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LONG TIME ASYMPTOTICS FOR FAST DIFFUSION VIA DYNAMICAL SYSTEMS METHODS

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Abstract. For the fast diffusion equation in the mass preserving parameter range, we obtain sharp asymptotic convergence rates to the Barenblatt solution with respect to the relative L^∞ norm from spectral gaps by establishing a nonlinear differentiable semiflow in Hölder spaces on a Riemannian manifold called cigar manifold. On this manifold, the equation becomes uniformly parabolic. It is possible to obtain faster rates than $O(1/\tau)$ when the reference Barenblatt solution is appropriately scaled. To this end, the interplay between weights in the function space, the spectrum of the linearized operator and growth of its (formal) eigenfunctions needs to be investigated carefully, leading to estimates in appropriately weighted relative L^∞ norms.

1. Introduction.

We are considering the porous medium family of equations

$$\rho_\tau = \frac{1}{m}\Delta\rho^m \tag{1}$$

in $\mathbb{R}^n$ with nonnegative initial data in the range $(1-\frac{2}{n})_+ := \max\{0, 1-\frac{2}{n}\} < m < 1$. It was originally conceived as the porous medium equation with parameters $m > 1$, in which case the equation features finite speed of propagation (weak solutions with compact support) due to the vanishing diffusion coefficient ρ^{m-1} for vanishing density ρ. In contrast, the fast diffusion range $m < 1$ features solutions that have a tail behavior like $O(|\boldsymbol{x}|^{-2/(1-m)})$. The further restriction on m guarantees that the typical solutions have a conserved finite mass, whereas for $m < 1 - \frac{2}{n}$ solutions become extinct in finite time.

The long-time asymptotic behavior of solutions to (1) is of particular interest to us. On the one hand there are indications for a similarity of behavior for the fast diffusion and the porous medium range, on the other hand the technical treatment

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seems more accessible for the fast diffusion range, which has drawn considerable interest in the last decade partly for this reason.

It has been shown by Herrero and Pierre [9] that the initial value problem for (1) with $0 < m < 1$ has a unique (global in time) solution for nonnegative initial data that are locally integrable. Their estimate (1.5) can also be used to show mass preservation if $m > 1 - \frac{2}{n}$ also holds. One family of solutions, the Barenblatt solution, is known explicitly. It is a spreading profile whose initial data are

$$u_B(\boldsymbol{x}) = \Big(B + \frac{1-m}{2}|\boldsymbol{x}|^2\Big)^{-\frac{1}{1-m}} \tag{2}$$

where $B > 0$ can be any constant and is in a one-to-one correspondence to the mass (L^1 norm) of the solution. A similar solution $(B - \frac{1-m}{2}|\boldsymbol{x}|^2)_+^{-\frac{1}{1-m}}$ applies for $m > 1$. One can capture the diffusive spreading by means of a time dependent rescaling of space

$$\begin{gathered} \boldsymbol{x} = (1+\tfrac{\tau}{\alpha})^{-\alpha}\boldsymbol{y}, \qquad t = \alpha\ln(1+\tfrac{\tau}{\alpha}), \\ \alpha = \tfrac{1}{2-(1-m)n}, \\ u(t,\boldsymbol{x}) = e^{nt}\rho(\alpha e^{t/\alpha} - \alpha, e^t\boldsymbol{x}) \end{gathered} \tag{3}$$

which transforms (1) into

$$\frac{\partial}{\partial t}u = \frac{1}{m}\Delta u^m + \nabla\cdot(\boldsymbol{x}u). \tag{4}$$

and u_B is a stationary solution to this rescaled equation. It has been shown by Friedman and Kamin [8] that all solutions of finite mass converge to the Barenblatt solution of the same mass as $t \to \infty$. (They showed it for $m > 1$, noting that the proof carries over to the range $\max\{0, 1-\frac{2}{n}\} < m < 1$ as well; see also Vázquez [16].) Our concern is to quantify the rate of this convergence sharply. In the unrescaled variables $(\boldsymbol{y}, \tau)$ these rates will be powers of τ; for the rescaled variables $(\boldsymbol{x}, t)$, this corresponds to exponential rates. While the original equation (1) is translation invariant, in the rescaled coordinates, this translation invariance is lost, and we have in particular convergence of a translated Barenblatt to the original Barenblatt a convergence that measures the increasing overlap of the spreading solutions in the unrescaled coordinates.

The first approach to such convergence questions has used the Bakry-Emery formalism [4] [13] [6], which amounts to deriving second order ordinary differential inequalities for an entropy functional. In this formalism, Carrillo and Vázquez [5] proved an $O(\tau^{-1/2})$ rate in the L^1 norm for the entire range and for data that have second moments (or a somewhat weaker 'finite relative entropy condition' that is relevant for $m \le 1 - \frac{2}{n+2}$, where the Barenblatt solution lacks second moments). This rate is optimal as a uniform rate for all m, but short of optimal for any single $m < 1$. The convergence rate of a translated Barenblatt is $O(\tau^{-1/(2-n(1-m))})$, and it turns out other data do not converge more slowly than this either. In the same work, Carrillo and Vázquez also established an $O(1/\tau)$ convergence rate for radially symmetric data (subject to a decay condition), not only in the L^1 norm, but also in the stronger relative L^∞ norm.

We will in particular generalize their result to general data, subject to the constraint that the center of mass be at the origin. (In the case $m \le 1 - \frac{1}{n}$, this center of mass condition is not needed, hence there is no problem in the range where Barenblatt doesn't have finite first moments.) Having modded out the space translation symmetry, the $O(1/\tau)$ convergence rate is natural. It is the convergence rate of a scaled (or equivalently: time-translated) Barenblatt to the original. The significance of these rates in terms of symmetry has already been pointed out in [5]. It had been observed in the 1D porous medium case by Vázquez [15],[14] and Witelski and Bernoff [17], and has been used for more recent purposes by Carrillo, Di Francesco and Toscani in [3].

It should be pointed out that the $O(1/\tau)$ rate has already been obtained by Kim and McCann for the very fast diffusion range $m < 1 - \frac{2}{n+2}$ [10] by potential methods (applying Δ^{-1} to (1)). Moreover a rate of $O(1/\tau^{1-\varepsilon})$ has been obtained in the range $1 - \frac{1}{n} < m < 1$ by McCann and Slepčev [12]; in this range the notion of displacement convexity can be used to boost the Bakry Emery approach, and obviously the result must be subject to the center of mass condition.

A recent preprint [2] by Blanchet, Bonforte, Dolbeault, Grillo, Vazquez should be noted for extending strong convergence results even into parameter range of finite time extinction. However, the precise rate of convergence in this preprint is quite implicit, and we have not extracted this quantitative information at the present time.

The present paper focuses on a different approach to this same question: namely to make the formal correspondence between eigenvalues of the linearization and the convergence rates to the equilibrium rigorous. This approach has been prepared by two of the present authors in [7], where the full spectrum of the linearization was calculated in a geometric linearization framework based on mass transport, an approach to the porous medium and fast diffusion equation pioneered by Otto [13]. Our dynamical systems approach not only recovers the $O(1/\tau)$ convergence rate (modulo translation) in the above-mentioned cases where it was already established, filling in the gap between $1 - \frac{2}{n+2}$ and $1 - \frac{1}{n}$, but it also allows, to some extent, to go beyond these results to yet faster convergence rates when the time translation is also modded out. The spectral approach had already be used by Angenent [1] for the porous medium equation in 1D, who amended a linearized approach by Zel'dovitch and Barenblatt [18] by including possible logarithmic terms in the higher asymptotics due to eigenvalue resonances. It is noteworthy that the eigenvalues found by Angenent indeed extrapolate our eigenvalues [7] from the fast diffusion into the porous medium range (there is no continuous spectrum for $m > 1$). Angenent's approach relies on a reference interval for the compact support and can apparently not be generalized to higher dimensions. A similar approach in higher dimensions by Koch [11] was limited by the fact that, at the time, he did not have the spectrum at his disposal.

2. **Review of spectral information.** Some readers may wish to skip directly to the next section with the results, but we find it easier to interpret these results in full view of the spectral information that generates them.

In [7], we found the spectrum of the linearization of (4) within Otto's framework that views (1) as a gradient flow on a formal infinite dimensional Riemannian manifold whose geodesic distance is the Wasserstein metric. The underlying function space was a weighted Sobolev space $W^{1,2}$. It turns out that this analysis carries over to a weighted L^2 space, for basically algebraic reasons. There is essential (continuous) spectrum and finitely many eigenvalues. These eigenvalues (of small absolute value) should determine convergence rates governed by certain (slow) modes that describe what kind of data actually exhibit this convergence rate. The numbers in the essential spectrum are larger in absolute value and describe how arbitrary data approximate the slow modes.

The Sobolev spaces from [7] do not seem particularly suited for carrying out the nonlinear theory. We will therefore come to linearize in the traditional functional analytic setting rather than using Otto's approach, and work with Hölder spaces. A closer analysis, relying on the explicit calculation from [7] shows that even in this different setting, the eigenvalues are the same, even though some of them may be 'swallowed' by an essential spectrum that becomes larger in the new setting. However, the geometrically significant eigenvalues that give the convergence rates are not swallowed by essential spectrum. The change in essential spectrum is basically due to the fact that the critical growth for eigenfunctions is different in the weighted (Hilbert) space and the unweighted (Hölder) space. Once we consider weighted Hölder spaces, the location of the spectrum remains unchanged with respect to [7], provided the critical growth enforced by the weight is the same. (A change in the nature of the spectrum from continuous to dense point is inconsequential.)

In our notation of eigenvalues, the first index indicates to which spherical harmonics the eigenvalue belongs, the second is just an enumeration index. There is the eigenvalue $\lambda_{00} = 0$ corresponding to a change of mass, which we may ignore, and there is an eigenvalue $\lambda_{10} = -1$ corresponding to a translation, which we may also ignore if we fix the center of mass at the origin. The first significant eigenvalue is then $\lambda_{01} = -(2-n(1-m))$. The convergence rate τ^{-1} arises from $e^{\lambda_{01}t}$ under the transformation (3). If we divide out the scaling that gives rise to λ_{01}, the next significant eigenvalues are λ_{20} and λ_{02}. It depends on m which of the two dominates. However, the eigenfunctions corresponding to these eigenvalues are unbounded, so they can only be retrieved in a weighted Hölder space. This is the reason why finer estimates are of necessity carried out in a weighted norm.

3. **Results.** We prove:

Theorem 1 (Exact leading-order asymptotics in the relative L^∞ norm)**.** *Fix $m \in \,]\frac{n-2}{n}, 1[$. Suppose $\rho(\tau, \boldsymbol{y})$ satisfies* (1) *and the condition*

$$\lim_{\tau\to\infty} \sup \sup_{\boldsymbol{y}} \left| \frac{\rho(\tau,\boldsymbol{y})}{\rho_B(\tau,\boldsymbol{y})} - 1 \right| = 0. \tag{5}$$

holds for some $B > 0$. If $m = m_2 = \frac{n}{n+2}$, we assume in addition

$$\int_{\mathbb{R}^n} \left| 1 - \frac{\rho(0,\boldsymbol{y})}{u_B(\boldsymbol{y})} \right|^2 (1+|\boldsymbol{y}|^2)^{-n/2} d\boldsymbol{y} < \infty. \tag{6}$$

Then there exists $\boldsymbol{z} \in \mathbb{R}^n$ such that

$$\sup_{\tau>0, \boldsymbol{y}\in\mathbb{R}^n} \tau \left| \frac{\rho(\tau, \boldsymbol{y}-\boldsymbol{z})}{\rho_B(\tau,\boldsymbol{y})} - 1 \right| < \infty. \tag{7}$$

Without the additional condition (6) *in the case $m = m_2$, we still get* (7)*, but with the leading factor τ replaced by with $\tau^{1-\varepsilon}$ for any ε.*

The interpretation of this result has been explained already in the previous section. It should be noted that the extra hypothesis (6) is needed exactly for that value of $m = 1 - \frac{2}{n+2} =: m_2$ where the eigenvalue λ_{01} meets the continuous spectrum in the Hilbert space spectral theory.

For $m < m_2$, we can get a slightly finer asymptotic in an even stronger norm that enforces some extra decay; however, since we are using the sharp continuous spectrum from the Hilbert space spectral theory in a Hölder space setting, we need to require a weighted L^2 integrability hypothesis:

Theorem 2 (Fine Asymptotics for $m < m_2$)**.** *Assume that $n \geq 2$ and $m_0 = \frac{n-2}{n} < m < m_2 = \frac{n}{n+2}$, or else $n = 1$ and $0 < m < m_2 = \frac{1}{3}$. Further assume that the mass of ρ_0 is one, and if $n = 1$ also that the center of mass of ρ_0 is 0. Let $\gamma := (2-p)/4 = \frac{n}{4} - \frac{1}{2}\frac{m}{1-m}$. If $(1+r^2)^\gamma |1 - \rho_0/u_B| \in L^\infty \cap L^2_{\rho^m}$, then*

$$\sup_{\tau>1, \boldsymbol{y}\in\mathbb{R}^n} \tau^{-\beta} \rho_B^{(m-1)\gamma}(\tau, \boldsymbol{y}) \left| \frac{\rho(\tau, \boldsymbol{y})}{\rho_B(\tau, \boldsymbol{y})} - 1 \right| < \infty \tag{8}$$

where $-\beta = (p+2)^2/8p < -1$ and $p = \frac{2}{1-m} - n$.

In this theorem, we have made use of the convenient parameter p, which determines the number of moments the Barenblatt solution has: namely all its moments of order $< p$ are finite. For given p, the corresponding m-value $1 - \frac{2}{p+n}$ is denoted as m_p.

In the case $m > m_2$, it makes sense to consider convergence towards a Barenblatt solution that has not only its center of mass shifted appropriately, but is also shifted conveniently in time. Then the eigenvalue λ_{01} becomes irrelevant and we get higher convergence rates. However to do this, we need to use a slightly weaker norm. We obtain

Theorem 3 (Second order asymptotics modulo translations and dilations)**.** *Suppose that $m > m_2 = \frac{n}{n+2}$ and that ρ_0 satisfies the assumptions of Theorem 1, and that the center of mass of ρ_0 is at the origin. If $m \leq m_6$, we assume in addition condition* (6)*. Then there exist τ_0 such that*

$$\limsup_{\tau\to\infty} \sup_{\boldsymbol{y}} \tau^\gamma \left(\tau^{\frac{n}{2-n(1-m)}} \rho_B(\tau, \boldsymbol{y}) \right)^\delta \left| \frac{\rho(\tau, \boldsymbol{y})}{\rho_B(\tau - \tau_0, \boldsymbol{y})} - 1 \right| < \infty$$

where, for $n \geq 2$, we have

$$\gamma = \frac{(p+2)^2}{8p} = \frac{[2-(n-2)(1-m)]^2}{8(1-m)[2-n(1-m)]} \qquad \text{if } m_2 < m \leq m_6$$

$$\gamma = \frac{2(p-2)}{p} = \frac{4-2(n+2)(1-m)}{2-(1-m)n} \qquad \text{if } m_6 \leq m \leq m_{n+4}$$

$$\gamma = \frac{n+p}{p} = \frac{2}{2-(1-m)n} \qquad \text{if } m_{n+4} \leq m < 1$$

and

$$\delta = \frac{1}{n+p}\left(\frac{p}{2} - 1\right) = \frac{1}{4}(2m - n(1-m)) \qquad \text{if } m_2 < m \leq m_6$$

$$\delta = \frac{1}{n+p}\left(\frac{p}{2} - 1 - \sqrt{(\tfrac{p}{2}+1)^2 - 4(p-2)}\right) = 1 - m \qquad \text{if } m_6 \leq m \leq m_{n+4}$$

$$\delta = \frac{1}{n+p}\left(\frac{p}{2} - 1 - \sqrt{(\tfrac{p}{2}-1)^2 - 2n}\right) \qquad \text{if } m_{n+4} \leq m < 1$$

(Recall $m_2 = \frac{n}{n+2}$, $m_n = \frac{n-1}{n}$, $m_{n+4} = \frac{n+1}{n+2}$.)

For $n = 1$, the first case applies to $m_2 < m \leq m_{p_*}$, and the third case to $m_{p_*} \leq m < 1$, with $p_* = 2(\sqrt{2}+1)$, the middle case being omitted.

4. **Proof ideas.** The detailed proofs will be published elsewhere. The first step is to write (4) in terms of the ratio $v = u/u_B$ or the deviation $w = v - 1$. The principal part of the differential operator thus obtained has a cylindrical end and is $u_B^{m-1}\Delta$. This is the same as the principal symbol of the Laplace Beltrami operator on $\mathbb{R}^n$ with respect to a certain (conformally flat) Riemannian metric, namely u_B^{1-m} times the Euclidean metric. The Riemannian manifold thus obtained is called cigar manifold, because it can be embedded into $\mathbb{R}^{n+1}$ in exactly this shape. With respect to this metric, the PDE becomes *uniformly* parabolic. To benefit from this fact,

all Hölder spaces need to be defined in terms of the geodesic distance on the cigar manifold, rather than in terms of the euclidean distance. Then standard (local) flat parabolic Schauder estimates, combined with the maximum principle, become global estimates on the cigar manifold. This takes care of the linear theory.

The local semiflow is constructed from a careful balance between the regularity loss from the nonlinearity (which contains itself second derivatives) and the smoothing by the linear equation. To extend the short-term dynamics globally in time, a-priori estimates are needed: they are basically a comparison principle telling that eventually, every solution can be nested between two Barenblatt solutions. This is a consequence of a key estimate by Vázquez (Thm. 21.1 of [16]). The concern is to get a *differentiable* semiflow.

Since we have abandoned the Hilbert space setting from [7], we lose self-adjointness, and it is not a-priori clear that a spectral radius of λ gives rise to an estimate $O(e^{\lambda t})$ rather than $O(e^{(\lambda+\varepsilon)t})$ for the semigroup generated. The analysis of the nature of the spectrum does actually indicate the presence of residual spectrum, which could give rise to such non-boundedness of the semigroup. Nevertheless it turns out that an estimate with the sharp exponent holds in almost all cases. The one (possible) exception occurs exactly when we study the semigroup in a weighted Hölder space that enforces the same critical growth as the weighted Hilbert space from [7].

The proof involves a shifting of the essential spectrum by conjugating with an exponential weight, together with an invocation of the maximum principle that controls the growth of solutions more tightly than could be anticipated from soft invocation of functional analysis and semigroups. The exceptions in which an extra L^2 bound is needed in the hypothesis arise from those cases where we were not able to carry this interplay between semigroup estimates and maximum principle through. At present we do not know whether this L^2 hypothesis is essential or just required by the method.

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