}(t)) - V^*(t, -\dot{u}(t))] \, dt \Big\}.$$

It is now convenient to identify A^2_H with $H \times L^2_H$ via the correspondence:

$$\begin{aligned} (c, x) \in H \times L^2_H \quad &\mapsto \quad u(t) = c + \int_t^T x(s) \, ds \in A^2_H \\ u \in A^2_H \quad &\mapsto \quad \big(u(T), -\dot{u}(t)\big) \in H \times L^2_H. \end{aligned}$$

We finally obtain

$$\begin{aligned} \Psi^*[v] \quad &\geq \quad -|u_0|^2 + \sup_{\tilde{a}\in H} \sup_{c\in H} \left\{ \langle \tilde{a}, v(0) + 2u_0\rangle + \langle -c, v(T)\rangle - \frac{1}{2}(|\tilde{a}|^2 + |c|^2) \right\} \\ &\quad + \sup_{\tilde{y}\in L^2_H} \sup_{x\in L^2_H} \left\{ \int_0^T (\langle \tilde{y}, \dot{v}\rangle + \langle x, -v\rangle - V(t, \tilde{y}(t)) - V^*(t, x(t)) dt \right\} \\ &= \quad -|u_0|^2 + \frac{1}{2}(|v(0) + 2u_0|^2 + |v(T)|^2) + \int_0^T [V^*(t, \dot{v}(t)) + V(t, -v(t))] \, dt \\ &= \quad \frac{1}{2}(|v(0)|^2 + |v(T)|^2) + 2\langle v(0), u_0\rangle + |u_0|^2 + \int_0^T [V(t, -v(t)) + V^*(t, \dot{v}(t))] \, dt \\ &= \quad \Phi[-v]. \end{aligned}$$

Here we have used that $V(t, \cdot)^{**} = V(t, \cdot)$ and that for any $L : [0, T] \times H \times H \to \mathbf{R} \cup \{+\infty\}$ convex and lower semicontinuous, we have:

$$\int_0^T L^*(t, s(t), v(t)) dt = \sup_{x,y\in L^2_H} \int_0^T [\langle y(t), s(t)\rangle + \langle x(t), v(t)\rangle - L(t, y(t), x(t))] dt$$

where L^* is the Legendre dual of L in both state variables. □

End of proof of Theorem 2.1. First we prove that the convex functional Ψ is subdifferentiable at $(0, 0)$. For that, it is sufficient to show that Ψ is bounded on neighborhoods of zero in $H \times L^2_H$. Note that

$$\begin{aligned} \Psi[a, y] \quad &\leq \quad 2|u_0| \cdot |a| + \frac{|a|^2_H}{2} + |u_0|^2 + \int_0^T [V(t, y(t)) + V^*(t, 0)] dt \\ &\leq \quad 2|u_0| \cdot |a| + \frac{|a|^2_H}{2} + |u_0|^2 + \int_0^T \big[\alpha(t)(|y(t)|^2 + 1) + V^*(t, 0)\big] \, dt \end{aligned}$$

which means that Ψ is bounded in a neighborhood of $(0, 0)$ in the space $H \times L^2_H$, hence it is subdifferentiable at $(0, 0)$.

Now take $-v \in \partial\Psi[0, 0] \in A^2_H$. Then again by Young's inequality,

$$\Psi[0, 0] + \Psi^*[-v] = 0 = \Phi[v] + \inf_{u\in A^2_H} \Phi[u]$$

It follows that:

$$-\inf_{A^2_H} \Phi = \Phi[v] \geq \inf_{A^2_H} \Phi$$

which means that $\inf_{A^2_H} \Phi \leq 0$. In view of (8), we must have $\inf_{A^2_H} \Phi = 0 = \Phi[v]$. Thus the minimum is zero and is attained at v. □

3. Yosida regularization and gradient flow of a semiconvex potential

Consider again a measurable function $v : [0,T] \times H \to \mathbf{R} \cup \{+\infty\}$ such that $V(t,\cdot)$ is convex and lower semicontinuous on H for every $t \in [0,T]$, but without the bound (4) of Theorem 2.1, which is quite restrictive and not satisfied by most potentials of interest. One way to remedy this is to regularize V by using inf-convolution. That is, for each $\lambda > 0$ we define

$$V_\lambda(t,x) = \inf\{V(t,y) + \frac{1}{2\lambda}|x-y|_H^2;\ y \in H\},$$

so that

$$V_\lambda(t,x) \le V(t,0) + \frac{1}{2\lambda}|x|^2$$

while its conjugate is given by

$$V_\lambda^*(t,y) = V^*(t,y) + \frac{\lambda}{2}|y|^2. \tag{13}$$

The V_λ now satisfy the hypothesis of Theorem 2.1 (as long as $\int_0^T V(t,0) + V^*(t,0)dt < \infty$) and therefore the corresponding evolution equations

$$\begin{cases} \dot v(t) + \partial V_\lambda(t, v(t)) &= 0 \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ v(0) &= u_0 \end{cases} \tag{14}$$

have unique solutions $v_\lambda(t)$ in A_H^2 that minimize

$$\Phi_\lambda[v] := |v(0) - u_0|^2 + \int_0^T [V_\lambda(t, v(t)) + V_\lambda{}^*(t, -\dot v(t)) + \langle v(t), \dot v(t)\rangle]\, dt. \tag{15}$$

Now we need to argue that $(v_\lambda)_\lambda$ converges as $\lambda \to 0$ to a solution of the original problem. This analysis is reminiscent of the approach via the resolvent theory of Hille-Yosida, but is much easier here since the variational argument does not require uniform convergence of $(v_\lambda)_\lambda$ or their time-derivatives.

One still needs an upper bound on the L^2-norm of $(\dot u_\lambda(t))_\lambda$ however. This is straightforward when V is a time-independent convex potential (as shown for example in [**8**]), but not always possible for general time-dependent potentials. However, we shall be able to provide such an estimate in the special case when the time-dependent potential is of the form $V(t,x) = e^{-2kt}\overline{W}(e^{kt}x)$ with $\overline{W}$ being an appropriate convex function.

Let us summarize the properties of infimal convolution recapitulated from Evans [**7**, §9.6.1].

LEMMA 3.1. *Let $\overline{W} : H \longrightarrow \mathbf{R} \cup \{+\infty\}$ be convex lower semicontinuous, $\overline{W}(u_0) < \infty$, and $\lambda > 0$. Then*

$$\overline{W}_\lambda(w) := \inf\{\overline{W}(u) + \frac{1}{2\lambda}|w-u|_H^2;\ u \in H\}, \tag{16}$$

is convex on H and bounded by

$$\overline{W}_\lambda(w) \le \overline{W}(u_0) + \frac{1}{2\lambda}|w - u_0|^2. \tag{17}$$

There exist Lipschitz maps $\nabla\overline{W}_\lambda : H \longrightarrow H$ and $J_\lambda : H \longrightarrow \operatorname{dom}\partial\overline{W}$ such that for each $w \in H$:
(i) (differentiability): $\partial\overline{W}_\lambda(w) = \{\nabla\overline{W}_\lambda(w)\}$;
(ii) (nonlinear resolvent): the infimum (16) is uniquely attained at $u = J_\lambda(w)$;
(iii) (first order condition):

$$\nabla\overline{W}_\lambda(w) = \frac{w - J_\lambda(w)}{\lambda} \in \partial\overline{W}(J_\lambda(w)); \tag{18}$$

(iv) (uniform gradient bound):

$$|\nabla\overline{W}_\lambda(w)| \le \inf_{u \in \partial\overline{W}(w)} |u|; \tag{19}$$

(v) (Lipschitz contractions): both J_λ and $\nabla(\lambda\overline{W}_\lambda/2)$ are contractions on H.

We shall call $\overline{W}_\lambda$ the Yosida λ-regularization of $\overline{W}$. Now we note the following useful property which says that the Yosida regularization actually commutes with our rescaling of space.

LEMMA 3.2 (Rescaling commutes with dualization and Yosida regularization). *Let $\overline{W}(u)$ be a lower semicontinuous convex function on H and consider the time dependent convex potential $V(t,v) := e^{-2kt}\overline{W}(e^{kt}v)$ and its Legendre-Fenchel transform $V^*(t,u)$ for each time t. Then*

$$V^*(t,u) = e^{-2kt}(\overline{W})^*(e^{kt}u) \tag{20}$$

where $(\overline{W})^*$ is the Legendre-Fenchel dual of $\overline{W}$. Moreover, if $V_\lambda(t,v)$ denotes the Yosida λ-regularization of $V(t,v)$ for each time t, and if $J_\lambda(t,v)$ is the corresponding attainment map, then

$$V_\lambda(t,v) := e^{-2kt}\overline{W}_\lambda(e^{kt}v) \quad \text{and} \quad J_\lambda(t,v) = e^{-kt}\tilde{J}_\lambda(e^{kt}v) \tag{21}$$

where $\overline{W}_\lambda$ is the Yosida λ-regularization of $\overline{W}$ and $\tilde{J}_\lambda$ its corresponding attainment map.

PROOF. Note

$$\begin{aligned}
V^*(t,v) &= \sup\{\langle v,x\rangle - e^{-2kt}\overline{W}(e^{kt}x); x \in H\} \\
&= e^{-2kt}\sup\{\langle e^{kt}v, e^{kt}x\rangle - \overline{W}(e^{kt}x); x \in H\} \\
&= e^{-2kt}\sup\{\langle e^{kt}v, w\rangle - \overline{W}(w); w \in H\} \\
&= e^{-2kt}(\overline{W})^*(e^{kt}v).
\end{aligned}$$

which proves (20). On the other hand,

$$\begin{aligned}
(V_\lambda)^*(t,x) &= V^*(t,x) + \frac{\lambda}{2}|x|^2 \\
&= e^{-2kt}(\overline{W})^*(e^{kt}x) + \frac{\lambda}{2}|x|^2 \\
&= e^{-2kt}\left((\overline{W})^*(e^{kt}x) + \frac{\lambda}{2}|e^{kt}x|^2\right) \\
&= e^{-2kt}(\overline{W}_\lambda)^*(e^{kt}x).
\end{aligned}$$

Use now (20) with $(\overline{W}_\lambda)^*$ instead of $\overline{W}$ to conclude that

$$V_\lambda(t,v) := e^{-2kt}\overline{W}_\lambda(e^{kt}v) \tag{22}$$

which means Yosida regularization commutes with our rescaling of space. □

Next we establish the required a priori estimate.

PROPOSITION 3.3 (Uniform speed limit). *Let $\overline{W}(u)$ be a lower semicontinuous convex function on H with $u_0 \in \operatorname{dom}\partial W$. Let $\overline{W}_\lambda$ be its Yosida regularization for each $\lambda > 0$ and define the time dependent potential $V_\lambda(v,t) := e^{-2kt}\overline{W}_\lambda(e^{kt}v)$ and its corresponding energy functional $\Phi_\lambda[\,\cdot\,]$ as in (15). If $v_\lambda \in A^2_H$ satisfies $\Phi_\lambda[v_\lambda] = 0$, then for a.e. $t \in [0,T]$, we have*

$$|\dot{v_\lambda}(t)| \le C_0 := 2(k|u_0| + \inf_{u\in\partial\overline{W}(u_0)}|u|). \tag{23}$$

PROOF. Suppose $v_\lambda \in A^2_H$ satisfies $\Phi_\lambda[v_\lambda] = 0$. As in Theorem (2.1) we conclude that:

$$\left\{\begin{array}{rcll}
-\dot{v_\lambda}(t) &=& \nabla V_\lambda(v_\lambda(t),t) & \text{for a.e. } t\in[0,T] \\
v_\lambda(0) &=& u_0. &
\end{array}\right. \tag{24}$$

To obtain the desired estimate, it is convenient to exploit our dynamical rescaling of space to find an autonomous gradient flow equivalent to (24). Indeed, note that if $\tilde{W}(w) := \overline{W}_\lambda(w) - k|w|^2/2$, then $v_\lambda(t) = e^{-kt}w_\lambda(t)$ satisfies (24) if and only if

$$\left\{\begin{array}{rcll}
-\dot{w_\lambda}(t) &=& \nabla\tilde{W}(w_\lambda(t)) & \text{for a.e. } t\in[0,T] \\
w_\lambda(0) &=& u_0. &
\end{array}\right. \tag{25}$$

Next we establish that if $w_\lambda \in A^2_H$ satisfies (25) then

$$|\dot{w_\lambda}(t)| \le e^{kt}|\nabla\tilde{W}(u_0)| \qquad \text{a.e. on } [0,T]. \tag{26}$$

Indeed, consider $f(t) = |\nabla \tilde{W}(w_\lambda(t))|^2/2$ which is absolutely continuous on $[0,T]$. Using $D^2\tilde{W} \geq -kI$ (assuming $\tilde{W}$ is C^2), we compute

$$\begin{aligned} f'(t) = & \quad \langle \nabla\tilde{W}(w_\lambda(t)), D^2\tilde{W}(w_\lambda(t))\dot{w_\lambda}(t)\rangle \\ = & - \langle \nabla\tilde{W}(w_\lambda(t)), D^2\tilde{W}(w_\lambda(t))\nabla\tilde{W}(w_\lambda(t))\rangle \\ \leq & \quad 2kf(t). \end{aligned}$$

Gronwall's inequality yields $f(t) \leq e^{2kt}f(0)$ whence $|\nabla\tilde{W}(w_\lambda(t))| \leq e^{kt}|\nabla\tilde{W}(w_\lambda(0))|$. This establishes (26) in light of (25).

Finally, we can address our main claim (23). Let $v_\lambda \in A^2_H$ satisfy (24) so that $w_\lambda(t) := e^{kt}v_\lambda(t)$ satisfies (26). Integrating that estimate yields

$$|w_\lambda(t) - w_\lambda(0)| \leq \int_0^t |\dot{w_\lambda}(\tau)|d\tau \tag{27}$$

$$\leq \frac{e^{kt}-1}{k}|\nabla\tilde{W}(u_0)|. \tag{28}$$

Applying (26) and (28) to $\dot{v_\lambda}(t) = e^{-kt}(\dot{w_\lambda}(t) - kw_\lambda(t))$ gives:

$$\begin{aligned} |\dot{v_\lambda}(t)| &\leq e^{-kt}(e^{kt}|\nabla\tilde{W}(u_0)| + k|w_\lambda(0)| + (e^{kt}-1)|\nabla\tilde{W}(u_0)|) \\ &= (2-e^{-kt})|\nabla\tilde{W}(u_0)| + ke^{-kt}|u_0| \\ &\leq 2|\nabla\overline{W}_\lambda(u_0)| + 2k|u_0| \\ &=: C_\lambda(u_0), \end{aligned}$$

where $\nabla\tilde{W}(u_0) = \nabla\overline{W}_\lambda(u_0) - ku_0$ has been used. Finally, the constants $C_\lambda(u_0) \leq C_0$ are bounded independently of λ by Lemma 3.1(iv), completing the proposition. □

Proof of Theorem 1.1. Start with $W : H \longrightarrow \mathbf{R} \cup \{+\infty\}$ semiconvex, meaning that for some $k \geq 0$ the function $\overline{W}(u) := W(u) + k|u|^2/2$ is strictly convex, lower semicontinuous on H, and not identically infinity. Taking k larger if necessary ensures $\overline{W}(u)$ grows quadratically and hence attains its minimum. Set $V(t,v) := e^{-2kt}\overline{W}(e^{kt}v)$ and let $V^*(t,u)$ denote its Legendre-Fenchel transform for each time t. For any $u_0 \in \operatorname{dom}\partial W$, consider the functional

$$\Phi[u] = \frac{1}{2}(|u(0)|^2 + |u(T)|^2) - 2\langle u(0), u_0\rangle + |u_0|^2 + \int_0^T [V(t,u(t)) + V^*(t,-\dot{u}(t))]\,dt$$

on the space of curves A^2_H. We need to show that there exists v in A^2_H such that:

$$\Phi[v] = \inf_{u\in A^2_H} \Phi[u] = 0.$$

Uniqueness of v then follows from strict convexity of $\Phi[\,\cdot\,]$, and it is easy to see that the path $w(t) := e^{kt}v(t)$ satisfies

$$\begin{cases} \dot{w}(t) + \partial W(w(t)) = 0 \quad \text{a.e. on } [0,T] \\ w(0) = u_0 \end{cases}$$

from the equality conditions in Young's inequality (10).

Let $\overline{W}_\lambda$ be the Yosida regularization of $\overline{W}$ for each $\lambda > 0$ and its associated map $\tilde{J}_\lambda$ from Lemma 3.1. We know from Lemma 3.2 that the λ-regularization of V satisfies $V_\lambda(v,t) := e^{-2kt}\overline{W}_\lambda(e^{kt}v)$ and that its corresponding attainment map $J_\lambda(t,v) = e^{-kt}\tilde{J}_\lambda(e^{kt}v)$ where $\tilde{J}_\lambda$ is the attainment map for $\overline{W}_\lambda$. If Φ_λ denotes the energy functional (15), then Theorem 2.1 yields $v_\lambda \in A^2_H$ such that $\Phi_\lambda[v_\lambda] = 0$ for each $\lambda > 0$. That is

$$\begin{cases} -\dot{v_\lambda}(t) = \nabla V_\lambda(t, v_\lambda(t)) \quad \text{for a.e. } t \in [0,T] \\ v_\lambda(0) = u_0. \end{cases} \tag{29}$$

By Proposition 3.3, we have for a.e. $t \in [0,T]$, the estimate

$$|\dot{v_\lambda}(t)| \leq C_0 := 2(k|u_0| + \inf_{u\in\partial\overline{W}(u_0)} |u|). \tag{30}$$

It follows that a subsequence $(v_{\lambda_j})_j$ is converging weakly in A^2_H to a path v. The projection of this path onto any vector in H lies in the real Sobolev space $A^2_{\mathbf{R}} \subset L^2[0,T]$ of Hölder-1/2 functions, hence converges pointwise. For each $t \in [0,T]$, it follows that $v_{\lambda_j}(t) \to v(t)$ weakly in H as $\lambda_j \to 0$. From Lemma 3.1, we have for any $\lambda > 0$ and any $t \geq 0$,

$$|v_\lambda(t) - J_\lambda(t, v_\lambda(t))| = \lambda|\nabla V_\lambda(t, v_\lambda)| = \lambda|\dot{v_\lambda}(t)| \leq \lambda C_0,$$

thus $J_{\lambda_j}(t, v_{\lambda_j}(t)) \to v(t)$ weakly in H for every $t \in [0,T]$.

Now $V(t,\cdot)$ and $V^*(t,\cdot)$ are weakly lower semi-continuous on H, and $V(t,\cdot)$ is bounded below by $\inf_H \overline{W}(u) > -\infty$. Using Fatou's lemma one easily deduces:

$$\begin{aligned} \int_0^T V(t, \quad v(t))dt &\leq \varliminf_j \int_0^T V(t, J_{\lambda_j}(t, v_{\lambda_j}(t)))dt, \\ \int_0^T V^*(t, -\dot{v}(t))dt &\leq \varliminf_j \int_0^T V^*(t, -\dot{v}_{\lambda_j}(t))dt, \end{aligned} \tag{31}$$

where $(u_0, p_0) \in \partial\overline{W}$, (20), convexity of $V^*(t,\cdot)$, and the bound

$$\begin{aligned} V^*(t, -\dot{v}(t)) &\geq e^{-2kt}[(\overline{W})^*(p_0) + \langle u_0, -e^{kt}\dot{v}(t) - p_0\rangle] \\ &\geq -|(\overline{W})^*(p_0) - \langle u_0, p_0\rangle| - C_0|u_0| \end{aligned}$$

have been used to establish strong and hence weak lower semicontinuity (31). Moreover,

$$|v(0)|^2 \leq \varliminf_j |v_{\lambda_j}(0)|^2 \quad \text{and} \quad |v(T)|^2 \leq \varliminf_j |v_{\lambda_j}(T)|^2,$$

$$\int_0^T \frac{|v_{\lambda_j}(t) - J_{\lambda_j}(t, v_{\lambda_j}(t))|^2}{\lambda_j} \to 0,$$

and

$$\int_0^T \lambda_j |\dot{v}_{\lambda_j}(t)|^2 dt \leq C_0^2 T \lambda_j \to 0.$$

Since for every $t \in [0,T]$ and any $\lambda > 0$ and any $v, x \in H$, we have:

$$V(t, J_\lambda(t, v(t))) = V_\lambda(t, v(t)) + \frac{|v(t) - J_\lambda(t, v(t))|^2}{2\lambda}$$

and

$$V_\lambda^*(t,x) = V^*(t,x) + \frac{\lambda}{2}|x|^2,$$

it follows that

$$\begin{aligned} \Phi[v] &= \frac{1}{2}(|v(0)|^2 + |v(T)|^2) - 2\langle v(0), u_0\rangle + |u_0|^2 + \int_0^T [V(t, v(t)) + V^*(t, -\dot{v}(t))]\, dt \\ &\leq |u_0|^2 + \varliminf_j \frac{1}{2}(|v_{\lambda_j}(0)|^2 + |v_{\lambda_j}(T)|^2) - 2\langle v_{\lambda_j}(0), u_0\rangle \\ &\quad + \varliminf_j \int_0^T (\frac{|v_{\lambda_j}(t) - J_{\lambda_j}(t, v_{\lambda_j}(t))|^2}{2\lambda_j} + \frac{\lambda_j}{2}|\dot{v}_{\lambda_j}(t)|^2)dt \\ &\quad + \varliminf_j \int_0^T V(t, J_{\lambda_j}(t, v_{\lambda_j}(t)))dt + \varliminf_j \int_0^T V^*(t, -\dot{v}_{\lambda_j}(t))dt \\ &\leq |u_0|^2 + \varliminf_j \frac{1}{2}(|v_{\lambda_j}(0)|^2 + |v_{\lambda_j}(T)|^2) - 2\langle v_{\lambda_j}(0), u_0\rangle \\ &\quad + \varliminf_j \int_0^T V_{\lambda_j}(t, v_{\lambda_j}(t)) + (V_{\lambda_j})^*(t, -\dot{v}_{\lambda_j}(t))dt \\ &\leq \varliminf_j \Bigg(\frac{1}{2}(|v_{\lambda_j}(0)|^2 + |v_{\lambda_j}(T)|^2) - 2\langle v_{\lambda_j}(0), u_0\rangle + |u_0|^2 \\ &\qquad + \int_0^T V_{\lambda_j}(t, v_{\lambda_j}(t)) + V_{\lambda_j}^*(t, -\dot{v}_{\lambda_j}(t))dt\Bigg) \\ &= 0 \end{aligned}$$

From (9) we have the opposite inequality $\Phi[\,\cdot\,] \geq 0$ so the theorem is proved. □

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