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STABLE ROTATING BINARY STARS AND FLUID IN A TUBE

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Communicated by Giles Auchmuty

Dedicated to the memory of John Joseph McCann (1914–2004).

Abstract. This paper considers compressible fluid models for a Newtonian rotating star. For fixed mass and large angular momentum, stable solutions to the associated Navier-Stokes-Poisson system are constructed in the form of slow, uniformly rotating binary stars with specified mass ratio. The variational method employed was suggested by Elliott Lieb; it predicts uniform rotation as a consequence rather than an assumption. The density profiles of the solutions are local energy minimizers in the Wasserstein L^∞ metric; no global energy minimum can be achieved. A one-dimensional toy model admitting explicit solution is also introduced which caricatures the situation: to any specified number of components and their masses corresponds a single family of solutions, parameterized by angular velocity up to the point of equatorial break-up; here the equilibrium model breaks down as the atmosphere of the lightest star in the system begins to drift into space.

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1. The stability of rotating stars

In a simple class of astrophysical models, a star is represented as a fixed mass of gravitating fluid, obeying an equation of state in which the pressure $P(\varrho)$ depends on the density only; with an appropriate equation of state this can be shown to be a reasonable model for a cold white dwarf star, as in Lieb and Yau [25]. The investigation of rotating equilibria for such a fluid has been of mathematical and physical interest since the time of Newton: the history of the homogeneous incompressible case alone was chronicled by Chandrasekhar [12]. More recently, compressible fluid models have enjoyed a revival of interest since Auchmuty and Beals [3][4] demonstrated the existence of axisymmetric equilibria in which infinitesmal concentric cylinders of fluid rotate differentially; see e.g. works [5][6][9][17][18][19][22][13] of Auchmuty, Caffarelli, Chanillo, Friedman, Li, and Turkington. Either the angular velocity or angular momentum profile of differential rotation was specified a priori, and satisfied decay conditions which — as Li pointed out [22] — precluded the possibility of uniform rotation. Thus the equilibria of Auchmuty and Beals, though they solve the inviscid Euler equations, do not represent ground states of the physical system: differential rotation implies that there is excess energy waiting to be dissipated through viscous friction.

A more fundamental problem is to determine the stable equilibrium states of the system, subject only to the physical constraints of specified fluid mass, linear momentum and angular momentum $\mathbf{J}$ about the center of mass. This is the problem addressed here. It is formulated as a variational minimization of the energy $E(\rho, \mathbf{v})$, which depends on the fluid density $\rho(\mathbf{x}) \geq 0$ and velocity field $\mathbf{v}(\mathbf{x})$ on $\mathbf{R}^3$. The problem is peculiar in that the energy — although bounded below — does not attain its constrained minimum except in the non-rotating case $\mathbf{J} = 0$. As Morgan describes [33], for $\mathbf{J} \neq 0$ the 'energy minimizing state is not an oblate spheroid, but a stationary ball with a small distant planet'; the smaller the planet, the more energetically favorable. As a result, one is forced to settle for *local* energy minimizers, where local must be suitably defined. Such minimizers prove to be stable, uniformly rotating solutions of the Euler- or Navier-Stokes-Poisson system:

$$\nabla P(\rho) = \rho \left\{ \nabla(V_\rho) + \omega^2 r(\mathbf{x})\, \hat{\mathbf{e}}_r \right\}; \tag{1}$$

$$-\Delta V_\rho = 4\pi\rho. \tag{2}$$

Here cylindrical co-ordinates have been chosen for the center of mass frame; the angular momentum $\mathbf{J} = J\hat{\mathbf{e}}_z$ selects the z-axis. This axis will be a principal axis of inertia for ρ, and the corresponding moment of inertia $I(\rho)$ determines the angular velocity $\omega := J/I(\rho)$. The bounded gravitational potential V_ρ is given by (2).

Regarding the astrophysical relevance of this formulation, we concede that for many applications relaxation to uniform rotation takes place on unreasonably long time scales. Nevertheless, there are contexts in which it may be a dominant effect. For example, observational evidence indicates that in ancient close binary systems, the rotational periods of the component stars coincide with the system's orbital period; the two stars rotate as a solid body; see Olson [34].

On physical grounds, it is evident that the system (1-2) should have solutions for any prescribed fluid mass M and angular momentum J. However, solutions have been proven to exist only for J small by Li [22]; his analysis assumes axisymmetry, and although formulated for prescribed angular velocity ω, is straightforward to adapt for prescribed angular momentum instead. He also shows no solutions persist if ω is too large. In the following pages, the existence of solutions is demonstrated in a complementary regime: for large angular momentum J. These solutions take the form of binary stars, in which the fluid mass is divided into two disjoint regions widely separated relative to their size. The ratio of masses between the two regions may be specified a priori. It is clear that these solutions will not be axisymmetric, but they do have $z = 0$ as a symmetry plane. The approach is a variation on the methods of Auchmuty, Beals [3] and Li [22], using comparison with a Kepler system of two-point masses orbiting each other to generate the necessary energy estimates a priori.

Since they are constructed as local energy minimizers, these binary stars will be stable. However, the stationarity condition they satisfy (15) differs slightly from the Euler-Lagrange equation for a global energy minimizer, in that the *chemical potential* — usually thought of as a Lagrange multiplier conjugate to the constraint of fixed mass — need not be constant throughout the set $\{\rho > 0\}$; instead, each connected component of $\{\rho > 0\}$ has its own chemical potential. This possibility is of particular relevance if one is interested in counting connected components of a solution as in Caffarelli and Friedman [9] or Chanillo and Li [13].

It also makes perfect sense physically: one would not expect particles at the earth's surface to be as tightly bound as at the surface of the sun, even if the system were in equilibrium.

Unfortunately, intermediate values of the angular momentum J remain inaccessible to us. However, some global features of the problem may be demonstrated in the context of a one-dimensional toy model proposed in §4. This model represents an interacting compressible fluid, constrained to live in a long light tube, and rotating end-over-end about its center of mass. It has the virtue of being exactly solvable: for a given mass, the solutions come in an uncountable number of disjoint families, each parameterized continuously by the angular velocity ω. The solutions with connected support — single stars — form a family which persists as long as J is not too large. The remaining families persist for J not too small, and represent binary stars or stellar systems in the astrophysical analogy. All families terminate with equatorial break-up, at which point the atmosphere drifts away from the surface of the lightest star in the system and a non-equilibrium transfer of mass between different components may ensue. It would be interesting to know which among these many equilibrium states actually represent W_∞ local energy minimizers, but that question is not addressed here.

In the following section, the three-dimensional problem and results are formulated precisely. Section 3 collects results which, although not original, are required for the analysis; the reduction to uniform rotation is due to Elliott Lieb. In §4 the one-dimensional model is introduced and analyzed, while the stationarity and regularity properties of local energy minimizers for the real problem are discussed in §5, after the Wasserstein L^∞ metric W_∞ has been introduced. The last section contains the proof that for large angular momentum, local minimizers exist in the form of binary stars.

Epilog

It is interesting to note that the Wasserstein L^∞ metric was subsequently used by Carrillo, Gualdani and Toscani to bound the growth of the wetted region in porous medium flow [10]; Vazquez showed this argument was limited to one-dimension [41]. The displacement convexity inequalities [30] [31] [32], which have proven to have far-reaching consequences [1] [2] [7] [8] [11] [14] [15] [16] [27] [29] [35] [36] [37] [38] [40], were first discovered in the context of toy model below, for a non-rotating star gravitating under one-dimensional Coulombic attraction.

2. Variational formulation

The state of a fluid may be represented by its mass density $\rho(\mathbf{x}) \geq 0$ and velocity vector field $\mathbf{v}(\mathbf{x})$ on $\mathbf{R}^3$. If the fluid interacts with itself through Newtonian gravity and satisfies an equation of state in which the pressure $P(\varrho)$ is an increasing function of the density only, then its energy $E(\rho, \mathbf{v})$ is given as the sum of three terms: the internal energy $U(\rho)$, gravitational interaction energy $G(\rho, \rho)$, and kinetic energy $T(\rho, \mathbf{v})$. Each is expressed as an integral over $\mathbf{x} \in \mathbf{R}^3$:

$$E(\rho, \mathbf{v}) := U(\rho) - G(\rho, \rho)/2 + T(\rho, \mathbf{v}); \tag{3}$$

$$U(\rho) := \int A(\rho(\mathbf{x}))\, d^3\mathbf{x}; \tag{4}$$

$$G(\sigma, \rho) := \int V_\sigma \, d\rho(\mathbf{x}); \tag{5}$$

$$T(\rho, \mathbf{v}) := \frac{1}{2} \int |\mathbf{v}|^2 d\rho(\mathbf{x}). \tag{6}$$

We hope no confusion is caused by our use of the same notation to denote both the measure $d\rho(\mathbf{x}) = \rho(\mathbf{x})\, d^3\mathbf{x}$ and its Lebesgue density. Here $A(\varrho)$ is a convex function obtained from the equation of state by integrating $dU = -Pdv$ from infinite to unit volume,

$$A(\varrho) := \int_1^\infty P(\varrho/v)\, dv, \tag{7}$$

while V_ρ represents the gravitational potential of the mass density $\rho(\mathbf{x})$:

$$V_\rho(\mathbf{x}) := \int \frac{d\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}. \tag{8}$$

Units are chosen so that the total mass of fluid $M = 1$ and the gravitational constant $G = 1$, and a frame of reference is chosen in which the center of mass

$$\overline{\mathbf{x}}(\rho) := \Big(\int d\rho\Big)^{-1} \int \mathbf{x}\, d\rho(\mathbf{x}) \tag{9}$$

is at rest. One is then interested in finding minimum energy configurations subject to constraints of fixed mass and angular momentum $\mathbf{J}$ about the center of mass $\overline{\mathbf{x}}(\rho)$. The fluid angular momentum is given by $\mathbf{J}(\rho, \mathbf{v})$:

$$\mathbf{J}(\rho, \mathbf{v}) := \int \Big(\mathbf{x} - \overline{\mathbf{x}}(\rho)\Big) \times \mathbf{v}\, d\rho(\mathbf{x}). \tag{10}$$

Before addressing the rotating problem $\mathbf{J} \neq 0$, further assumptions and results will be required of the non-rotating energy $E_0(\rho) := U(\rho) - G(\rho, \rho)/2$. The pressure $P(\varrho)$ may take a quite general form, including the polytropic equations of state $P(\varrho) = \varrho^q$ with $q > 4/3$ and the Chandrasekhar equation from Lieb and Yau [25], which models the quantum degeneracy pressure of relativistic fermions. Following Auchmuty and Beals [3], the tacit assumptions on $P(\varrho)$ will be:

(F1) $P : [0, \infty) \longrightarrow [0, \infty)$ continuous and strictly increasing;

(F2) $\lim_{\varrho \to 0} P(\varrho)\varrho^{-4/3} = 0$;

(F3) $\liminf_{\varrho \to \infty} P(\varrho)\varrho^{-4/3} > K(M)$.

In (F3), the constant $K(M) > 0$ must be sufficiently large to prevent gravitational collapse: $U(\rho)$ must control $G(\rho, \rho)$ at large densities so that the Chandrasekhar mass for the model is assumed greater than $M = 1$. (F1)–(F2) ensure that $A(\varrho)$ is C^1 and strictly convex with $A'(\varrho)\varrho - A(\varrho) = P(\varrho)$ on $[0, \infty)$. (F2) also ensures that a diffuse gas does not disperse to ∞. Under these assumptions, $E_0(\rho)$ will be bounded below on

$$\mathcal{R}(\mathbf{R}^3) = \{\rho \in L^{4/3}(\mathbf{R}^3) \mid \rho \geq 0 \quad \int \rho = 1\} \tag{11}$$

assuming its minimum there [3]. The problem is formulated in $L^{4/3}(\mathbf{R}^3)$ because $E_0(\rho) \leq C$ and (F3) imply a bound on $\|\rho\|_{4/3}$. Results are also required regarding the non-rotating minimizer σ_m of $E_0(\rho)$ among configurations of mass $m < 1$, and the corresponding minimum energy

$$e_0(m) := E_0(\sigma_m) = \inf_{\rho \in \mathcal{R}(\mathbf{R}^3)} E_0(m\rho). \tag{12}$$

Drawn from Auchmuty and Beals [3] and Lieb and Yau [25], these are summarized in Theorem 3.5 below.

In the presence of rotation, it is convenient to formulate the variational problem on a centered subset $\mathcal{R}_0(\mathbf{R}^3)$ of $\mathcal{R}(\mathbf{R}^3)$:

$$\mathcal{R}_0(\mathbf{R}^3) := \{\rho \in \mathcal{R}(\mathbf{R}^3) \mid \overline{\mathbf{x}}(\rho) = 0; \quad \operatorname{spt} \rho \text{ is bounded}\}. \tag{13}$$

Here $\operatorname{spt} \rho$ denotes the *support* of ρ, the smallest closed set carrying the full mass of ρ. Bounded support ensures that ρ has a center of mass and finite moments of inertia. It is not implausible that solutions to (1) will have bounded support, since that equation is trivially satisfied where ρ vanishes. The velocity fields $\mathbf{v}$ will be taken to lie in $\mathcal{V}(\mathbf{R}^3) := \{\mathbf{v} : \mathbf{R}^3 \longrightarrow \mathbf{R}^3 \text{ Lebesgue measurable}\}$. For prescribed angular momentum $\mathbf{J} \neq 0$, the energy $E(\rho, \mathbf{v})$ is bounded from below on $\mathcal{R}_0(\mathbf{R}^3)$ by the non-rotating energy $e_0(1)$. However, as in Morgan [33], Example 3.6 demonstrates that this bound — although approached — will not be attained [30]. Thus the search for a global energy minimizer will be futile, and one is forced to settle for *local minimizers* of $E(\rho, \mathbf{v})$ in an appropriate topology. However, the choice of topology on $\mathcal{R}(\mathbf{R}^3)$ is quite delicate: Remark 3.7 shows that local energy minimizers will not exist if this topology is inherited from a topological vector space. Instead, $\mathcal{R}(\mathbf{R}^3)$ is topologized via the Wasserstein L^∞ metric of the probability literature. This metric is defined in §5 and denoted by W_∞. The velocity fields $\mathbf{v}$ may be topologized in any way which makes $\mathcal{V}(\mathbf{R}^3)$ a topological vector space. *Local* and *continuous* refer to these topologies hereafter.

With these definitions, Theorem 2.1 may be stated; it collects the results of §3 and §5. Its conclusions apply to energy minimizers subject only to a constraint on the z-component $J_z(\rho, \mathbf{v}) := \hat{\mathbf{e}}_z \cdot \mathbf{J}(\rho, \mathbf{v})$ of the angular momentum, but are extended to the case of physical interest by the corollary and remark following. That local minimizers exist in the form of binary stars for large $\mathbf{J}$ is the content of Theorem 6.1 and its corollary. Both theorems are proved by adapting the variational approach of Auchmuty and Beals [3] to the context of W_∞-local energy minimizers.

Two final definitions are required: let $[\lambda]_+ := \max\{\lambda, 0\}$; and for $\delta > 0$ define the *δ-neighbourhood* of $\Omega \subset \mathbf{R}^3$ to be the set

$$\Omega + B_\delta(0) := \bigcup_{\mathbf{y} \in \boldsymbol{\Omega}} \{\mathbf{x} \in \mathbf{R}^3 \mid |\mathbf{x} - \mathbf{y}| < \delta\}. \tag{14}$$

Theorem 2.1 (Properties of W_∞-Local Energy Minimizers)**.**
Let $J > 0$. If $(\rho, \mathbf{v})$ minimizes $E(\rho, \mathbf{v})$ locally on $\mathcal{R}_0(\mathbf{R}^3) \times \mathcal{V}(\mathbf{R}^3)$ subject to the constraint $J_z(\rho, \mathbf{v}) = J$ then:

- *(i) the z-axis is a principal axis of inertia for ρ, with a moment of inertia $I(\rho)$ from (16) which is maximal and non-degenerate;*
- *(ii) the rotation is uniform: $\mathbf{v}(\mathbf{x}) := (J\hat{\mathbf{e}}_z \times \mathbf{x})/I(\rho)$;*
- *(iii) ρ is continuous on $\mathbf{R}^3$;*
- *(iv) on each connected component Ω_i of $\{\rho > 0\}$, ρ satisfies*

$$A'\Big(\rho(\mathbf{x})\Big) \;=\; \left[\frac{J^2}{2I^2(\rho)} r^2(\mathbf{x}) + V_\rho(\mathbf{x}) + \lambda_i\right]_+ \tag{15}$$

  for some chemical potential $\lambda_i < 0$ depending on the component;
- *(v) the equations (15) continue to hold on a δ-neighbourhood of the Ω_i;*
- *(vi) where ρ is positive, it has as many derivatives as the inverse of $A'(\varrho)$;*
- *(vii) if $P(\varrho)$ is continuously differentiable on $[0, \infty)$ then ρ satisfies (1);*
- *(viii) this solution is stable with respect to L^∞-small perturbations of the Lagrangian fluid variables.*

Corollary 2.2 (Local Minimizers with $J_z = J$ have $\mathbf{J}(\rho, \mathbf{v}) = J\hat{\mathbf{e}}_z$)**.**
Let $J > 0$. Suppose $(\rho, \mathbf{v})$ minimizes $E(\rho, \mathbf{v})$ locally on $\mathcal{R}_0(\mathbf{R}^3) \times \mathcal{V}(\mathbf{R}^3)$ subject to the constraint $J_z(\rho, \mathbf{v}) = J$. Then $(\rho, \mathbf{v})$ also minimizes $E(\rho, \mathbf{v})$ locally subject to the constraints $\mathbf{J}(\rho, \mathbf{v}) = J\hat{\mathbf{e}}_z$.

Proof. Theorem 2.1(i-ii) shows that the angular momentum of $(\rho, \mathbf{v})$ satisfies the constraints $\mathbf{J}(\rho, \mathbf{v}) = J\hat{\mathbf{e}}_z$ of the more restricted minimization. □

Remark 2.3. (Converse) Although the proof is not given, a converse to Corollary 2.2 is also true: any local minimizer subject to the vector constraint $\mathbf{J}(\rho, \mathbf{v}) = J\hat{\mathbf{e}}_z$ also minimizes locally among the larger class of competitors with prescribed $J_z(\rho, \mathbf{v}) = J$, provided the topology on $\mathcal{V}(\mathbf{R}^3)$ is assumed to enjoy a little more structure: the map taking $\mathbf{w} \in \mathbf{R}^3$ to $\mathbf{v}(\mathbf{x}) := \mathbf{w} \times \mathbf{x} \in \mathcal{V}(\mathbf{R}^3)$ should be continuous. The proof requires Remark 3.3, and the observation that a local energy minimizer subject to the constraint $\mathbf{J}(\rho, \mathbf{v}) = \mathbf{J}$ must rotate about a principal axis with maximal moment of inertia. Otherwise a slight rotation would lower its energy. To exploit this observation, it is necessary to know that slight rotations are *local* perturbations in $\mathcal{R}_0(\mathbf{R}^3) \times \mathcal{V}(\mathbf{R}^3)$, but this follows from the topology on $\mathcal{V}(\mathbf{R}^3)$ and Lemma 5.1(iii).

Remark 2.4. (Stationarity Conditions for Energy Minimizers) The Euler-Lagrange equation for a global energy minimizer, or indeed any critical point of the functional $E(\rho, \mathbf{v})$, differs from Theorem 2.1(iv) in that (15) would be satisfied on all of $\mathbf{R}^3$ for a single chemical potential λ_i. Since the Navier-Stokes equation (1) follows from (15) by taking a gradient and multiplying by ρ, it will be satisfied whether or not $\lambda_i = \lambda_j$ on different connected components of $\{\rho > 0\}$. Conversely, for $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ to be a solution of (1), an integration shows that the conclusion of Theorem 2.1(iv) is necessary as well as sufficient.

3. Uniform rotation minimizes kinetic energy

This section recounts several results which, although not original, will be required for the analysis. In particular, it is shown that the problem of minimizing $E(\rho, \mathbf{v})$ is equivalent to a minimization in which the fluid rotates uniformly about its center of mass; the idea of this reduction is due to Elliott Lieb [23], and was implemented in the thesis upon which this paper is based [30]. Results regarding the minimization of the non-rotating energy are also recalled. Used here to demonstrate that the energy of a rotating star — though bounded below — cannot attain its minimum, they will also be required in §6 below.

Since the z-component of the angular momentum is specified, the moment of inertia $I(\rho)$ of $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ in the direction of $\hat{\mathbf{e}}_z$ will be relevant; in cylindrical co-ordinates $(r(\mathbf{x}), \phi(\mathbf{x}), z(\mathbf{x}))$ such that $\mathbf{x} = (r\cos\phi, r\sin\phi, z)$ it is given by

$$I(\rho) := \int r^2(\mathbf{x} - \overline{\mathbf{x}}(\rho))\, d\rho(\mathbf{x}). \tag{16}$$

Proposition 3.1 (Uniform Rotation around Center of Mass [23])**.**
Fix a fluid density $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ and $J \geq 0$. Among all velocities $\mathbf{v} \in \mathcal{V}(\mathbf{R}^3)$ for which $T(\rho, \mathbf{v}) < \infty$ and satisfying the constraint $J_z(\rho, \mathbf{v}) = J$, the kinetic energy

$T(\rho, \mathbf{v})$ is uniquely minimized by a uniform rotation $\mathbf{v}(\mathbf{x}) := \omega \hat{\mathbf{e}}_z \times \mathbf{x}$ with angular velocity $\omega := J/I(\rho)$.

Proof. Let $\mathcal{H} := L^2(\mathbf{R}^3, d\rho(\mathbf{x})) \subset \mathcal{V}(\mathbf{R}^3)$ denote the Hilbert space of vector fields on $\mathbf{R}^3$, with inner product $\langle \mathbf{v}, \mathbf{v} \rangle_{\mathcal{H}} := 2T(\rho, \mathbf{v})$. The uniform rotation $\hat{\mathbf{e}}_z \times \mathbf{x} \in \mathcal{H}$ since $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ compactly supported gives $I(\rho) < \infty$, while the velocities $\mathbf{v}$ of interest lie in the affine subspace $\mathcal{G} \subset \mathcal{H}$ where the constraint $\langle \mathbf{v}, \hat{\mathbf{e}}_z \times \mathbf{x} \rangle_{\mathcal{H}} = J$ is satisfied. Minimizing the norm $\langle \mathbf{v}, \mathbf{v} \rangle_{\mathcal{H}}$ over $\mathcal{G}$ yields $\mathbf{v}_g := \omega(\hat{\mathbf{e}}_z \times \mathbf{x})$: any other $\mathbf{v} \in \mathcal{G}$ differs from $\mathbf{v}_g$ by a vector orthogonal to $\hat{\mathbf{e}}_z \times \mathbf{x}$ in $\mathcal{H}$. □

Corollary 3.2 (Local Energy Minimizers Rotate Uniformly).
Let $J \geq 0$. If $(\rho, \mathbf{v})$ minimizes $E(\rho, \mathbf{v})$ locally on $\mathcal{R}_0(\mathbf{R}^3) \times \mathcal{V}(\mathbf{R}^3)$ subject to the constraint $J_z(\rho, \mathbf{v}) = J$, then $\mathbf{v}(\mathbf{x}) = \omega \hat{\mathbf{e}}_z \times \mathbf{x}$ with $\omega := J/I(\rho)$.

Proof. The curve $(1-t)\mathbf{v}(\mathbf{x}) + t\,\omega(\hat{\mathbf{e}}_z \times \mathbf{x})$ is continuous in the topological vector space $\mathcal{V}(\mathbf{R}^3)$, and the linear constraint is satisfied along it. Moreover, $T(\rho, \mathbf{v})$ and hence $E(\rho, \mathbf{v})$ is a quadratic function of t along this curve, assuming its minimum at $t = 1$ by Proposition 3.1. Thus $(\rho, \mathbf{v})$ cannot be a local minimum unless $\mathbf{v}(\mathbf{x}) = \omega \hat{\mathbf{e}}_z \times \mathbf{x}$. □

Remark 3.3. (Uniform Rotation when $\mathbf{J}(\rho, \mathbf{v})$ is Prescribed). The proofs of Proposition 3.1 and its corollary extend to the case where the linear constraint $J_z(\rho, \mathbf{v}) = J$ is replaced by three linear constraints $\mathbf{J}(\rho, \mathbf{v}) = J\hat{\mathbf{e}}_z$. The conclusion then is that $\mathbf{v}(\mathbf{x}) = \mathbf{w} \times \mathbf{x}$, where $\mathbf{w} \in \mathbf{R}^3$ is the unique angular velocity compatible with the given density ρ and angular momentum $\mathbf{J}$. Of course, the axis $\mathbf{w}$ of rotation may not coincide with the z-axis.

If $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ rotates with velocity $\mathbf{v}(\mathbf{x}) = (J\hat{\mathbf{e}}_z \times \mathbf{x})/I(\rho)$, then its kinetic energy $T(\rho, \mathbf{v})$ is given by

$$T_J(\rho) := \frac{J^2}{2I(\rho)}. \tag{17}$$

A second corollary shows that the minimization of Theorem 2.1 is equivalent to the minimization of

$$E_J(\rho) := U(\rho) - G(\rho, \rho)/2 + T_J(\rho). \tag{18}$$

Corollary 3.4 (Velocity-free Reformulation).
Let $J \geq 0$ and $\rho \in \mathcal{R}_0(\mathbf{R}^3)$, and define $\omega := J/I(\rho)$. Then $(\rho, \mathbf{v})$ minimizes $E(\rho, \mathbf{v})$ locally on $\mathcal{R}_0(\mathbf{R}^3) \times \mathcal{V}(\mathbf{R}^3)$ subject to the constraint $J_z(\rho, \mathbf{v}) = J$ if and only if ρ minimizes $E_J(\rho)$ locally on $\mathcal{R}_0(\mathbf{R}^3)$ and $\mathbf{v}(\mathbf{x}) = \omega \hat{\mathbf{e}}_z \times \mathbf{x}$.

Proof. Assume $(\rho, \mathbf{v})$ minimizes $E(\rho, \mathbf{v})$ locally on $\mathcal{R}_0(\mathbf{R}^3)$ subject to the constraint $J_z(\rho, \mathbf{v}) = J$. By Corollary 3.2, $\mathbf{v}(\mathbf{x}) = \omega \hat{\mathbf{e}}_z \times \mathbf{x}$, whence $T(\rho, \mathbf{v}) = T_J(\rho)$. Lemma 5.1(v) shows that $I(\rho)$ is continuous on $\mathcal{R}_0(\mathbf{R}^3)$, therefore ρ' sufficiently close to ρ ensures that $\omega' := J/I(\rho')$ differs little from ω. Because $\mathcal{V}(\mathbf{R}^3)$ is a topological vector space, $\mathbf{v}'(\mathbf{x}) := \omega' \hat{\mathbf{e}}_z \times \mathbf{x}$ can be made close to $\mathbf{v}$. Since $(\rho, \mathbf{v})$ is a local energy minimum, taking ρ' closer to ρ if necessary ensures $E(\rho, \mathbf{v}) \leq E(\rho', \mathbf{v}') = E_J(\rho')$, establishing one implication.

The other implication is easier. Assume ρ minimizes $E_J(\rho)$ locally, and define $\mathbf{v}(\mathbf{x}) := \omega \hat{\mathbf{e}}_z \times \mathbf{x}$. For ρ' near ρ and any $\mathbf{v}' \in \mathcal{V}(\mathbf{R}^3)$, Proposition 3.1 yields $E(\rho', \mathbf{v}') \geq E_J(\rho') \geq E_J(\rho) = E(\rho, \mathbf{v})$. □

The analysis will henceforth be devoted to $E_J(\rho)$. Some results regarding the non-rotating problem $J = 0$ are required. Implications of Auchmuty and Beals [3, Theorems A and B] and Lieb and Yau [25, Theorem 3(b,d,e)] are summarized here. Results from the latter are statedly explicitly for the Chandrasekhar equation of state, but apply equally well to all $A(\varrho)$ consistent with (F1)–(F3). If, in addition, $A'(\varrho^3)$ is convex, uniqueness of minimizer up to translation can also be shown as in Lieb and Yau [25, Lemma 11 and remark following].

Theorem 3.5 (Non-rotating Stars [3, 25])**.**
For $E_0(\rho)$ from (18), $e_0(m)$ from (12) and $m \in [0, 1]$:

(i) $E_0(\rho)$ attains its minimum $e_0(m)$ among ρ such that $m^{-1}\rho \in \mathcal{R}(\mathbf{R}^3)$;
(ii) $e_0(m)$ decreases continuously from $e_0(0) = 0$ and is strictly concave;

There are bounds $R_0(m)$ and $C_0(m)$ on the radius and central density, such that any mass m minimizer σ_m of $E_0(\rho)$ satisfies

(iii) σ_m is spherically symmetric and radially decreasing after translation;
(iv) $\|\sigma_m\|_\infty \leq C_0(m)$;
(v) $\operatorname{spt} \sigma_m$ is contained in a ball of radius $R_0(m)$;
(vi) σ_m is continuous; where positive it has as many derivatives as the inverse of $A'(\varrho)$;
(vii) σ_m satisfies (15) on all of $\mathbf{R}^3$ for $J = 0$ and a single $\lambda < 0$;
(viii) the left and right derivatives of $e_0(m)$ bound λ: $e_0'(m^+) \leq \lambda \leq e_0'(m^-)$.

For a rotating star $J > 0$, it has already been asserted that the lower bound $E_J(\rho) \geq e_0(1)$ is approached but not attained on $\mathcal{R}(\mathbf{R}^3)$. That it cannot be attained is now clear: $E_0(\rho) \geq e_0(1)$ while $T_J(\rho) \geq 0$; when the first inequality is saturated, Theorem 3.5(v) forces the second inequality to be strict. The following example uses Theorem 3.5(ii) to construct $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ with $E_J(\rho)$ arbitrarily close to $e_0(1)$. This observation of Lieb [23] and [30] was also developed by Morgan [33].

Example 3.6. (No Constrained Minimum of $E(\rho, \mathbf{v})$ is Attained). Let $J > 0$ and σ_m and σ_{1-m} be the non-rotating energy minimizers of masses m and $1-m$ respectively. From Theorem 3.5(ii), $e_0(m) + e_0(1-m)$ approximates $e_0(1)$ for $m > 0$ sufficiently small. Since σ_m has a finite radius, $|\mathbf{y}|$ sufficiently large yields a trial function $\rho(\mathbf{x}) := \sigma_m(\mathbf{x}) + \sigma_{1-m}(\mathbf{x} - \mathbf{y})$ with energy

$$E_J(\rho) = e_0(m) + e_0(1-m) - G(\sigma_{1-m}, \sigma_m) + T_J(\rho). \tag{19}$$

Taking $|\mathbf{y}|$ larger if necessary forces $T_J(\rho)$ to be small since

$$I(\rho) = I(\sigma_m) + I(\sigma_{1-m}) + m(1-m)|\mathbf{y}|^2. \tag{20}$$

Thus $E_J(\rho)$ can be made to approach the energy $e_0(1)$ of the non-rotating minimizer.

Remark 3.7. (No Local Minimizers in a Vector Space Topology). The preceding example showed that the search for a global minimizer will be fruitless. More is true: for $E_J(\rho)$ to have even local minimizers, the topology on $\mathcal{R}_0(\mathbf{R}^3)$ must not be inherited from a topological vector space. Otherwise, a local minimum $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ would be stable with respect to all perturbations $\rho + t\sigma \in \mathcal{R}_0(\mathbf{R}^3)$; that is, $t > 0$ sufficiently small would imply $E_J(\rho + t\sigma) \geq E_J(\rho)$. The resulting stationarity condition would be (15), satisfied on on all of $\mathbf{R}^3$ for a fixed λ_i independent of i. But this is absurd: it implies $\rho(\mathbf{x}) \to \infty$ as $r(\mathbf{x}) \to \infty$. Stated physically, it is energetically favorable to slow down a rotating star by removing a small bit of mass to a far away orbit, where it carries little kinetic energy but great angular momentum.

4. Fluid in a tube: A toy model

Before proceeding with the analysis of the three-dimensional problem, a one-dimensional toy model is introduced which illustrates a number of subtleties. This model represents an interacting fluid, constrained to live in a long light tube, and rotating end-over-end about its center of mass. The interaction is one-dimensional Coulomb attraction — force independent of distance — while the equation of state is taken to be $P(\varrho) = c\varrho^2$ for simplicity. As in the three-dimensional problem, the energy (21) of a mass of fluid carrying angular momentum J assumes its minimum only in the non-rotating case $J = 0$. However, the (one-dimensional) Euler-Poisson system (28) is explicitly soluble for this model, and a complete catalog of solutions may be obtained. These fall into an uncountable number of disjoint families or *sequences*, each parameterized continuously by the angular velocity $\omega = J/I(\rho) > 0$ up to some *critical value* ω_c. Beyond ω_c the sequence fails to exist. The solutions with connected support — single stars — begin with the non-rotating minimizer and persist as long as J is not too large. Each remaining family persists for J not too small, and consists of configurations in which a number of components with fixed masses 'orbit' each other; these represent binary stars or stellar systems in the astrophysical analogy.

The absence of bifurcations in this model should be emphasized. In problems of stellar evolution, bifurcations along equilibrium sequences raise interesting cosmological possibilities. For example, a theory of formation of double stars known as the fission hypothesis asserts that as ω is increased by gravitational contraction, a single star may deform *quasi-statically* into a binary system; see Lyttleton [28] or Tassoul [39]. Proposed by Kelvin and Tait before the turn of the century, this conjecture has not yet been rigorously resolved even in the context of the homogeneous incompressible model in $\mathbf{R}^3$. On the other hand, numerical studies of James [21] show that bifurcations do not occur in axisymmetric uniformly rotating models with polytropic equations of state $P(\varrho) = \varrho^q$ in which $q < 2.24$. Instead, the axisymmetric equilibria remain stable up to a point of 'equatorial break-up'. This is also the case in our toy model. There cooling or contraction may be represented by decreasing c at fixed J, which, after rescaling units, is equivalent to increasing J at fixed c. For a single star, ω increases with J; the radius grows, and the atmosphere near the surface becomes thinner and thinner until it is no longer gravitationally bound. For larger J there is no nearby equilibrium and the family ends. The same mechanism is responsible for the demise of all other equilibrium sequences as well. In these sequences however, ω varies *inversely* with J at large angular momentum: $\omega \to 0$ as $J \to \infty$. In this limit, the components approximate non-rotating minimizers of the same masses, placed so far apart that the system rotates very slowly. For larger ω, the stars draw closer together and the stellar material becomes less concentrated; equilibrium persists only as long as the atmosphere of the lightest star (or planet) continues to be bound.

In our one-dimensional model, the state of the fluid is represented by its mass density $\rho(x) \geq 0$ on the line; its total mass is is taken to be M and its center of mass to lie at the origin. If the whole tube rotates about this center of mass, the energy of the fluid is given by

$$E_J(\rho) := \int_{\mathbf{R}} \rho^2(x)\, dx \; + \; \frac{1}{2} \iint d\rho(x)\, |x-y|\, d\rho(y) \; + \; \frac{J^2}{2I(\rho)}. \tag{21}$$

Units of mass, length and energy may be fixed to ensure $c = M = G = 1$, where G is the 'gravitational' constant — the coefficient of the potential energy. The angular momentum J scales with $(M^6c^3/G)^{1/4}$, and the moment of inertia $I(\rho)$ is given by

$$I(\rho) := \int_{\mathbf{R}} x^2 d\rho(x). \tag{22}$$

The energy $E_J(\rho)$ is defined on the space $\mathcal{R}_0(\mathbf{R}) \subset L^2(\mathbf{R})$ of densities $\rho(x)$ with bounded support and satisfying the constraints

$$\rho(x) \geq 0, \tag{23}$$

$$\int d\rho(x) = 1, \tag{24}$$

$$\int x\, d\rho(x) = 0. \tag{25}$$

Defining $[x]_+ := \max\{x, 0\}$, any minimizer of $E_J(\rho)$ on $\mathcal{R}_0(\mathbf{R})$ must be a pointwise a.e. solution to the Euler-Lagrange equation

$$2\rho(x) = \left[\frac{J^2x^2}{2I^2(\rho)} - V_\rho(x) + \lambda\right]_+. \tag{26}$$

Here λ is the Lagrange multiplier conjugate to the mass constraint, while V_ρ is the gravitational potential

$$V_\rho(x) := \int |x - y|\, d\rho(y). \tag{27}$$

For the real model, (26) is established rigorously in Auchmuty and Beals [3] (see also §5 below); for the toy model, the proof would be similar.

However, unless $J = 0$, (26) can have no solutions in $\mathcal{R}_0(\mathbf{R})$, thus $E_J(\rho)$ is not minimized there: since $V_\rho(x)$ grows no faster than linearly for $\rho \in \mathcal{R}_0(\mathbf{R})$, any solution of (26) would diverge quadratically as $|x| \to \infty$, violating the mass constraint.

On the other hand, the equations corresponding to the Euler-Poisson system (1) are quite easy to solve; they are obtained by differentiating (26)–(27) to yield:

$$2\rho'(x) + 2M(x) - M(\infty) - \frac{J^2x}{I^2(\rho)} = \xi \quad \text{where } \rho(x) > 0 \tag{28}$$

and $M(x) := \int_{-\infty}^{x} \rho$. Here $\xi = 0$ if ρ has its center of mass at the origin.

The theorem below classifies all continuous solutions $\rho \in \mathcal{R}_0(\mathbf{R})$ to these equations; their properties are immediate from the exact formulas. The remainder of this section is occupied by its proof.

Theorem 4.1 (Catalog of One-dimensional Equilibria).
Choose the number of components $n \geq 1$ and their masses $(m_1, \ldots, m_n)$, ordered from left to right and with $\sum m_i = 1$. The radius $r \in (\pi/2, \pi]$ of the lightest component may also be specified. Then there is unique solution to (28) in $\mathcal{R}_0(\mathbf{R})$ with the prescribed parameters; it is given by (30)–(31). The angular velocity ω and radii r_i of any heavier components are determined by (32), while the locations z_i of the components are determined by (33); $\xi = 0$ and $J = \omega I(\rho)$. All continuous $\rho \in \mathcal{R}_0(\mathbf{R})$ which solve (28) with $J > 0$ are of this form. The r_i and ω increase continuously with r while the $|z_i|$ decrease. As $r \to \pi$, all tend to finite limiting values determined by the masses.

4.1. **Solutions with Connected Support.**

Proof. All continuous solutions $\rho \in \mathcal{R}_0(\mathbf{R})$ of (28), differentiable where positive, must be C^∞ there: $M(x)$ gains regularity from ρ and the result follows by a bootstrap. Thus ρ satisfies

$$\rho''(x) + \rho(x) = \omega^2/2 \quad \text{where } \rho(x) > 0, \tag{29}$$

for some ω. Conversely, any solution ρ to (29) with $\{\rho > 0\}$ connected also solves (28) for $J = \omega I(\rho)$ and some ξ. Such a star, if it has radius r and center of mass at the origin, can only be of the form

$$\rho_r(x) := \frac{\eta(r)}{2}\left(1 - \frac{\cos(x)}{\cos(r)}\right) \quad \text{if } x \in [-r, r], \qquad 0 \text{ otherwise.} \tag{30}$$

r must lie in $[\pi/2, \pi]$, while the normalization constant $\eta(r) := (r - \tan r)^{-1}$ for unit mass. The angular velocity required to sustain ρ_r is related to r by $\omega^2 = 2\rho_r''(r) + 2\rho_r(r) = \eta(r)$. Thus (28) is satisfied for $J = \omega I(\rho_r)$ and some ξ, while $\rho_r'(0) = 0$ and $M(0) = 1/2$ imply $\xi = 0$. Finally, $\eta(r)$ increases from 0 to π^{-1} on $[\pi/2, \pi]$, so ω parameterizes the sequence as it ranges from 0 to $\omega_c(1) = \pi^{-1/2}$.

For single stars it remains to demonstrate that $J = I(\rho_r)\omega$ varies directly with ω. Since ω increases with r, it suffices to show that $I(\rho_r)$ is also increasing. For $r < r'$, $\rho_r(x) = \rho_{r'}(x)$ is solved at a unique value of $|x| < r'$; $I(\rho_r) < I(\rho_r')$ therefore follows from (22). Thus J attains its maximal value for $r = \pi$. At this value, the density gradient at the star's surface vanishes: $\rho_r'(r) = \eta(\pi)\tan(\pi) = 0$; the pressure gradient must vanish as well, so the fluid at the surface goes 'into orbit'. For larger angular momentum this outermost fluid cannot remain contiguous with the star in equilibrium.

4.2. **Solutions with Disconnected Support.** Having enumerated the solutions corresponding to single stars, it remains to consider solutions ρ to (28) for which $\{\rho > 0\}$ is disconnected. These must also satisfy (29). If the interval $(z-r, z+r)$ is a connected component of $\{\rho > 0\}$, then the restriction of ρ to this interval must be $m\rho_r(x-z)$ for some mass $m < 1$. As before, $r \in [\pi/2, \pi]$. One can then ask: given an ordered n-tuple of masses satisfying $m_1 + \ldots + m_n = 1$, for which angular velocities will there be a solution $\rho \in \mathcal{R}_0(\mathbf{R})$ given by

$$\rho(x) = \sum m_i \rho_{r_i}(x - z_i) \tag{31}$$

for some radii r_i and centers z_i, ordered so that $z_i + r_i \leq z_{i+1} - r_{i+1}$. All solutions to (28) in $\mathcal{R}_0(\mathbf{R})$ must be of this form: a star cannot have infinitely many planets with radii $r \geq \pi/2$ and also have bounded support. Below it is demonstrated that exactly one such solution exists for each $\omega > 0$ up to some critical value $\omega_c(m_1, m_2, \ldots, m_n) < \infty$. The sizes of the components vary inversely with their masses, and it is easiest to parameterize the sequence in terms of the radius r of the lightest component. $r = \pi$ at the critical value $\omega = \omega_c(m_1, \ldots, m_n)$, while $r \to \pi/2$ (the radius of the non-rotating minimizer) and $J \to \infty$ as $\omega \to 0$.

If (31) is to satisfy (29), it is necessary that $\rho''(z_i + r_i) = \omega^2/2$ independently of i. Thus the radii must satisfy

$$\eta(r_i) = \omega^2/m_i. \tag{32}$$

Since $\eta(r)$ increases from 0 to π^{-1} on $[\pi/2, \pi]$, these equations are soluble provided $\omega^2/m \leq \pi^{-1}$ for the lightest mass m. Conversely, $\omega > 0$ may selected by prescribing the radius $r \in (\pi/2, \pi]$ of the lightest component, in which case the remaining radii are uniquely determined. (29) will be satisfied, provided the centers z_i are chosen far enough apart so that the components do not overlap. This will be verified a posteriori. With $J/I(\rho)$ replaced by ω, (28) will also be satisfied on each component separately if the constant of integration ξ is allowed to depend on the component. The trick is to choose the centers so that $\xi = 0$ for all i. Computing (28) at $x = z_i$ where $\rho'(z_i) = 0$, it is clear that $\xi = 0$ is equivalent to

$$z_i := \omega^{-2} \left(\sum_{j<i} m_j - \sum_{j>i} m_j \right). \tag{33}$$

$\sum m_i z_i = 0$ follows, proving that ρ has its center of mass at the origin. At this point $I(\rho)$ may be determined, and (28) will be satisfied with $J = I(\rho)\omega$. A posteriori, one notes that $z_{i+1} - z_i = \omega^{-2}(m_{i+1} + m_i) \geq 2\pi$; since $r_i \leq \pi$ there is no danger of overlapping components. This concludes the proof of Theorem 4.1. Note that if the lightest component has radius π and the *same mass* as one of its neighbours, these components nearly touch: they are separated by a single point. □

Having shown these solutions to exist, it is natural to remark upon their relationship to the energy functional $E_J(\rho)$. Fix a solution ρ to (28) from Theorem 4.1. While (26) cannot be satisfied for a global choice of λ, it is satisfied on each component of ρ separately: permitting λ to vary from component to component, differentiating makes this clear. Thus the mass within each component of ρ is in equilibrium. There will be perturbations $\rho + t\sigma$ in $\mathcal{R}_0(\mathbf{R})$ which lead to a linear decrease in $E_J(\rho)$, but these involve either a transfer of mass between components, or from some component(s) into the vacuum $\{\rho = 0\}$. Such perturbations involve 'tunneling' of mass from one region to another through a potential energy barrier, and as such are unphysical. If they could be precluded, ρ would be a critical point for the functional (21). This is also the nature of the local minima for the three dimensional model which are investigated in the remaining sections.

5. W_∞-local energy minimizers

If a stability analysis is to explore local minima, a topology must be specified which determines a precise meaning for *local*. The examples of the preceding sections illustrate that for rotating stars this choice will be delicate: for $E_J(\rho)$ to have local minima the topology must be strong enough to preclude tunneling of mass; for such minima to be meaningful, it must be weak enough so that physical flows are continuous. A topology enjoying these properties is found in the probability literature: it is induced by the Wasserstein L^∞ metric on $\mathcal{R}(\mathbf{R}^3)$, described e.g. by Givens and Shortt [20]. This metric is recalled in the sequel, where results from Auchmuty and Beals [3] are applied to show that a local minimizer $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ of $E_J(\rho)$ must be continuous everywhere, smooth where positive, and satisfy the stationarity condition of Theorem 2.1(iv). It follows by taking a gradient that ρ represents a stable solution to the Navier-Stokes-Poisson system (1).

Viewed as a measure, $\rho \in \mathcal{R}(\mathbf{R}^3)$ has unit mass. It may be represented in many ways as the probability distribution or *law* of a vector-valued random variable

$\mathbf{x} : S \longrightarrow \mathbf{R}^3$ on a probability space (S, Σ, ν). The relationship between $\mathbf{x}$ and ρ, here denoted by $\mathbf{x}_{\#}\nu = \rho$, is that $\nu[\mathbf{x}^{-1}(\Omega)] = \rho[\Omega]$ for Borel $\Omega \subset \mathbf{R}^3$; $\mathbf{x}$ is said to *push-forward* the measure ν to ρ. The Wasserstein L^∞ distance between two measures $\rho, \kappa \in \mathcal{R}(\mathbf{R}^3)$ may now be defined as an infimum over all random variable representations of ρ and κ on a space (S, Σ, ν):

$$W_\infty(\rho, \kappa) := \inf_{\substack{\mathbf{x}_{\#}\nu=\rho \\ \mathbf{y}_{\#}\nu=\kappa}} \|\mathbf{x} - \mathbf{y}\|_{\infty,\nu}. \tag{34}$$

Here $\|\mathbf{x} - \mathbf{y}\|_{\infty,\nu}$ denotes the supremum of $|\mathbf{x} - \mathbf{y}|$ over S, discarding sets of ν-measure zero. Whether the infimum in (34) ranges over all probability spaces (S, Σ, ν), or is restricted to (say) $S = [0, 1]$ with Lebesgue measure, is irrelevant. That W_∞ is a metric follows from Strassen's Theorem, as explained in Givens and Shortt [20]. Note that although $W_\infty(\rho, \kappa)$ may be infinite on $\mathcal{R}(\mathbf{R}^3)$, it is finite whenever ρ and κ are of bounded support.

It is clear from the Lagrangian formulation of fluid mechanics that the Wasserstein L^∞ metric is not unphysically strong. In that formulation, the state of a fluid system is specified by its original density profile $\rho \in \mathcal{R}(\mathbf{R}^3)$, together with the positions of the fluid particles as a function of time. At time t, $\mathbf{Y}_t(\mathbf{y}) \in \mathbf{R}^3$ represents the position of the fluid which originated at $\mathbf{Y}_0(\mathbf{y}) = \mathbf{y}$. The density profile ρ_t after time t is obtained as the push-forward of ρ through $\mathbf{Y}_t$. From its definition,

$$W_\infty(\rho_s, \rho_t) \leq \|\mathbf{Y}_s - \mathbf{Y}_t\|_{\infty,\rho}. \tag{35}$$

If the fluid particles move with bounded velocities, then $\mathbf{Y}_t(\mathbf{y})$ will be a Lipshitz function of t uniformly in $\mathbf{y}$, and it is evident that (35) will be controlled by a multiple of $|s - t|$. Thus $\rho_t \in \mathcal{R}(\mathbf{R}^3)$ evolves continuously as a function of time, at least for bounded fluid velocities. The same argument shows Lemma 5.1(iii): an L^∞-small perturbation of the Lagrangian fluid variables produces only a W_∞-small perturbation of the density: a local energy minimum $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ must be physically stable.

The next lemma collects elementary properties required of W_∞. The proofs are immediate from the definition (34). Here $\operatorname{spt}(\rho - \kappa) \subset \mathbf{R}^3$ denotes the support of the signed measure $\rho - \kappa$, while a *δ-neighbourhood* is defined as in (14).

Lemma 5.1 (Simple Properties of the Wasserstein L^∞ Metric)**.**
Let $\rho, \kappa \in \mathcal{R}(\mathbf{R}^3)$. Then

(i) $W_\infty(\rho, \kappa)$ *does not exceed the diameter of* $\operatorname{spt}(\rho - \kappa)$*;*
(ii) if $W_\infty(\rho, \kappa) < \delta$*, each connected component of the δ-neighbourhood of* $\operatorname{spt}\rho$ *has the same mass for κ as for ρ;*

(iii) $W(\rho, \mathbf{y}_{\#}\rho) \leq \|\mathbf{y} - id\|_{\infty,\rho}$ *for* $\mathbf{y} : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ *measurable and* $id(\mathbf{x}) := \mathbf{x}$*;*
(iv) the centers of mass $|\overline{\mathbf{x}}(\rho) - \overline{\mathbf{x}}(\kappa)| \leq W_\infty(\rho, \kappa)$*;*
(v) the moment of inertia $I(\rho)$ *depends continuously on* ρ*.*

Lagrange multipliers conjugate to the center of mass constraint do not appear in (15). This is because a local energy minimizer ρ on $\mathcal{R}_0(\mathbf{R}^3)$ is also stable under perturbations which shift its center of mass:

Corollary 5.2. *If* ρ *minimizes* $E_J(\rho)$ *locally on* $\mathcal{R}_0(\mathbf{R}^3)$*, then it minimizes* $E_J(\rho)$ *locally on* $\mathcal{R}(\mathbf{R}^3)$*.*

Proof. There exists $\delta > 0$ such that $E_J(\rho) \leq E_J(\kappa)$ whenever $\kappa \in \mathcal{R}_0(\mathbf{R}^3)$ with $W_\infty(\rho, \kappa) < 2\delta$. Now, suppose $\kappa \in \mathcal{R}(\mathbf{R}^3)$ with $W_\infty(\rho, \kappa) < \delta$. Part (iv) of the lemma shows that $|\overline{\mathbf{x}}(\kappa)| < \delta$; part (iii) then shows that the translate of κ by $-\overline{\mathbf{x}}(\kappa)$ lies within 2δ of ρ in $\mathcal{R}_0(\mathbf{R}^3)$. By translation invariance, $E_J(\rho) \leq E_J(\kappa)$. □

Therefore, suppose ρ minimizes $E_J(\rho)$ locally on $\mathcal{R}_0(\mathbf{R}^3)$ and let $\sigma \in L^\infty(\mathbf{R}^3)$. Even if the perturbation $\rho + t\sigma \in \mathcal{R}(\mathbf{R}^3)$ for $t \in [0, 1]$, it may not be W_∞-continuous as function of t; nevertheless, Lemma 5.1(i) guarantees that $E_J(\rho + t\sigma)$ is minimized by ρ provided σ is supported on a small enough set. σ will then be a useful variation of $E_J(\rho)$.

The *variational derivative* $E'_J(\rho)$ of the energy $E_J(\rho)$ is formally given by

$$E'_J(\rho)(\mathbf{x}) := A'(\rho(\mathbf{x})) - V_\rho(\mathbf{x}) - \frac{J^2}{2I^2(\rho)} r^2\Big(\mathbf{x} - \overline{\mathbf{x}}(\rho)\Big). \tag{36}$$

For $J = 0$ and a restricted class of perturbations $\sigma \in P_0$, a more general result [3] not quite including the kinetic energy $T_J(\rho)$ shows that

$$\lim_{t \to 0} t^{-1} \left(E_J(\rho + t\sigma) - E_J(\rho)\right) = \int E'_J(\rho)\sigma. \tag{37}$$

The admissable perturbations depend on ρ:

$$P_0 := \bigcup_{R<\infty} \left\{ \sigma \in L^\infty(\mathbf{R}^3) \;\middle|\; \begin{array}{l} \sigma = 0 \text{ where } \rho > R \text{ or } |\mathbf{x}| > R \\ \sigma \geq 0 \text{ where } \rho < R^{-1} \end{array} \right\}. \tag{38}$$

(37) may be immediately extended to positive angular momentum $J > 0$ by the following observation: even though σ may not have its center of mass at the origin, $I(\rho)$ is cubic in ρ; a direct computation shows that $\lim_{t \to 0} t^{-1} \left(I(\rho + t\sigma) - I(\rho)\right) = I(\sigma)$.

The next pair of propositions show that local minimizers of $E_J(\rho)$ satisfy the stationarity conditions of Theorem 2.1(iv-v).

Proposition 5.3 (Locally Constant Chemical Potential). *Let $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ minimize $E_J(\kappa)$ among $\kappa \in \mathcal{R}(\mathbf{R}^3)$ for which $W_\infty(\rho,\kappa) < 2\delta$. Let M be an open set with diameter no greater than 2δ which intersects $\operatorname{spt}\rho$. There is a unique $\lambda_i \in \mathbf{R}$ depending on M such that (15) holds on M a.e.*

Proof. The proof is an application of a constrained minimization argument found in Auchmuty and Beals [3]. We rely on intermediate results formulated there. Therefore, define the convex cone $P_{loc} := \{\sigma \in P_0 \mid \operatorname{spt}\sigma \subset M\}$, and let $\mathcal{U} = \{\rho + \sigma \geq 0 | \sigma \in P_{loc}\}$, so that P_{loc} is the tangent cone of $\mathcal{U}$ at ρ. It is noted above that $E_J(\rho)$ is differentiable at ρ in the directions σ of P_{loc}. On $\mathcal{W}_{loc} = \mathcal{R}(\mathbf{R}^3) \cap \mathcal{U}$, where the mass constraint is satisfied, Lemma 5.1(i) shows that ρ minimizes $E_J(\kappa)$. Moreover, since the open set M intersects $\operatorname{spt}\rho$, it must carry positive mass under ρ. Thus there is a smaller subset $C \subset M$ of positive measure on which $\rho(\mathbf{x})$ is bounded away from zero and infinity. If χ_C is the characteristic function of this set, both $\pm\chi_C \in P_{loc}$, although $\rho \pm \chi_C \notin \mathcal{W}_{loc}$. These conditions imply that there is a unique Lagrange multiplier $\lambda \in \mathbf{R}$ such that

$$\int E_J'(\rho)\sigma \geq \lambda \int \sigma \tag{39}$$

for all $\sigma \in P_{loc}$ [3, Proposition 2]. If $E_J'(\rho) < \lambda$ on a subset $K \subset M$ which had positive measure, this subset may be taken slightly smaller so that ρ is bounded on K; $\chi_K \in P_{loc}$ would then contradict (39). On the other hand, if $E_J'(\rho) > \lambda$ on a subset $K \subset M$ with positive measure and where $\rho > 0$, then K may be taken slightly smaller so that ρ is bounded away from zero and infinity on K; in this case $-\chi_K \in P_{loc}$ contradicts (39). Since $A'(\rho)$ in (36) vanishes precisely where ρ does, these two inequalities show that (15) holds for almost all $\mathbf{x} \in M$ with $\lambda_i := \lambda$. □

Proposition 5.4 (Componentwise Constant Chemical Potential). *Let $\rho \in \mathcal{R}_0(\mathbf{R}^3)$ minimize $E_J(\kappa)$ among $\kappa \in \mathcal{R}(\mathbf{R}^3)$ for which $W_\infty(\rho,\kappa) < 2\delta$. Choose one of the connected components Ω_i of the δ-neighbourhood (14) of $\operatorname{spt}\rho$. Then there is a constant $\lambda_i < 0$ such that (15) holds a.e. on Ω_i.*

Proof. For $\mathbf{y} \in \Omega_i$ the ball $B_\delta(\mathbf{y})$ intersects $\operatorname{spt}\rho$. Thus Proposition 5.3 guarantees a unique $\lambda(\mathbf{y})$ such that (15) holds a.e. on $B_\delta(\mathbf{y})$ when $\lambda_i := \lambda(\mathbf{y})$. The claim is that $\lambda(\mathbf{y})$ is independent of $\mathbf{y}$. Therefore, fix $\mathbf{y} \in \Omega_i$. Since $B_\delta(\mathbf{y})$ is open, it will also be true that a slightly smaller ball $B_{\delta-\epsilon}(\mathbf{y})$ intersects $\operatorname{spt}\rho$. If $|\mathbf{x} - \mathbf{y}| < \epsilon$, then $M = B_\delta(\mathbf{x}) \cap B_\delta(\mathbf{y})$ intersects $\operatorname{spt}\rho$. In Proposition 5.3, the uniqueness of λ corresponding to M forces $\lambda(\mathbf{x}) = \lambda(\mathbf{y})$. Thus $\lambda(\mathbf{y})$ is locally constant. As a result, the disjoint sets $C = \{\mathbf{x} \in \Omega_i \mid \lambda(\mathbf{x}) = \lambda(\mathbf{y})\}$ and

$D = \{\mathbf{x} \in \Omega_i \mid \lambda(\mathbf{x}) \neq \lambda(\mathbf{y})\}$ are both open. Since $\Omega_i = C \cup D$ is connected, $C = \Omega_i$. Defining $\lambda_i := \lambda(\mathbf{y})$, (15) must be satisfied a.e. on Ω_i.

An additional argument shows $\lambda < 0$: Any point on the boundary of Ω_i cannot lie within δ of $\operatorname{spt}\rho$. Since $\operatorname{spt}\rho$ is bounded, Ω_i has non-empty boundary, and it follows that $\rho(x) = 0$ on a set of positive measure in Ω_i. On the other hand, $A(\varrho)$ is strictly convex so $A'(\rho(\mathbf{x}))$ vanishes only if $\rho(\mathbf{x}) = 0$. $\lambda \geq 0$ in (15) would imply $\rho > 0$ a.e. on Ω_i, a contradiction. □

The arguments of Auchmuty and Beals [3] now apply to local minimizers on $E_J(\rho)$, yielding:

Proposition 5.5 (Regularity of W_∞-local Energy Minimizers)**.**
Let ρ minimize $E_J(\rho)$ locally on $\mathcal{R}_0(\mathbf{R}^3)$. Then ρ is continuous everywhere; where positive it has as many derivatives as the inverse of $A'(\varrho)$.

Proof. Proposition 5.4 applies by Corollary 5.2. The stationarity condition (15) must be used to control ρ with V_ρ at large densities. The chemical potential $\lambda_i < 0$ may be discarded, while $r^2(\mathbf{x})$ cannot be too large on the bounded support of ρ, so $A'(\rho(\mathbf{x})) \leq V_\rho(\mathbf{x}) + C$ for $C < \infty$ depending on ρ but independent of $\mathbf{x}$. Wherever $A'(\rho) \geq 2C$, the bound $A'(\rho) \leq 2V_\rho$ holds. Thus V_ρ is continuous on $\mathbf{R}^3$ as in Auchmuty and Beals [3, Lemma 3 and Theorem A].

Continuity of ρ on Ω_i follows from that of V_ρ through (15) because $A'(\varrho)$ is continuously invertible. Ω_i was a component of some δ-neighbourhood of $\operatorname{spt}\rho$, so it is clear that ρ will be compactly supported on it. Because V_ρ gains a derivative from ρ, smoothness of ρ where positive follows from a bootstrap in (15). □

Only Theorem 2.1(i) remains to be proven:

Lemma 5.6 (Principal Axis of Inertia)**.**
Let ρ minimize $E_J(\rho)$ locally on $\mathcal{R}_0(\mathbf{R}^3)$. Then the z-axis is a principal axis of inertia for ρ, with a moment of inertia $I(\rho)$ which is maximal and non-degenerate.

Proof. Let $I_{ij}(\rho) := \int (\delta_{ij}|\mathbf{x}|^2 - x_j x_i) d\rho(\mathbf{x})$ denote the moment of inertia tensor $\underline{\mathrm{I}}(\rho)$ of ρ, and $\hat{\mathbf{l}} \in \mathbf{R}^3$ denote the eigenvector of $\underline{\mathrm{I}}(\rho)$ corresponding to its maximal eigenvalue. Then $I(\rho) = \langle \hat{\mathbf{e}}_z, \underline{\mathrm{I}}(\rho)\hat{\mathbf{e}}_z \rangle \leq \langle \hat{\mathbf{l}}, \underline{\mathrm{I}}(\rho)\hat{\mathbf{l}} \rangle$. The first claim is that the inequality is saturated. If not, a slight rotation of ρ bringing $\hat{\mathbf{l}}$ toward the z-axis would increase $I(\rho)$: letting $\hat{\mathbf{k}}(\theta) := \cos(\theta)\hat{\mathbf{l}} + \sin(\theta)\hat{\mathbf{k}}$ where $\hat{\mathbf{k}}$ and $\hat{\mathbf{l}}$ are orthonormal, either $\langle \hat{\mathbf{k}}(\theta), \underline{\mathrm{I}}(\rho)\, \hat{\mathbf{k}}(\theta) \rangle$ is constant or it attains a unique local maximum at $\theta = 0$. Since $E_0(\rho)$ is rotation invariant, $E_J(\rho)$ would be decreased. But ρ

minimizes $E_J(\rho)$ locally. A contradiction is produced since for ρ with bounded support, a slight rotation is a W_∞-local perturbation by Lemma 5.1(iii).

Now suppose that $\langle \hat{\mathbf{e}}_z, \underline{\mathrm{I}}(\rho)\hat{\mathbf{e}}_z\rangle$, although maximal, is not unique. Then a slight rotation of ρ (about an axis other than $\hat{\mathbf{e}}_z$) is also a W_∞ local minimizer of $E_J(\rho)$ on $\mathcal{R}_0(\mathbf{R}^3)$. By Propositions 5.4 and 5.5, $A'(\rho) - V_\rho$ must be constant along line segments parallel to the z-axis where $\rho > 0$, and cannot be constant along line segments with other orientations. This cannot be true for both ρ and its rotate. □

Proof of Theorem 2.1. Let $(\rho, \mathbf{v})$ locally minimize $E(\rho, \mathbf{v})$ subject to the constraint $J_z(\rho, \mathbf{v}) = J$. Corollary 3.4 proves (ii) and implies that ρ locally minimizes $E_J(\rho)$. Parts (i), (iii, vi) and (iv-v) then follow from Lemma 5.6, and Propositions 5.5 and 5.4 respectively. If $P(\varrho)$ is continuously differentiable, then $A''(\varrho) = P'(\varrho)/\varrho$ and (vii) follows by taking the gradient of (15). By Lemma 5.1(iii), the energy cannot be decreased by perturbations of ρ which result from L^∞-small perturbations in the Lagrangian fluid variables. Perturbations of the velocity field $\mathbf{v}$ are irrelevant: if consistent with the constraint, Proposition 3.1 shows that $E(\rho, \mathbf{v})$ can only increase relative to $E_J(\rho)$. □

6. Existence of binary stars

This section is devoted to establishing the existence of local minimizers for $E_J(\rho)$ carrying large angular momentum J. Such minimizers represent stable, uniformly rotating solutions to the Navier-Stokes-Poisson system (1). They are constructed in the form of binary stars, which is to say that the fluid mass is divided into two disjoint regions Ω^- and Ω^+, widely separated relative to their size. The mass ratio $m : 1 - m$ between the two regions is specified a priori.

The $\Omega^\pm \subset \mathbf{R}^3$ will be closed balls centered on the plane $z = 0$, whose size and separation scale with J^2 as in (42)–(43); the relevant fluid configurations are

$$\mathcal{W}_J := \{\rho^- + \rho^+ \in \mathcal{R}(\mathbf{R}^3) \mid \int \rho^- = m, \quad \operatorname{spt} \rho^\pm \subseteq \Omega^\pm\}. \tag{40}$$

The following theorem will be proved:

Theorem 6.1 (Existence of Binary Stars).
Given $m \in (0, 1)$, choose the angular momentum J to be sufficiently large depending on m. Then any global minimizer of $E_J(\rho)$ on $\mathcal{W}_J$ will, after a rotation about the z-axis and a translation, have support contained in the interior of $\Omega^- \cup \Omega^+$. It will also be symmetric about the plane $z = 0$ and a decreasing function of $|z|$.

Since a global energy minimizer on $\mathcal{W}_J$ exists by the arguments of Auchmuty and Beals [3] or Li [22] summarized below, this theorem has as its consequence:

Corollary 6.2. *Given $m \in (0,1)$, let $J > J(m)$ as in Theorem 6.1. Then the energy $E_J(\rho)$ admits a local minimizer ρ on $\mathcal{R}(\mathbf{R}^3)$ in the form of a global energy minimizer on $\mathcal{W}_J$. Uniformly rotating, ρ minimizes $E(\rho, \mathbf{v})$ locally on $\mathcal{R}(\mathbf{R}^3) \times \mathcal{V}(\mathbf{R}^3)$ subject to the constraint $J_z(\rho, \mathbf{v}) = J$ or $\mathbf{J}(\rho, \mathbf{v}) = J\hat{\mathbf{e}}_z$.*

Proof. Let ρ be the minimizer on $\mathcal{W}_J$. The theorem shows that $\operatorname{spt} \rho$ is compact in the interior of $\Omega^- \cup \Omega^+$, therefore separated from the boundary by a positive distance δ. Lemma 5.1(ii) shows that if $\kappa \in \mathcal{R}(\mathbf{R}^3)$ with $W_\infty(\rho, \kappa) < \delta$, then κ in $\mathcal{W}_J$. Thus $E_J(\rho) \leq E_J(\kappa)$. Since $E_J(\rho)$ is locally minimized, Corollary 3.4 provides a local minimizer $(\rho, \mathbf{v})$ of $E(\rho, \mathbf{v})$ subject to the constraint on J_z. Corollary 2.2 shows that $(\rho, \mathbf{v})$ satisfies the constraint on the vector angular momentum as well. $\square$

The separation of the domains $\Omega^\pm$ is determined by the Kepler problem for two point masses m and $1 - m$, rotating with angular momentum $J > 0$ about their fixed center of mass. The reduced mass of that system is denoted by $\mu := m(1 - m)$. As a function of the radius of separation d, the gravitational plus kinetic energy

$$-\frac{\mu}{d} + \frac{J^2}{2\mu d^2} \geq -\frac{\mu^3}{2J^2} \tag{41}$$

assumes its minimum at separation $\eta := \mu^{-2} J^2$. This is the radius of the circular orbit. Therefore, choose two points $\mathbf{y}^\pm \in \mathbf{R}^3$ from the plane $z = 0$, separated by η, to be the centers of $\Omega^\pm$:

$$\eta := \mu^{-2} J^2 = |\mathbf{y}^- - \mathbf{y}^+|; \tag{42}$$

$$\Omega^\pm := \left\{ \mathbf{x} \in \mathbf{R}^3 \mid |\mathbf{x} - \mathbf{y}^\pm| \leq \eta/4 \right\}. \tag{43}$$

Here and throughout the following, the superscripts $\pm$ denote an implicit dependence on J, or equivalently η. When η is large, one expects a stable, slowly rotating equilibrium to exist in which fluid components with masses m and $1-m$ lie near $\mathbf{y}^-$ and $\mathbf{y}^+$. The distance separating $\Omega^\pm$ and the diameter of their union is given by:

$$\operatorname{dist}(\Omega^-, \Omega^+) = \eta/2; \tag{44}$$

$$\operatorname{diam}(\Omega^- \cup \Omega^+) = 3\eta/2. \tag{45}$$

It follows that for $\rho \in \mathcal{W}_J$ rotating uniformly with angular momentum J, the fluid velocities will not be too large:

Lemma 6.3 (Velocities Decrease as Angular Momentum Grows)**.**
Fix $m \in (0,1)$ and let $\epsilon > 0$. For $J \geq \epsilon$, there is a maximum velocity $v(m,\epsilon)$ which does not depend on J, such that if $\rho \in \mathcal{W}_J$ and $\mathbf{x} \in \Omega^- \cup \Omega^+$ then $Jr(\mathbf{x} - \overline{\mathbf{x}}(\rho))/I(\rho) \leq v(m,\epsilon)$. Moreover $v(m,\epsilon) \to 0$ as $\epsilon \to \infty$.

Proof. Let $\rho = \rho^- + \rho^+ \in \mathcal{W}_J$. The centers of mass $\overline{\mathbf{x}}(\rho^\pm) \in \Omega^\pm$, or rather their projections onto $z = 0$, are separated by at least $\eta/2$ (44). The moment of inertia $I(\rho)$ is bounded below by that of two point masses m and $1-m$ at this separation:

$$I(\rho) = \mu\, r^2\Big(\overline{\mathbf{x}}(\rho^-) - \overline{\mathbf{x}}(\rho^+)\Big) + I(\rho^-) + I(\rho^+) \tag{46}$$

$$\geq \mu\, \eta^2/4. \tag{47}$$

At the same time $\mathbf{x} \in \Omega^\pm$ implies $r(\mathbf{x} - \overline{\mathbf{x}}(\rho)) \leq 3\eta/2$. Since $\eta = \mu^{-2} J^2$, these two estimates show $r(\mathbf{x} - \mathbf{x}(\rho))/I(\rho) \leq O(J^{-2})$ as $J \to \infty$, proving the lemma. $\square$

Before addressing the proof of Theorem 6.1, the direct argument for existence of a global energy minimizer on $\mathcal{W}_J$ is recalled following Auchmuty and Beals [3] and Li [22]. For constants $\lambda^\pm$, such a minimizer satisfies the Euler-Lagrange equations

$$A'\Big(\rho(\mathbf{x})\Big) = \left[\frac{J^2}{2I^2(\rho)} r^2\Big(\mathbf{x} - \overline{\mathbf{x}}(\rho)\Big) + V_\rho(\mathbf{x}) + \lambda^\pm\right]_+ \text{ a.e. on } \Omega^\pm, \tag{48}$$

much like (15) before setting $\overline{\mathbf{x}}(\rho) = 0$. To prove existence of a minimizer, one first imposes a large bound $\|\rho\|_\infty \leq R$ on the configurations in $\mathcal{W}_J$. $E_J(\rho)$ is then lower semi-continuous in the weak topology on $\mathcal{W}_J \subset L^{4/3}(\mathbf{R}^3)$; the kinetic term $T_J(\rho)$ is continuous. Since $E_J(\rho)$ diverges with $\|\rho\|_{4/3}$, the Banach-Alaoglu compactness theorem guarantees a minimizer ρ_R. Because ρ_R was constrained to be bounded, it satisfies a version of (48) in which the truncation $[x]_+$ is modified so that $[x]_+ = A'(R)$ when $x > A'(R)$. From this equation, an additional argument of Li [22, Proposition 1.4] using the bound on $Jr/I(\rho_R)$ from Lemma 6.3 shows

$$\|\rho_R\|_\infty < C(m) \tag{49}$$

independent of R. This R-independent ρ_R is the desired minimizer. As in Lemma 6.3, the constant $C(m)$ is J-independent for J bounded away from zero.

Theorem 6.1 controls the support of the global minimizer $\rho = \rho^- + \rho^+$ on $\mathcal{W}_J$. Its proof begins with a series of estimates on $\rho^\pm$, the components of ρ supported in $\Omega^\pm$ respectively. Using the symmetry in m and $1-m$, it is sufficient to establish these estimates for ρ^- only. The first proposition relies on an energetic comparison with the configuration $\sigma^- + \sigma^+$ obtained from suitable translations of non-rotating minimizers σ_m from Theorem 3.5:

$$\sigma^-(\mathbf{x}) := \sigma_m(\mathbf{x} - \mathbf{y}^-); \quad \sigma^+(\mathbf{x}) := \sigma_{1-m}(\mathbf{x} - \mathbf{y}^+). \tag{50}$$

Proposition 6.4 (Energy Converges to Non-rotating Minimum).
Given $\epsilon > 0$, if J is sufficiently large and $\rho^- + \rho^+$ minimizes $E_J(\rho)$ on $\mathcal{W}_J$, then $E_0(\rho^-) \leq e_0(m) + \epsilon$. Here $e_0(m)$ is the mass m infimum (12) of $E_0(\rho)$.

Proof. By Theorem 3.5(v), taking J large enough will ensure that $\sigma^\pm$ is supported in $\Omega^\pm$. Then $\sigma^- + \sigma^+ \in \mathcal{W}_J$, so its energy decomposes as

$$E_J(\sigma^- + \sigma^+) - E_0(\sigma^-) - E_0(\sigma^+) = -G(\sigma^-, \sigma^+) + T_J(\sigma^- + \sigma^+). \tag{51}$$

The gravitational interaction and kinetic energy may be estimated by comparison with the point masses (41): $G(\sigma^-, \sigma^+) = \mu\eta^{-1}$ by Newton's Theorem and (42), while $I(\sigma^- + \sigma^+) \geq \mu\eta^2$ as in (46). Thus the right side of (51) is less than $-\mu^3 J^{-2}/2$, yielding

$$\begin{aligned} E_0(\sigma^-) + E_0(\sigma^+) &> E_J(\sigma^- + \sigma^+) \\ &\geq E_J(\rho^- + \rho^+) \\ &> E_0(\rho^-) + E_0(\rho^+) - G(\rho^-, \rho^+) \end{aligned}$$

The last inequality follows from $T_J(\rho) > 0$, since $E_J(\rho^- + \rho^+)$ also decomposes as in (51). Taking J large forces the separation $\eta/2$ between Ω^- and Ω^+ to diverge. Taking $G(\rho^-, \rho^+) \leq 2\mu\eta^{-1} < \epsilon$ proves the proposition because $E_0(\rho^+) \geq E_0(\sigma^+)$. □

Thus $E(\rho^-)$ converges to the minimum energy for a non-rotating mass m as $J \to \infty$. In this case the concentration-compactness lemma of Lions [26, Theorem II.2 and Corollary II.1] provides a subsequence of the ρ^-, which, after translation, converges strongly in $L^{4/3}(\mathbf{R}^3)$ to a minimizer for the non-rotating problem. The next two results exploit this convergence.

Lemma 6.5 (Bound for the Chemical Potential).
Given $\epsilon > 0$ and J large enough, if $\rho^- + \rho^+$ minimizes $E_J(\rho)$ on $\mathcal{W}_J$, then the chemical potential λ^- in (48) satisfies $\lambda^- \leq e_0'(m^-) + \epsilon$. Here $e_0'(m^-) < 0$ is the bound for the non-rotating chemical potential from Theorem 3.5(viii).

Proof. The proposition can only fail if there exists a sequence of angular momenta $J_n \to \infty$ together with minimizers $\rho_n^- + \rho_n^+$ for $E_{J_n}(\rho)$ on $\mathcal{W}_{J_n}$, for which the chemical potentials λ_n^- have a limit greater than $e_0'(m^-)$. The Euler-Lagrange equation (48) implies

$$A'(\rho_n^-) \geq V_{\rho_n^-} + \lambda_n^- \text{ a.e. on } \Omega^-, \tag{52}$$

an invariant statement under translations of ρ_n^-. Proposition 6.4 and [26] imply — after translating each ρ_n^- and extracting a subsequence also denoted ρ_n^- — that one has $L^{4/3}(\mathbf{R}^3)$ convergence to a non-rotating minimizer σ_m for $E_0(\rho)$. Since V_ρ is the convolution of ρ with a weak $L^3_w(\mathbf{R}^3)$ function, the Generalized Young's Inequality shows that $V_{\rho_n^-} \to V_{\sigma_m}$ strongly in $L^{12}(\mathbf{R}^3)$ (here $3/4 + 1/3 = 1 + 1/12$). Extracting another subsequence, one has pointwise convergence a.e. of both ρ_n^- and $V_{\rho_n^-}$. A contradiction follows from (52) on the set $\{\sigma_m > 0\}$, where by Theorem 3.5(vii)–(viii):

$$A'(\sigma_m) - V_{\sigma_m} = \lambda_m \leq e_0'(m^-).$$

□

Proposition 6.6 (Bound on the Radius of Support)**.**
There exists a radius $R(m)$ independent of J, such that if $\rho^- + \rho^+$ minimizes $E_J(\rho)$ on $\mathcal{W}_J$ for J sufficiently large, then spt ρ^- *is contained in a ball of radius $R(m)$.*

Proof. Take J large enough that $\lambda > e_0'(m^-)$ bounds λ^- by Proposition 6.5, while the velocity bound $v(m)$ of Proposition 6.3 satisfies $v^2(m) \leq -\lambda$. In the Euler-Lagrange equation (48) these estimates yield

$$A'(\rho^-) \leq \left[V_{\rho^-} + V_{\rho^+} + \lambda/2\right]_+ \text{ a.e. on } \Omega^-. \tag{53}$$

Strict convexity of $A(\varrho)$ forces $\rho = 0$ where $A'(\rho) = 0$, so ρ^- must vanish where the gravitational potential is less than $-\lambda/2$. V_{ρ^+} is easily controlled: for J large enough, $V_{\rho^+} < -\lambda/6$ on Ω^- since the distance to spt $\rho^+ \subset \Omega^+$ will be large (44). Therefore, consider V_{ρ^-}. For $\rho \in L^1(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$, there is a pointwise bound

$$\|V_\rho\|_\infty \leq k\|\rho\|_1^{2/3}\|\rho\|_\infty^{1/3} \tag{54}$$

saturated when ρ is supported on the smallest ball consistent with $\|\rho\|_\infty$. Since $\|\rho^-\|_\infty \leq C(m)$ from (49), choose $\delta > 0$ such that $\|\rho\|_1 \leq \delta$ and $\|\rho\|_\infty \leq C(m)$ imply $\|V_\rho\|_\infty \leq -\lambda/6$. Now, let $R_0(m)$ from Theorem 3.5(v) bound the support radii of all mass m non-rotating minimizers σ_m, and choose $R(m) \geq R_0(m)$ large enough so that $m/(R(m) - R_0(m)) \leq -\lambda/6$. Using Proposition 6.4 and [26] once again, J large enough implies that ρ^- is $L^{4/3}(\mathbf{R}^3)$ close to a translate of some σ_m; in particular, all but mass δ of ρ^- is forced into a ball of radius $R_0(m)$. Neither the restriction of ρ^- to this ball, nor the remaining mass δ, contributes more than

$-\lambda/6$ to V_{ρ^-} outside the larger ball of radius $R(m)$. Thus (53) establishes the proposition. □

The following lemma and proposition essentially prove Theorem 6.1.

Lemma 6.7. *For $\epsilon > 0$ define $g_\epsilon(x) := (x^2 + \epsilon^2/\mu)^{-1} - 2(x - 2\epsilon)^{-1}$. If ϵ is sufficiently small, the function $g_\epsilon(x)$ is uniquely minimized on the interval* $(1/2, 3/2)$ *and has no local maxima there.*

Proof. For ϵ sufficiently small, the functions $g_\epsilon(z)$ are analytic and uniformly bounded on $\{z \in \mathbf{C} \mid |z| > 1/4\}$. It follows that $g_\epsilon(z)$ converges uniformly to $g_0(z) := z^{-2} - 2z^{-1}$ as $\epsilon \to 0$ on $|z| \geq 1/2$. The derivatives converge also. $g_0'(x)$ vanishes on $(0, \infty)$ only at $x = 1$, while $g_0''(x) > 0$ for $x < 3/2$. Therefore, if $\delta < 1/2$, sufficiently small ϵ ensures: $g_\epsilon''(x) > 0$ where $|x - 1| < \delta$, while $g_\epsilon'(x) < 0$ for $1/2 \leq x \leq 1 - \delta$ and $g_\epsilon'(x) > 0$ for $x \geq 1 + \delta$. The lemma is proved. □

Proposition 6.8 (Estimate for the Center of Mass Separation)**.**
Let $0 < \delta < 1/2$. For J sufficiently large, if $\rho = \rho^- + \rho^+$ minimizes $E_J(\rho)$ on $\mathcal{W}_J$ then the ratio $|\overline{\mathbf{x}}(\rho^-) - \overline{\mathbf{x}}(\rho^+)| \,/\, \eta$ lies within δ of 1. Here $\eta = \mu^{-2}J^2$.

Proof. Take J large enough so that Proposition 6.6 provides bounds $R(m)$ and $R(1 - m)$ for the support of $\rho^\pm$. Taking J larger if necessary ensures

$$R := 2\max\{R(m), R(1 - m)\} < \eta/4. \tag{55}$$

Since $\operatorname{spt}\rho^\pm$ must be contained within radius R of $\overline{\mathbf{x}}(\rho^\pm)$, there is room in $\Omega^\pm$ to translate ρ^- and ρ^+ independently so that $\overline{\mathbf{x}}(\rho^\pm) = \mathbf{y}^\pm$ lie at separation η. Denote these translates by κ^- and κ^+, so that $\kappa = \kappa^- + \kappa^+ \in \mathcal{W}_J$. As in (51), $E(\kappa)$ differs only from $E(\rho)$ by terms of the form $-G(\rho^-, \rho^+) + T_J(\rho)$. These terms may be estimated using the center of mass separation d between the translates of ρ^- and ρ^+; with an abuse of notation, they are denoted by $G(d)$ and $T_J(d)$, and the moment of inertia by $I(d)$:

$$\begin{aligned} \frac{\mu}{d + 2R} &< G(d) < \frac{\mu}{d - 2R} \\ \mu d^2 &< I(d) < \mu d^2 + R^2 \\ \frac{J^2}{2(\mu d^2 + R^2)} &< T_J(d) < \frac{J^2}{2\mu d^2}. \end{aligned}$$

If ρ minimizes $E_J(\rho)$ on $\mathcal{W}_J$, comparison with κ forces $d := |\overline{\mathbf{x}}(\rho^-) - \overline{\mathbf{x}}(\rho^+)|$ to satisfy

$$-G(d) + T_J(d) \leq -G(\eta) + T_J(\eta). \tag{56}$$

Using the preceding estimates and $J^2 = \mu^2\eta$, the implication of (56) for the dimensionless parameter $x := d/\eta$ in terms of $\epsilon := R/\eta$ is

$$-\frac{2}{x-2\epsilon} + \frac{1}{x^2+\epsilon^2\mu^{-1}} \leq -\frac{2}{1+2\epsilon} + 1. \tag{57}$$

This condition is satisfied for $x = 1$. However, it fails to be satisfied at $x = 1 \pm \delta$ for large J, because it does not hold in the $\epsilon \to 0$ limit. Lemma 6.7 then guarantees that for large J, (57) can hold on $x \in [1/2, 3/2]$ only when $|x-1| < \delta$. This range includes all relevant separations by (44), thus proving the proposition. □

Proof of Theorem 6.1. First it is shown that any minimizer $\rho = \rho^- + \rho^+$ for $E_J(\rho)$ on $\mathcal{W}_J$ may be translated so that both $\overline{\mathbf{x}}(\rho^\pm)$ lie in the plane $z = 0$. Since the $\Omega^\pm$ are convex and symmetric about $z = 0$, it is enough to know that ρ enjoys a plane of symmetry $z = c$. This follows from a strong rearrangement inequality in Lieb [24, Lemma 3] and Fubini's Theorem: the symmetric decreasing rearrangement of ρ along lines parallel to the z-axis leaves $U(\rho)$ and $I(\rho)$ unchanged; however, since the potential $(r^2+z^2)^{-1/2}$ is strictly decreasing as a function of $|z|$, the rearrangement increases $G(\rho,\rho)$ unless ρ is already symmetric decreasing about a plane $z = c$. Since ρ minimizes $E_J(\rho)$ and its rearrangement is in $\mathcal{W}_J$, $G(\rho,\rho)$ cannot be increased.

Now, take J large enough so that Proposition 6.6 provides a bound R such that $\operatorname{spt}\rho^\pm \subset B_R(\overline{\mathbf{x}}(\rho^\pm))$ if $\rho^- + \rho^+$ minimizes $E_J(\rho)$ on $\mathcal{W}_J$. Translate ρ so that its symmetry plane is $z = 0$ and let $d := |\overline{\mathbf{x}}(\rho^-) - \overline{\mathbf{x}}(\rho^+)|$. Then (43)–(45) show that if $d - 2R > \eta/2$ and $d + 2R < 3\eta/2$, a translation and rotation of ρ yields a minimizer in $\mathcal{W}_J$ supported away from the boundary of $\Omega^- \cup \Omega^+$. By Proposition 6.8, this is certainly true when J and hence η is sufficiently large. □

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