Existence, uniqueness, concavity and geometry of the monopolist's problem facing consumers with nonlinear price preferences

by

Kelvin Shuangjian Zhang

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Abstract

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A monopolist wishes to maximize her profits by finding an optimal price menu. After she announces a menu of products and prices, each agent will choose to buy that product which maximizes his utility, if positive. The principal's profits are the sum of the net earnings produced by each product sold. These are determined by the costs of production and the distribution of products sold, which in turn are based on the distribution of anonymous agents and the choices they make in response to the principal's price menu.

In this thesis, two existence results will be provided, assuming each agent's disutility is a strictly increasing but not necessarily affine (i.e., quasilinear) function of the price paid. This has been an open problem for several decades before the first multi-dimensional result obtained here and independently by Nöldeke and Samuelson in 2015.

Additionally, a necessary and sufficient condition for the convexity or concavity of this principal's (bi-level) optimization problem is investigated. Concavity when present, makes the problem more amenable to computational and theoretical analysis; it is key to obtaining uniqueness and stability results for the principal's strategy in particular. Even in the quasilinear case, our analysis goes beyond previous work by addressing convexity as well as concavity, by establishing conditions which are not only sufficient but necessary, and by requiring fewer hypotheses on the agents' preferences. Moreover, the analytic and geometric interpretations of a specific condition relevant to the concavity of the problem has been explored.

Finally, various examples are given to explain the interaction between preferences of agents' utility and monopolist's profit which ensure statements equivalent to the concavity of the principal-agent problem. In particular, an example with quasilinear preferences on n-dimensional hyperbolic spaces is given with explicit solutions to show uniqueness without concavity. Similar results on spherical and Euclidean spaces are also provided. Additionally, the solutions of hyperbolic and spherical cases converge to those of Euclidean spaces as curvature goes to 0.

To my parents
Shuying Yu and Muli Zhang
and my sister
Qiuyu Zhang

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Contents

Chapter 1

Introduction

1.1 Problem formulation

As one of the central problems in microeconomic theory, the *principal-agent framework* characterizes the type of non-competitive decision-making problems which involve aligning incentives so that one set of parties (the agents) finds it beneficial to act in the interests of another (the principal) despite holding private information. Such problem arises in a variety of different contexts. Besides nonlinear pricing [1, 29, 43, 46], economists also use this framework to model many different types of transactions, including tax policy [14, 26, 35], contract theory [33], regulation of monopolies [3], product line design [37], labour market signaling [42], public utilities [34], and mechanism design [17, 23, 24, 27, 31, 45]. Many of these share the same mathematical model. In this thesis, we use nonlinear pricing to motivate the discussion, although our conclusions may be equally pertinent to many other areas of application. We only consider the case where both agent types and product attributes are continuous.

Consider the problem for a multiproduct monopolist who sells indivisible products to a population of consumers, who each buy at most one unit. Assume there is neither cooperation nor competition between agents. Additionally, assume the monopolist is able to produce enough of each product such that there are neither product supply shortages nor economies of scale. Taking into account participation constraints and incentive compatibility, the monopolist would like to find the optimal menu of prices to maximize her total profit.

Let X, Y be open and bounded subsets in $\mathbf{R}^m$ and $\mathbf{R}^n$ $(m \geq n)$, respectively, with closures $cl(X)$ and $cl(Y)$. Suppose the monopolist wants to maximize her profits by selecting the dependence of the price $v(y)$ on each type $y \in cl(Y)$ of product sold. An agent of type $x \in X$ will choose to buy that product which maximizes his benefit

$$u(x) := \max_{y \in cl(Y)} G(x, y, v(y)), \tag{1.1.1}$$

where $(x, y, z) \in X \times cl(Y) \times \mathbf{R} \longmapsto G(x, y, z) \in \mathbf{R}$, is the given direct utility function for agent type x to choose product type y at price z.

After agents, whose distribution $d\mu(x)$ is known to the monopolist, have chosen their favorite items to buy, the monopolist calculates her profit given by the functional

$$\Pi(v, y) := \int_X \pi(x, y(x), v(y(x))) d\mu(x), \tag{1.1.2}$$

where $y(x)$ denotes the product type y which agent type x chooses to buy (and which maximizes (1.1.1)), $v(y(x))$ denotes the selling price of type $y(x)$ and $\pi \in C^0(cl(X \times Y) \times \mathbf{R})$ denotes the principal's net profit of selling product type $y \in cl(Y)$ to agent type $x \in X$ at price $z \in \mathbf{R}$. The monopolist wants to maximize her net profit among all lower semicontinuous pricing policies.

In economic models, incentive compatibility is needed to ensure that all the agents report their preferences truthfully. According to the revelation principle (see [30]), this costs no generality. Decisions made by monopolist according to the information collected from agents then lead to the expected market reaction (as in [5, 37]). Individual rationality is required to ensure full participation so that each agent will choose to play, possibly by accepting the outside option. Individual agents accept to contract only if the benefits they earn are no less than their outside option. We model this by assuming the existence of a distinguished point $y_\emptyset \in cl(Y)$ which represents the outside option, and whose price cannot exceed some fixed value $z_\emptyset \in \mathbf{R}$ beyond the monopolist's control. This removes any incentive for the monopolist to raise the prices of other options too high. (We can choose normalizations such as $\pi(x, y_\emptyset, z_\emptyset) = 0 = G(x, y_\emptyset, z_\emptyset)$ and $(y_\emptyset, z_\emptyset) = (0, 0)$, or not, as we wish.)

The following is a table of notation:

Table 1.1: Notation

Mathematical Expression	Economic Meaning
x	agent type
y	product type
$X \subset \mathbf{R}^m$	(open, bounded) domain of agent types
$cl(Y) \subset \mathbf{R}^n$	domain of product types, closure of Y
$v(y)$	selling price of product type y (in Chapter 3 we use $p(y)$ instead)
$v(y_\emptyset) \leq z_\emptyset$	price normalization of the outside option $y_\emptyset \in cl(Y)$
$u(x)$	indirect utility of agent type x
$\mathrm{dom}Du$	points in X where u is differentiable
$G(x, y, z)$	direct utility of buying product y at price z for agent x
$H(x, y, u)$	price at which y brings x value u, so that $H(x, y, G(x, y, z)) = z$
$\pi(x, y, z)$	the principal's profit for selling product y to agent x at price z
$d\mu(x)$	Borel probability measure giving the distribution of agent types on X
$\mu \ll \mathcal{L}^m$	μ vanishes on each subset of $\mathbf{R}^m$ having zero Lebesgue volume $\mathcal{L}^m$
$\Pi(v, y)$	monopolist's profit facing agents' responses $y(\cdot)$ to her chosen price policy $v(\cdot)$
$\mathbf{\Pi}(u)$	monopolist's profit, viewed instead as a function of agents' indirect utilities $u(\cdot)$

Definition 1.1.1 (Incentive compatible and individually rational)**.** A measurable map $x \in X \longmapsto$

$(y(x), z(x)) \in cl(Y \times Z)$ of agents to (product, price) pairs is called *incentive compatible* if and only if $G(x, y(x), z(x)) \geq G(x, y(x'), z(x'))$ for all $(x, x') \in X^2$. Such a map offers agent x no incentive to pretend to be x'. It is called *individually rational* if and only if $G(x, y(x), z(x)) \geq G(x, y_\emptyset, z_\emptyset)$ for all $x \in X$, meaning no individual x strictly prefers the outside option $(y_\emptyset, z_\emptyset)$ to his assignment $(y(x), z(x))$.

Proposition 1.1.2. *The principal's program is as follows:*

$$(P_0) \begin{cases} \sup \Pi(v, y) = \int_X \pi(x, y(x), v(y(x))) d\mu(x) \quad \textit{subject to} \\ x \in X \longmapsto (y(x), v(y(x))) \ \textit{incentive compatible, individually rational,} \\ \textit{and } v : cl(Y) \longrightarrow cl(Z) \ \textit{lower semicontinuous with } v(y_\emptyset) \leq z_\emptyset. \end{cases}$$

1.2 Background

We study a general version of a multidimensional nonlinear pricing model, which is a natural extension of the models studied by Mussa-Rosen [29], Mirrlees [26], Spence [42, 43], Myerson [31], Baron-Myerson [3], Maskin-Riley [23], Wilson [46], Rochet-Choné [37], Monteiro-Page [27] and Carlier [5]. A major distinction lies in whether the agents' private type is one-dimensional (such as [29, 23]), or multidimensional (such as [33, 37, 27, 5]). Another distinction is whether preferences are quasilinear on price (such as [1, 5]) or fully nonlinear (such as [32, 25]), especially for multidimensional models.

For the quasilinear case, where the utility $G(x, y, z)$ depends linearly on its third variable, and net profit $\pi(x, y, z) = z - a(y)$ represents difference of selling price z and manufacturing cost a of product type y, theories of existence [4, 38, 5, 27], uniqueness [6, 11, 29, 37] and robustness [4, 11] have been well studied.

When parameterization of preferences is linear in agent types and price, where $cl(X) = cl(Y) = [0, \infty)^n$, $G(x, y, z) = \langle x, y \rangle - z$, and $(y_\emptyset, z_\emptyset) = (0, 0)$, Rochet and Choné (1998, [37]) not only obtain existence results but also partially characterize optimal solutions and expound their economic interpretations, given that monopolist profits can be characterized by the aggregate difference between selling prices and quadratic manufacturing costs. Here $\langle\ ,\ \rangle$ denotes the Euclidean inner product.

More generally, Carlier ([5]) has proved existence results for general quasilinear utility $G(x, y, z) = b(x, y) - z$, where agent type and product type are not necessarily of the same dimension and monopolist profit equals selling price minus some linear manufacturing cost.

Figalli-Kim-McCann [11] reveals the equivalence of function space convexity to a non-negative fourth order cross-curvature condition, and conditions of functional concavity, where uniqueness and stability of the monopolist's maximizing strategy follow from strict concavity.

1.3 Motivation

Starting from the celebrated work of Nobel laureates Mirrlees [26] and Spence [42], there are two main types of generalizations. One generalization is regarding dimension, from one-dimensional to multidimensional. The other generalization is in the form of utility functions, to beyond quasilinear.

The generalization of quasilinear to nonlinear preferences has many potential applications. For example, the benefit function $G(x, y, v(y)) = b(x, y) - v^2(y)$ models agents who are more sensitive to higher prices, while the function $G(x, y, v(y)) = b(x, y) - v^{\frac{1}{2}}(y)$ models agents who are less sensitive to higher prices, and utility $G(x, y, v(y)) = b(x, y) - f(x, v(y))$ describes the scenario when different agents might have different sensitivities to the same price. See Wilson [46, Chapter 7] for the importance of taking income effects into account. Very few results are known for such nonlinearities, due to the complications which they entail.

In 2013, Trudinger's lecture at the optimal transport program at MSRI inspired us to try generalizing Carlier [5] and Figalli-Kim-McCann [11] to the non-quasilinear case. With the tools developed by Trudinger [44] and others [2, 41], we are able to provide existence, convexity, and concavity theorems for general utility and net profit functions.

The generalized existence problem was also mentioned as a conjecture by Basov [4, Chapter 8]. Independently of the present work, Nöldeke and Samuelson (2015, [32]) provided a general existence result assuming that $cl(X)$, $cl(Y)$ are compact and the utility G is decreasing with respect to its third variable, by implementing a duality argument based on Galois connections.

The equivalence of concavity to the corresponding non-negative cross-curvature condition revealed by Figalli-Kim-McCann [11] directly motivates our work. In addition to the quasilinearity of $G(x, y, z) = b(x, y) - z$ essential to their model, they require additional restrictions such as $m = n$ and $b \in C^4(cl(X \times Y))$ which are not economically motivated and which we shall relax or remove. However, we shall eventually show that under certain conditions the concavity or convexity of G and π (or their derivatives) with respect to v tends to be reflected by the concavity or convexity of Π, not with respect to v or y, but rather with respect to the agents' indirect utility u, in terms of which the principal's maximization is reformulated. Moreover, our results allow for the monopolist's profit π to depend in a general way both on monetary transfers and on the agents' types x, revealed after contracting. Such dependence plays an important role in applications such as insurance marketing.

Inspired by Kim-McCann [18], which expressed the fourth-order Ma-Trudinger-Wang condition in optimal transportation theory as the non-negativity of the sectional curvature in a specific pseudo-Riemannian geometry, we would like to explore the geometric interpretations of the (G3) hypothesis, a Ma-Trudinger-Wang type condition, for our concavity results.

The work [11] by Figalli-Kim-McCann provides a non-negative definiteness condition of a certain fourth order differential expression (B3), which not only is equivalent to the convexity of some function space, but also implies concavity of the maximization functional, and thus uniqueness follows from a strict version of (B3). One may wonder what happens if this curvature condition (B3) is violated. Inspired by Loeper [20], which claims that, for quasilinear Riemannian quadratic utility, (B3) is satisfied only if the Riemannian sectional curvature is non-negative, some part of the thesis aims to investigate uniqueness without concavity on the hyperbolic spaces with constant negative curvatures. Besides, previously there are few explicit results on spaces of dimensions greater than two.

It is worth mentioning that given the technical arguments exploited in this thesis, it may be very fruitful to study possible generalizations of other known results for convex functions to G-convex functions, which will be defined in Section 2.2.

1.4 Outline of the thesis

Chapter 2 provides some preliminaries and, in particular, a generalized notion of convexity: the G-convexity (c.f. [44, 2, 41]). We also see that the incentive compatibility is conveniently encoded by the G-convexity of the agents' indirect utility u, which is an analog of Carlier [5].

Initiated independently of [32], Chapter 3 provides a general existence result for the multidimensional monopolist model with general nonlinear preferences with less restriction on boundedness of the product domain, by extending Carlier [5] to fully nonlinear preferences. Due to the lack of natural compactness, the proof of this work is quite different from that of Nöldeke-Samuelson. Furthermore, G-convex analysis, which is strongly tied to Trudinger's theory on the regularity of nonlinear PDEs [44] developed for vastly different purposes, is employed to deal with the difficulty of non-quasilinear preferences.

Chapter 4 presents another general existence result given the generalized single-crossing condition and boundedness of the consumer-type and product-type spaces. This result is also shown using G-convex analysis, but the proof is different from Chapter 3, since most assumptions are different.

We will show convexity results in Chapter 5. In Chapter 5, we generalize uniqueness and concavity results of Figalli-Kim-McCann to the non-quasilinear case. In this work, we first give a necessary and sufficient condition (G3) under which the function space $\mathcal{U}_\emptyset$ is convex.

We then provide the equivalent conditions, respectively, to the concavity, convexity, uniform concavity, and uniform convexity of the functional $\mathbf{\Pi}$. We also give sufficient conditions for strict concavity, which implies uniqueness for this problem. Besides, the maximizers of $\mathbf{\Pi}$ may not be unique under convexity but are attained at some extreme point(s) (the elements that cannot be represented by a convex combination of other elements) of the function space $\mathcal{U}_\emptyset$.

We also show that the concavity condition is equivalent to the non-positive definiteness of some quadratic form on $\mathbf{R}^{n+1}$.

The condition (G3) is so crucial to the concavity result that we want to investigate it a bit more. Chapter 6 shows that (G3) is equivalent to the non-positive definiteness of some fourth order differential expression along affinely parametrized line segments, which is an analog of the non-negative definiteness of the fourth order condition given in Trudinger [44] for regularity of prescribed Jacobian equations. It also coincides in the quasilinear case with the fourth order condition provided by Figalli-Kim-McCann in [11], which strengthens to the Ma-Trudinger-Wang condition [21] in regularity theory of Optimal Transport.

Motived by Kim-McCann [18], in Chapter 7, we will show that (G3) is equivalent to the non-negativity of the sectional curvature in a natural pseudo-Riemannian geometry associated to the economic problem at hand.

Oriented by Loeper's work [20], Chapter 8 proves uniqueness by showing (in exact form) the unique solutions of special examples with quasilinear preferences where domains are symmetric disks on the n-dimensional hyperbolic spaces $\mathbf{H}^n$, and the utilities are quasilinear quadratic hyperbolic distances. It also shows solutions on the spheres $\mathbf{S}^n$ and Euclidean spaces $\mathbf{R}^n$, where the utilities are quasilinear quadratic spherical and Euclidean distances, respectively. Moreover, the solutions on $\mathbf{S}^n$ and $\mathbf{H}^n$ converge to those on $\mathbf{R}^n$ as curvature goes to 0.

For non-quasilinear preferences, we specialize the form obtained from Chapter 5 into various examples and give the equivalent conditions to the concavity/convexity of the maximization problem.

Remark 1.4.1. Chapter 4, 5, 6, 7 and the second part of Chapter 8 are joint work with my advisor Robert J. McCann (see McCann-Zhang [25]). It should be mentioned here that neither the convexity work, nor the earlier two existence results, require the monopolist's profit to take on a particular form, which is a generalization from much of the literature. We believe that the (G3) condition and the concavity results are critical steps for the computational analysis of the multidimensional nonlinear principal-agent problem. It will be exciting to investigate the situations when the monopolist has less or inaccurate information, as well as how our results apply to the duopoly and other oligopoly games. The G-convexity method in this thesis is potentially applicable to other problems under the same principal-agent framework, such as the study of tax policy ([26]) and other regulatory policies ([3]). For an application of G-convexity to geometric optics, see [16].

Chapter 2

Preliminaries and G-convexity

2.1 Preliminaries

Let X be a subset of $\mathbf{R}^m$.

Definition 2.1.1 (Subdifferential)**.** Recall that the subdifferential of a function $u : X \longrightarrow \mathbf{R}$ at $x_0 \in X$ is defined as the set:

$$\partial u(x_0) := \{y \in \mathbf{R}^m \mid u(x) - u(x_0) \geq \langle x - x_0, y \rangle, \text{ for all } x \in X\}. \tag{2.1.1}$$

Here $\langle\ ,\ \rangle$ denotes the Euclidean inner product.

Lemma 2.1.2. *The set defined in* (2.1.1) *is nonempty for every $x_0 \in X$ if and only if u is convex.*

We give a proof below for a generalized version of this lemma.

For any vectors $p, w \in \mathbf{R}^n$, we denote $p \parallel w$ if p and w are parallel.

We use $\mathcal{L}^m$ to denote Lebesgue measure on $\mathbf{R}^m$, which characterizes the m-dimensional volume. A non-negative measure μ is said to be absolutely continuous with respect to $\mathcal{L}^m$ if for every measurable set A, $\mathcal{L}^m(A) = 0$ implies $\mu(A) = 0$. This is written as $\mu \ll \mathcal{L}^m$.

Let f be a function on X. We say $f \in L^1(X)$ if $\int_X |f(x)| d\mathcal{L}^m(x) < \infty$. Denote by $W^{1,1}(X)$ the Sobolev space of $L^1(X)$ functions whose first derivatives exist in the weak sense and belong to $L^1(X)$. For more properties of Sobolev spaces and weak derivatives, see Evans [10, Chapter 5]. If ω is a subset of X, the notation $\omega \subset\subset X$ means that the closure of ω is also included in X.

Here we use $G_x = \left(\frac{\partial G}{\partial x^1}, \frac{\partial G}{\partial x^2}, ..., \frac{\partial G}{\partial x^m}\right)$, $G_y = \left(\frac{\partial G}{\partial y^1}, \frac{\partial G}{\partial y^2}, ..., \frac{\partial G}{\partial y^n}\right)$, $G_z = \frac{\partial G}{\partial z}$ to denote derivatives with respect to $x \in X \subset \mathbf{R}^m$, $y \in Y \subset \mathbf{R}^n$, and $z \in \mathbf{R}$, respectively. Also, for second partial derivatives, we adopt the following notation

$$G_{x,y} = \begin{bmatrix} \frac{\partial^2 G}{\partial x^1 \partial y^1} & \frac{\partial^2 G}{\partial x^1 \partial y^2} & \cdots & \frac{\partial^2 G}{\partial x^1 \partial y^n} \\ \frac{\partial^2 G}{\partial x^2 \partial y^1} & \frac{\partial^2 G}{\partial x^2 \partial y^2} & \cdots & \frac{\partial^2 G}{\partial x^2 \partial y^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 G}{\partial x^m \partial y^1} & \frac{\partial^2 G}{\partial x^m \partial y^2} & \cdots & \frac{\partial^2 G}{\partial x^m \partial y^n} \end{bmatrix}$$

and $G_{x,z} = \big(\frac{\partial^2 G}{\partial x^1 \partial z}, \frac{\partial^2 G}{\partial x^2 \partial z}, ..., \frac{\partial^2 G}{\partial x^m \partial z}\big)$.

A function is said to be C^0 if it is continuous on its domain. We say $G \in C^1(cl(X \times Y \times Z))$, if all the partial derivatives $\frac{\partial G}{\partial x^1}$, ..., $\frac{\partial G}{\partial x^m}$, $\frac{\partial G}{\partial y^1}$, ..., $\frac{\partial G}{\partial y^n}$, $\frac{\partial G}{\partial z}$ exist and are continuous. Also, we say $G \in C^2(cl(X \times Y \times Z))$, if all the partial derivatives up to second order (i.e. $\frac{\partial^2 G}{\partial \alpha \partial \beta}$, where $\alpha, \beta = x^1, ..., x^m, y^1, ..., y^n, z$) exist and are continuous. Any bijective continuous function whose inverse is also continuous, is called a homeomorphism (a.k.a. bicontinuous).

We will use Einstein notation for simplifying expressions including summations of vectors, matrices, and general tensors for higher order derivatives. There are essentially three rules of Einstein summation notation, namely: 1. repeated indices are implicitly summed over; 2. each index can appear at most twice in any term; 3. both sides of an equation must contain the same non-repeated indices. For example, $a_{ij}v_i = \sum_i a_{ij}v_i$, $a_{ij}b^{kj}v_k = \sum_j \sum_k a_{ij}b^{kj}v_k$. We also use a comma to separate subscripts: the subscripts before the comma represent derivatives with respect to first variable and those after the comma represent derivatives with respect to the second variable. For instance, for $b = b(x, y)$, $b_{,kl}$ represents second derivatives with respect to y only. And for $G = G(x, y, z)$, where $z \in \mathbf{R}$, $G_{i,jz}$ denotes third order derivatives with respect to x, y and z, instead of using another comma to separate subscripts corresponding to y and z.

2.2 G-convex, G-subdifferentiability

In this section, we introduce some tools from convex analysis and the notion of G-convexity (c.f. [44, 2, 41]), which is a generalization of ordinary convexity.

Let X, Y, and Z be subsets of $\mathbf{R}^m$, $\mathbf{R}^n$, and $\mathbf{R}$ respectively. Assume $G : X \times Y \times Z \longrightarrow \mathbf{R}$ is any function which is strictly decreasing in its last variable. For each $(x, y) \in X \times cl(Y)$ and $u \in G(x, y, cl(Z))$, define $H(x, y, u) := z$ whenever $G(x, y, z) = u$, i.e., $H(x, y, \cdot) = G^{-1}(x, y, \cdot)$. In the context of nonlinear pricing, $G(x, y, z)$ represents the utility that consumer x obtains for purchasing product y at price z, while $H(x, y, u)$ denotes the price paid by agent x for product y when receiving value u.

From Lemma 2.1.2, for any convex function u on X and any fixed point $x_0 \in X$, there exists $y_0 \in \partial u(x_0)$, satisfying

$$u(x) \geq \langle x, y_0 \rangle - (\langle x_0, y_0 \rangle - u(x_0)), \text{ for all } x \in X, \tag{2.2.1}$$

where equality holds at $x = x_0$. On the other hand, if for any $x_0 \in X$, there exists y_0, such that (2.2.1) holds for all $x \in X$, then u is convex. The following definition is analogous to this property, which is a special case of G-convexity, when $G(x, y, z) = \langle x, y \rangle - z$. In this case, we have $H(x, y, u) = \langle x, y \rangle - u$.

Definition 2.2.1 (G-convexity)**.** A function $u \in C^0(X)$ is called *G-convex* if for each $x_0 \in X$, there exist $y_0 \in cl(Y)$, and $z_0 \in cl(Z)$ such that $u(x_0) = G(x_0, y_0, z_0)$, and $u(x) \geq G(x, y_0, z_0)$, for all $x \in X$.

Similarly, one can also generalize the definition of subdifferential from (2.2.1).

Definition 2.2.2 (G-subdifferentiability)**.** The G-subdifferential of a function $u : X \longrightarrow \mathbf{R}$ is a point-to-set mapping defined by

$$\partial^G u(x) := \{y \in cl(Y) | u(x') \geq G(x', y, H(x, y, u(x))), \text{ for all } x' \in X\}.$$

A function u is said to be *G-subdifferentiable* at x if and only if $\partial^G u(x) \neq \emptyset$.[1]

In particular, if $G(x, y, z) = \langle x, y \rangle - z$, then the G-subdifferential coincides with the subdifferential. There are other generalizations of convexity and subdifferentiability. For instance, h-convexity in Carlier [5], or equivalently, b-convexity in Figalli-Kim-McCann [11], or c-convexity in Gangbo-McCann [12], is a special form of G-convexity, where $G(x, y, z) = h(x, y) - z$, which plays an important role in the quasilinear case. For more references of convexity generalizations, see Kutateladze-Rubinov [19], Elster-Nehse [9], Balder [2], Dolecki-Kurcyusz [7], Singer [41], Rubinov [40], and Martínez-Legaz [22].

As mentioned above, a well-known result in convex analysis is that a function is convex if and only if it is subdifferentiable everywhere. The following lemma adapts this to G-convexity.

Lemma 2.2.3. *A function $u : X \to \mathbf{R}$ is G-convex if and only if it is G-subdifferentiable everywhere.*

Proof. Assume u is G-convex, we want to show that u is G-subdifferentiable everywhere, i.e., we need to prove $\partial^G u(x_0) \neq \emptyset$ for all $x_0 \in X$.

Since u is G-convex, by definition, for each x_0, there exists y_0, z_0, such that $u(x_0) = G(x_0, y_0, z_0)$, and for all $x \in X$,

$$u(x) \geq G(x, y_0, z_0) = G(x, y_0, H(x_0, y_0, u(x_0))).$$

By the definition of G-subdifferentiability, $y_0 \in \partial^G u(x_0)$, i.e. $\partial^G u(x_0) \neq \emptyset$.

On the other hand, assume u is G-subdifferentiable everywhere, then for each $x_0 \in X$, there exists $y_0 \in \partial^G u(x_0)$. Set $z_0 := H(x_0, y_0, u(x_0))$ so that $u(x_0) = G(x_0, y_0, z_0)$.

Since $y_0 \in \partial^G u(x_0)$, for all $x \in X$, we have

$$u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0))) = G(x, y_0, z_0).$$

By definition, u is G-convex. $\square$

Using Lemma 2.2.3, one can show the following result, which connects incentive compatibility in the economic context with G-convexity and G-subdifferentiability in mathematical analysis, generalizing the results of Rochet [36] and Carlier [5].

Proposition 2.2.4 (G-convex utilities characterize incentive compatibility)**.** *Let (y, z) be a pair of mappings from X to $cl(Y) \times cl(Z)$. This (product, price) pair is incentive compatible if and only if $u(\cdot) := G(\cdot, y(\cdot), z(\cdot))$ is G-convex and $y(x) \in \partial^G u(x)$ for each $x \in X$.*

Proof. " $\Rightarrow$ ". Suppose (y, z) is incentive compatible. For any fixed $x_0 \in X$, let $y_0 = y(x_0)$ and $z_0 = z(x_0)$. Then $u(x_0) = G(x_0, y(x_0), z(x_0)) = G(x_0, y_0, z_0)$. By incentive compatibility of the contract (y, z), for any $x \in X$, one has $G(x, y(x), z(x)) \geq G(x, y(x_0), z(x_0))$. This implies $u(x) \geq G(x, y_0, z_0)$, for any $x \in X$, since $u(x) = G(x, y(x), z(x))$, $y_0 = y(x_0)$ and $z_0 = z(x_0)$. By definition, u is G-convex.

Since $u(x_0) = G(x_0, y_0, z_0)$, by definition of function H, one has $z_0 = H(x_0, y_0, u(x_0))$. Combining with $u(x) \geq G(x, y_0, z_0)$, for any $x \in X$, which is concluded from above, we have $u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0)))$, for any $x \in X$. By definition of G-subdifferentiability, one has $y_0 \in \partial^G u(x_0)$, and thus $y(x_0) = y_0 \in \partial^G u(x_0)$.

[1] In Trudinger [44], this point-to-set mapping $\partial^G u$ is also called *G-normal* mapping; see this paper for more properties related to G-convexity.

" $\Leftarrow$ ". Assume that $u = G(x, y(x), z(x))$ is G-convex, and $y(x) \in \partial^G u(x)$, for any $x \in X$. For any fixed $x \in X$, since $y(x) \in \partial^G u(x)$, for any $x' \in X$, one has

$$u(x') \geq G(x', y(x), H(x, y(x), u(x))). \tag{2.2.2}$$

Since $u(x) = G(x, y(x), z(x))$, by definition of function H, one has $z(x) = H(x, y(x), u(x))$. Combining with the inequality (2.2.2), we have $u(x') \geq G(x', y(x), z(x))$. Noticing $u(x') = G(x', y(x'), z(x'))$, one has $G(x', y(x'), z(x')) = u(x') \geq G(x', y(x), z(x))$. By definition, (y, z) is incentive compatible. $\square$

Chapter 3

Existence: unbounded product spaces

3.1 Introduction

Recently, Nöldeke-Samuelson (2015, [32]) provided a general existence result assuming that the consumer and product space are compact, by implementing a duality argument based on Galois connections. In this chapter, we explore existence using G-convex analysis, which was introduced in Section 2.2, but with less restriction on boundedness of the product domain and without assuming the generalized single-crossing condition. As a result of the lack of natural compactness, the proof of this result is quite different from that of Nöldeke-Samuelson [32]. It should be mentioned here that the existence results from this chapter, Chapter 4, and Nöldeke-Samuelson require no restrictions on the monopolist profit to take on a special form, which is a generalization from much of the literature.

In Section 2.2, we identified incentive compatibility with a G-convexity constraint. In this chapter, we will rewrite the maximization problem by converting the optimization variables from a product-price pair of mappings to a product-value pair. It can then be shown that the product-value pair converges under the G-convexity constraint. The existence result follows.

The remainder of this chapter is organized as follows. Section 3.2 states the mathematical model and assumptions. Section 3.3 reformulates the monopolist's problem and prepares some propositions for the next section. In Section 3.4, we state the existence theorem as well as the convergence proposition.

3.2 Model

Our model of the principal-agent problem is a bilevel optimization. After a monopolist publishes her price menu, each agent maximizes his utility through the purchase of at most one product. Knowing only the distribution of agent types, the monopolist maximizes aggregate profits based on agents' choices, which are based on the price menus.

Suppose the agents' preferences are given by some parametrized utility function $G(x, y, z)$, where x is a M-dimensional vector of consumer characteristics, y is a N-dimensional vector of attributes of each product, and z represents the price of each product. Denote by X the space of agent types, by Y the space of products, by $cl(Y)$ the closure of Y, by Z the space of prices, and by $cl(Z)$ the closure of Z. In this chapter only, we use the letters M and N as dimensions of the spaces of agents and products. Other chapters adopt m and n as dimensions of the corresponding spaces.

The monopolist sells indivisible products to agents, i.e., she will sell neither a part/percentage of one product nor a product with some probability. Each agent buys at most one unit of product. We assume no competition, cooperation, or communication between agents. For any given price menu $p : cl(Y) \to cl(Z)$, an agent $x \in X$ knows his utility $G(x, y, p(y))$ for purchasing each product y at price $p(y)$. It follows that each agent solves the following maximization problem

$$u(x) := \max_{y \in cl(Y)} G(x, y, p(y)), \tag{3.2.1}$$

where $u(x)$ represents the maximal utility agent x can obtain, and $u : X \longrightarrow \mathbf{R}$ is also called the value function or indirect utility function. At this point, it is assumed that the maximum in (3.2.1) is attained for each agent x.

If agent x purchases product y at price $p(y)$, the monopolist would earn from this transaction a profit of $\pi(x, y, p(y))$. For example, monopolist profit can take the form $\pi(x, y, p(y)) = p(y) - c(y)$, where $c(y)$ is a variable manufacturing cost function. Summing over all agents in the distribution $d\mu(x)$, the monopolist's total profit is characterized by

$$\Pi(p, y) := \int_X \pi(x, y(x), p(y(x))) d\mu(x), \tag{3.2.2}$$

which depends on her price menu $p : cl(Y) \to cl(Z)$ and agents' choices $y : X \to cl(Y)$.[1]

Since the monopolist only observes the overall distribution of agent attributes and is unable to distinguish individual agent characteristics, the monopolist takes into account the following incentive-compatibility constraint when determining product-price pairs $(y, p(y))$, which ensures that no agent has the incentive to pretend to be another agent type.

In addition, we adopt a participation constraint in order to rule out the possibility of the monopolist charging exorbitant prices and the agents still having to make transactions despite this: each agent $x \in X$ will refuse to participate to the market if the maximum utility he can obtain is less than his reservation value $u_\emptyset(x)$, where the function $u_\emptyset : X \to \mathbf{R}$ is given in the form $u_\emptyset(x) := G(x, y_\emptyset, z_\emptyset)$, for some $(y_\emptyset, z_\emptyset) \in cl(Y \times Z)$, where $y_\emptyset$ represents the outside option, whose price equals to some fixed value $z_\emptyset \in \mathbf{R}$ beyond the monopolist's control.

[1] It is worth mentioning that in some literature, the monopolist's objective is to design a product line $\tilde{Y}$ (i.e. a subset of $cl(Y)$) and a price menu $\tilde{p} : \tilde{Y} \to \mathbf{R}$ that jointly maximize overall monopolist profit. Then, given $\tilde{Y}$ and $\tilde{p}$, an agent of type x chooses the product $y(x)$ that solves

$$\max_{y \in \tilde{Y}} G(x, y, \tilde{p}(y)) := u(x).$$

Allowing the price to take value $\bar{z}$ (which may be $+\infty$), and assuming Assumption 1 below, the effect of designing a product line $\tilde{Y}$ and price menu $\tilde{p} : \tilde{Y} \to \mathbf{R}$ is equivalent to that of designing a price menu $p : cl(Y) \to (-\infty, +\infty]$, which equals $\tilde{p}$ on $\tilde{Y}$ and maps $cl(Y) \setminus \tilde{Y}$ to $\bar{z}$, such that no agents choose to purchase any product from $cl(Y) \setminus \tilde{Y}$, which is less attractive than the outside option $y_\emptyset$ according to Assumption 1. In this paper, we use the latter as the monopolist's objective.

For any given price menu $p : cl(Y) \to (-\infty, +\infty]$, one can construct a mapping $y : X \to cl(Y)$ such that each $y(x)$ solves the maximization problem in (3.2.1). But such mapping is not unique, for some fixed price menu, without the single-crossing type assumptions. Therefore, we adopt in (3.2.2) the total profit as a functional of both price menu p and its corresponding mapping y.

For monopolist profit, some literature assumes $\pi(x, y_\emptyset, z_\emptyset) \geq 0$ for all $x \in X$ to ensure that the outside option is harmless to the monopolist. Here, it is not necessary to adopt such an assumption for the sake of generality.

The monopolist's problem can be described as follows:

$$(P_1)\begin{cases} \sup \Pi(p, y) = \int_X \pi(x, y(x), p(y(x)))\, d\mu(x) \\ s.t.\ (y, p(y)) \text{ is incentive compatible;} \\ s.t.\ G(x, y(x), p(y(x))) \geq u_\emptyset(x) \text{ for all } x \in X; \\ s.t.\ p \text{ is lower semicontinuous.} \end{cases} \tag{3.2.3}$$

We assume that p is lower semicontinuous, without which the maximum in (3.2.1) may not be attained.

The following assumptions are made. We use $C^0(X)$ to denote the space of all continuous functions on X, and use $C^1(X)$ to denote the space of all differentiable functions on X whose derivative is continuous. Note that, even in the 1 dimensional case, we assume no single-crossing type condition.

Assumption 1. *Agents' utility $G \in C^1(cl(X \times Y \times Z))$, where the space of agents X is a bounded open convex subset in $\mathbf{R}^M$ with C^1 boundary, the space of products $Y \subset \mathbf{R}^N$, and range of prices $Z = (\underline{z}, \bar{z})$ with $-\infty < \underline{z} < \bar{z} \leq +\infty$. Assume $G(x, y, \bar{z}) := \lim_{z \longrightarrow \bar{z}} G(x, y, z) \leq G(x, y_\emptyset, z_\emptyset)$, for all $(x, y) \in X \times cl(Y)$; and assume this inequality is strict when $\bar{z} = +\infty$.*

Here we do not necessarily assume X, Y, and Z are compact spaces; in particular, Y and Z are potentially unbounded (i.e. we do not set *a priori* bounds for product attributes or an *a priori* upper bound for price). However, we do specify a lower bound for the price range, since the monopolist has no incentive to set price close to negative infinity.

Assumption 2. *$G(x, y, z)$ is strictly decreasing in z for each $(x, y) \in cl(X \times Y)$.*

This assumption says that the higher the price paid to the monopolist, the lower the utility that will be left for the agent, for any given product.

Assumption 3. *G is coordinate-monotone in x. That is, for each $(y, z) \in cl(Y \times Z)$, and for all $(\alpha, \beta) \in X^2$, if $\alpha_i \geq \beta_i$ for all $i = 1, 2, ..., M$, then $G(\alpha, y, z) \geq G(\beta, y, z)$.*

In Assumption 3, we assume that agent utility increases along each consumer attribute coordinate.

In the following, we use $D_x G(x, y, z) := (\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \dots, \frac{\partial G}{\partial x_M})(x, y, z)$ to denote derivative with respect to x. For any vector in $\mathbf{R}^M$ or $\mathbf{R}^N$, we use $||\cdot||$ and $||\cdot||_\alpha$ to denote its Euclidean 2-norm and α-norm ($\alpha \geq 1$), respectively. For example, for $x \in \mathbf{R}^M$, we have $||x|| = \sqrt{\sum_{i=1}^M x_i^2}$ and $||x||_\alpha = (\sum_{i=1}^M |x_i|^\alpha)^{\frac{1}{\alpha}}$. We use H defined in Section 2.2 as the inverse of G with respect to the third variable, i.e., for each $(x, y) \in X \times cl(Y)$, $H(x, y, \cdot) = G^{-1}(x, y, \cdot)$. Here, $H(x, y, u)$ represents the price paid by agent x for product y when receiving value u.

In Rochet-Choné's model, $H(x, y, u) = x \cdot y - u$ and $\pi(x, y, z) = z - C(y)$, for some superlinear cost function C. In this case, $\pi(x, y, H(x, y, u)) = x \cdot y - u - C(y)$. Since C is superlinear and the space X is bounded, it is reasonable to assume the following:

Assumption 4. *$\pi(x, y, H(x, y, u))$ is (super-)linearly decreasing in y. That is, there exist $\alpha \geq 1$, $a_1, a_2 > 0$ and $b \in \mathbf{R}$, such that $\pi(x, y, H(x, y, u)) \leq -a_1||y||_\alpha^\alpha - a_2 u + b$ for all $(x, y, u) \in \{(x, y, G(x, y, z))|x \in X, y \in Y, z \in \mathbf{R}\}$; or equivalently, $\pi(x, y, z) + a_2 G(x, y, z) \leq -a_1||y||_\alpha^\alpha + b$ for all $(x, y, z) \in X \times cl(Y) \times \mathbf{R}$.*

As shown in the alternative formulation, Assumption 4 requires the existence of some weighted surplus which is superlinearly decreasing with respect to the product. In the case where Y is bounded, Assumption 4 is equivalent to the existence of some weighted surplus bounded from above.

Assumptions 5 - 7 are some technical assumptions on $D_x G$, which are automatically satisfied for X, Y, Z being compact.

Assumption 5. *$D_x G(x, y, z)$ is Lipschitz with respect to x, uniformly in (y, z), meaning there exists $k \in \mathbf{R}$ such that $||D_x G(x, y, z) - D_x G(x', y, z)|| \leq k||x - x'||$ for all $(x, x', y, z) \in X^2 \times cl(Y) \times cl(Z)$.*

Assumption 6. *$||D_x G(x, y, z)||_1$ increases sub-linearly with respect to y. More precisely, there exist $\beta \in (0, \alpha]$, $c > 0$, and $d \in \mathbf{R}$, such that $||D_x G(x, y, z)||_1 \leq c||y||_\beta^\beta + d$ for all $(x, y, z) \in X \times cl(Y) \times cl(Z)$.*

Assumption 7. *Coercivity of 1-norm of $(D_x G)$. For all $s > 0$, there exists $r > 0$, such that*

$$\sum_{i=1}^{M} |D_{x_i} G(x, y, z)| \geq s$$

for all $(x, y, z) \in X \times cl(Y) \times cl(Z)$, whenever $||y|| \geq r$.

Allowing Assumption 3, the derivatives $D_{x_i} G$ are always nonnegative; therefore, we no longer need to take absolute values of $D_{x_i} G$ in the inequality of Assumption 7. And then Assumption 7 says that the marginal utility of agents who select the same product y will increase to infinity as $||y||$ approaches infinity, uniformly for all agents and prices. For instance, when $M = N$, utility $G(x, y, z) = \sum_{i=1}^{M} x_i y_i^2 - f(z)$ satisfies Assumption 7, because $\sum_{i=1}^{M} |D_{x_i} G(x, y, z)| = \sum_{i=1}^{M} D_{x_i} G(x, y, z) = \sum_{i=1}^{M} y_i^2 \to +\infty$ as $||y|| \to +\infty$. In addition, this G also satisfies all the other assumptions.

In general, if Y is bounded, any G in the form of $G(x, y, z) = b(x, y) - f(y, z)$, with $b \in C^1(cl(X \times Y))$ and $f \in C^0(cl(Y \times Z))$, satisfies Assumption 5 - 7.

Assumptions 8 states constraints on the continuity of principal's profit function π, integrability of participation constraint $u_\emptyset$, and absolute continuity of the measure μ with respect to the Lebesgue measure.

Assumption 8. *The profit function π is continuous on $cl(X \times Y \times Z)$. The participation constraint $u_\emptyset$ is integrable with respect to $d\mu$, where the measure $d\mu$ is equivalent to the Lebesgue measure restricted on X.*

For $\alpha \geq 1$, denote $L^\alpha(X)$ as the space of functions for which the α-th power of the absolute value is Lebesgue integrable with respect to the measure $d\mu$. That is, a function $f : X \longrightarrow \mathbf{R}$ is in $L^\alpha(X)$ if and only if $\int_X |f|^\alpha d\mu < +\infty$. For instance, Assumption 8 implies $u_\emptyset \in L^1(X)$.

3.3 Reformulation of the monopolist's problem

The purpose of this section is to fix terminology and prepare the preliminaries for the main results in the next section. We also rewrite the monopolist's problem in Proposition 3.3.4, which is an equivalent form of (3.2.3).

We introduce implementability here, which is closely related to incentive-compatibility and can also be exhibited by G-convexity and G-subdifferential.

Definition 3.3.1 (implementability)**.** A function $y : X \to cl(Y)$ is called *implementable* if and only if there exists a function $z : X \to \mathbf{R}$ such that the pair (y, z) is incentive compatible.

Remark 3.3.2. Allowing Assumption 2, a function y is implementable if and only if there exists a price menu $p : cl(Y) \to \mathbf{R}$ such that the pair $(y, p(y))$ is incentive compatible.

Proof. One direction is easier: given p and y, define $z(\cdot) := p(y(\cdot))$. Then the conclusion follows directly.

Given an incentive-compatible pair $(y, z) : X \to cl(Y) \times \mathbf{R}$, we need to construct a price menu $p : cl(Y) \to \mathbf{R}$. If $y = y(x)$ for some $x \in X$, define $p(y) := z(x)$; for any other $y \in cl(Y)$, define $p(y) := \bar{z}$.

We first show p is well-defined. Suppose $y(x) = y(x')$ with $x \neq x'$, from incentive compatibility of (p, y), we have $G(x, y(x), z(x)) \geq G(x, y(x'), z(x')) = G(x, y(x), z(x'))$. Since G is strictly decreasing on its third variable, the above inequality implies $z(x) \leq z(x')$. Similarly, one has $z(x) \geq z(x')$. Therefore, $z(x) = z(x')$ and thus p is well-defined.

The incentive compatibility of $(y, p(y))$ follows from that of (y, z) and definition of p. □

As a corollary of Proposition 2.2.4, implementable functions can be characterized as G-subdifferential of G-convex functions.

Corollary 3.3.3. *Given Assumption 2, a function $y : X \to cl(Y)$ is implementable if and only if there exists a G-convex function $u(\cdot)$ such that $y(x) \in \partial^G u(x)$ for each $x \in X$.*

Proof. One direction is immediately derived from the definition of implementability and Proposition 2.2.4.

Suppose there exists some convex function u such that $y(x) \in \partial^G u(x)$ for each $x \in X$. Define $z(\cdot) := H(\cdot, y(\cdot), u(\cdot))$, then $u(x) = G(x, y(x), z(x))$. Proposition 2.2.4 implies (y, z) is incentive compatible, and thus y is implementable. □

When parameterization of preferences is linear in agent types and price, Corollary 3.3.3 says that a function is implementable if and only if it is monotone increasing. In general quasilinear cases, this coincides with Proposition 1 of Carlier [5].

From the original monopolist's problem (3.2.3), we replace product-price pair (p, y) by the value-product pair (u, y), using $u(\cdot) = G(\cdot, y(\cdot), p(y(\cdot)))$. Combining this with Proposition 2.2.4, the incentive-compatibility constraint $(y, p(y))$ is equivalent to G-convexity of $u(\cdot)$ and $y(x) \in \partial^G u(x)$ for all $x \in X$. Therefore, one can rewrite the monopolist's problem as follows.

Proposition 3.3.4. *Given Assumptions 1 and 2, the monopolist's problem (P_1) is equivalent to*

$$(P_2)\begin{cases}\sup \tilde{\Pi}(u,y) := \int_X \pi(x,y(x),H(x,y(x),u(x)))d\mu(x)\\ s.t.\ u \text{ is } G\text{-convex };\\ s.t.\ y(x)\in\partial^G u(x) \text{ and } u(x)\geq u_\emptyset(x) \text{ for all } x\in X.\end{cases} \tag{3.3.1}$$

Proof. We need to prove both directions for equivalence of (P_1) and (P_2).

1. For any incentive-compatible pair $(y, p(y))$, define $u(\cdot) := G(\cdot, y(\cdot), p(y(\cdot)))$. Then by Proposition 2.2.4, we have $u(\cdot)$ is G-convex and $y(x) \in \partial^G u(x)$ for all $x \in X$. From the participation constraint, $G(x, y(x), p(y(x))) \geq u_\emptyset(x)$ for all $x \in X$. This implies $u(x) \geq u_\emptyset(x)$ for all $x \in X$. Besides, two integrands are equal: $\pi(x, y(x), p(y(x))) = \pi(x, y(x), H(x, y(x), u(x)))$. Therefore, $(P_1) \leq (P_2)$.

2. On the other hand, assume $u(\cdot)$ is G-convex, $y(x) \in \partial^G u(x)$ and $u(x) \geq u_\emptyset(x)$ for all $x \in X$. From Corollary 3.3.3 and Remark 3.3.2, we know y is implementable and there exists a price menu $p : cl(Y) \to \mathbf{R}$, such that the pair $(y, p(y))$ is incentive compatible, where $p(y) = H(x, y(x), u(x))$ for $y = y(x) \in y(X) := \{y(x) \in cl(Y) | x \in X\}$; $p(y) = \bar{z}$ for other $y \in cl(Y)$. Firstly, the mapping p is well-defined, using the same argument as that in Remark 3.3.2. Secondly, the participation constraint holds since $G(x, y(x), p(y(x))) = u(x) \geq u_\emptyset(x)$ for all $x \in X$.

Thirdly, let us show this price menu p is lower semicontinuous. Let $\tilde{p}$ be the restriction of p to $y(X)$. Suppose that $\{y_k\} \subset y(X)$ converges $y_0 \in y(X)$ with $y_k = y(x_k)$ and $y_0 = y(x_0)$, satisfying $\lim_{k\to\infty} \tilde{p}(y_k) = \liminf_{y\to y_0} \tilde{p}(y)$. Let $z_k := \tilde{p}(y_k)$ and $z_\infty := \lim_{k\to\infty} z_k$. To prove lower semicontinuity of $\tilde{p}$, we need to show $\tilde{p}(y_0) \leq z_\infty$. Since $y_k \in \partial^G u(x_k)$, we have $u(x) \geq G(x, y_k, H(x_k, y_k, u(x_k))) = G(x, y_k, z_k)$. Taking $k \to \infty$, we have $u(x) \geq G(x, y_0, z_\infty)$. This implies $G(x_0, y_0, \tilde{p}(y_0)) = u(x_0) \geq G(x_0, y_0, z_\infty)$. By Assumption 2, we know $\tilde{p}(y_0) \leq z_\infty$. Thus $\tilde{p}$ is lower semicontinuous. Since p is an extension of $\tilde{p}$ from $y(X)$ to $cl(Y)$ as its lower semicontinuous hull, satisfying $v(y) = \bar{z}$ for all $y \in cl(Y) \setminus y(X)$, we know p is also lower semicontinuous.

Lastly, two integrands are equal: $\pi(x, y(x), p(y(x))) = \pi(x, y(x), H(x, y(x), u(x)))$. Therefore, $(P_1) \geq (P_2)$. □

In the next section, we will show the existence result of the rewritten monopolist's problem (P_2) given in (3.3.1). For the preparation of the main result, we introduce the following lemma and propositions.

Proposition 3.3.5 shows that the inverse function of G is also continuous, because G is continuous and monotonic on the price variable.

Proposition 3.3.5. *Given Assumption 1 and Assumption 2, the function H is continuous.*

Proof. (Proof by contradiction). Suppose H is not continuous, then there exists a sequence $(x_n, y_n, z_n) \subset cl(X \times Y \times Z)$ converging to (x, y, z) and $\varepsilon > 0$ such that $|H(x_n, y_n, z_n) - H(x, y, z)| > \varepsilon$ for all $n \in \mathbf{N}$. Without loss of generality, we assume $H(x_n, y_n, z_n) - H(x, y, z) > \varepsilon$ for all $n \in \mathbf{N}$. Therefore, we have $H(x_n, y_n, z_n) > H(x, y, z) + \varepsilon$. By Assumption 2, this implies $z_n < G(x_n, y_n, H(x, y, z) + \varepsilon)$ for all $n \in \mathbf{N}$. Taking limit $n \to \infty$ at both sides, since G is continuous from Assumption 1, we have $z \leq G(x, y, H(x, y, z) + \varepsilon)$. This implies $H(x, y, z) \geq H(x, y, z) + \varepsilon$, a contradiction. □

Given coordinate monotonicity of G in the first variable, one can show that all the G-convex functions are nondecreasing. Therefore, the value functions are also monotonic with respect to the agents' attributes.

Proposition 3.3.6. *Given Assumption 3, G-convex functions are nondecreasing in coordinates.*

Proof of Proposition 3.3.6. Let u be any G-convex function, and let α, β be any two agent types in X with $\alpha \geq \beta$. By G-convexity of u, for this β, there exist $y \in cl(Y)$ and $z \in cl(Z)$, such that $u(\beta) = G(\beta, y, z)$ and $u(x) \geq G(x, y, z)$, for any $x \in X$. Since $\alpha \geq \beta$, by Assumption 3, we have $G(\alpha, y, z) \geq G(\beta, y, z)$. Combining with $u(\alpha) \geq G(\alpha, y, z)$ and $u(\beta) = G(\beta, y, z)$, one has $u(\alpha) \geq u(\beta)$. Thus, u is nondecreasing. □

Proposition 3.3.7 presents that uniform boundedness of the agents' value functions on some compact subset implies uniform boundedness of the corresponding agents' choices of their favorite products.

Proposition 3.3.7. *Given Assumptions 1, 2, 3, 7, and let $u(\cdot)$ be a G-convex function on X, ω be a compact subset of X, $\delta > 0, R > 0$, satisfying $\omega + \delta\overline{B(0,1)} \subset X$ and $|u(x)| \leq R$ for all $x \in \omega + \delta\overline{B(0,1)}$ (here, $\overline{B(0,1)}$ denotes the closed unit ball of $\mathbf{R}^M$). Then, there exists $T = T(\omega, \delta, R) > 0$, such that $||y|| \leq T$ for any $x \in \omega$ and any $y \in \partial^G u(x)$.*

Proof. (Proof by contradiction).

By Assumption 3 and Assumption 7, for $s = \frac{4R\sqrt{M}}{\delta}$, there exists $r > 0$, such that for any $(x, y, z) \in X \times cl(Y) \times cl(Z)$, whenever $||y|| \geq r$, we have $\sum\limits_{i=1}^{M} D_{x_i} G(x, y, z) \geq \frac{4R\sqrt{M}}{\delta}$.

Assume the boundedness conclusion of this proposition is not true. Then for this r, there exist $x_0 \in \omega$ and $y_0 \in \partial^G u(x_0)$, such that $||y_0|| \geq r$. Thus,

$$\sum_{i=1}^{M} D_{x_i} G(x, y_0, z) \geq \frac{4R\sqrt{M}}{\delta}, \qquad \text{for all } x \in X, z \in \mathbf{R}. \tag{3.3.2}$$

Since $y_0 \in \partial^G u(x_0)$, by definition of G-subdifferential, we have $u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0)))$, for any $x \in X$. Take $x = x_0 + \delta x_{-1}$, where $x_{-1} := (\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}, \cdots, \frac{1}{\sqrt{M}})$ is a unit vector in $\mathbf{R}^M$ with each coordinate equal to $\frac{1}{\sqrt{M}}$. Then

$$u(x_0 + \delta x_{-1}) \geq G(x_0 + \delta x_{-1}, y_0, H(x_0, y_0, u(x_0))). \tag{3.3.3}$$

For any $x \in \omega + \delta\overline{B(0,1)}$, from conditions in the proposition, we have $||u(x)|| \leq R$. Therefore,

$$\begin{aligned}
2R &\geq |u(x_0 + \delta x_{-1})| + |u(x_0)| \\
&\geq |u(x_0 + \delta x_{-1}) - u(x_0)| && \text{(By the triangle inequality)} \\
&\geq u(x_0 + \delta x_{-1}) - u(x_0) \\
&\geq G(x_0 + \delta x_{-1}, y_0, H(x_0, y_0, u(x_0))) && \text{(By inequality (3.3.3)) and (by definition} \\
&\quad - G(x_0, y_0, H(x_0, y_0, u(x_0))) && \text{of } H,\ u(x_0) = G(x_0, y_0, H(x_0, y_0, u(x_0)))) \\
&= \int_0^1 \delta \langle x_{-1}, D_x G(x_0 + t\delta x_{-1}, y_0, H(x_0, y_0, u(x_0))) \rangle dt && \text{(By the fundamental theorem of Calculus)} \\
&= \frac{\delta}{\sqrt{M}} \int_0^1 \sum_{i=1}^{M} D_{x_i} G(x_0 + t\delta x_{-1}, y_0, H(x_0, y_0, u(x_0))) dt \\
&\geq \frac{\delta}{\sqrt{M}} \int_0^1 \frac{4R\sqrt{M}}{\delta} dt && \text{(By inequality (3.3.2))}
\end{aligned}$$

$$= \frac{\delta}{\sqrt{M}} \cdot \frac{4R\sqrt{M}}{\delta}$$
$$= 4R,$$

a contradiction. Thus, our assumption is wrong. Therefore, there exists $T > 0$, such that for any $x \in \omega$, $y \in \partial^G u(x)$, one has $||y|| \leq T$. In addition, here $T = T(\omega, \delta, R)$ is independent of u. In fact, from the above argument, we can see that $T \leq r$, which does not depend on u. □

The above two propositions will also be employed in the proof of Proposition 3.4.3.

3.4 Main result

In this section, we state the existence theorem, the proof of which is provided at the end of this section.

Theorem 3.4.1 (Existence)**.** *Under Assumptions 1 - 8, the monopolist's problem* (P_2) *admits at least one solution.*

Technically, in order to demonstrate existence, we start from a sequence of value-product pairs, whose total profits have a limit that is equal to the supremum of (P_2). Then we need to show that this sequence converges, up to a subsequence, to a pair of limit mappings. Then we show this limit value-product pair satisfies the constraints of (P_2), and its corresponding total payoff is no worse than those of any other admissible pairs.

In the following, the notation $\omega \subset\subset X$ represents the closure of ω is also included in X.

Lemma 3.4.2 provides convergence results of a sequence of convex functions, which are uniformly bounded in Sobolev spaces on open convex subsets. We state this classical result without proof, which can be found in Carlier [5].

Lemma 3.4.2. *Let* $\{u_n\}$ *be a sequence of convex functions on* X *such that, for every open convex set* $\omega \subset\subset X$*, the following holds:*

$$\sup_n ||u_n||_{W^{1,1}(\omega)} < +\infty.$$

Then there exists a function u^* *which is convex in* X*, a measurable subset* A *of* X *and a subsequence again labeled* $\{u_n\}$ *such that*

1. $\{u_n\}$ *converges to* u^* *uniformly on compact subsets of* X*;*

2. $\{\nabla u_n\}$ *converges to* ∇u^* *pointwise in* A *and* $dim_H(X \setminus A) \leq M - 1$*, where* $dim_H(X \setminus A)$ *is the Hausdorff dimension of* $X \setminus A$*.*

We extend the above convergence result to G-convex functions in the following proposition, which is required in the proof of the Existence Theorem, as it extracts a limit function from a converging sequence of value functions.

Proposition 3.4.3. *Assume Assumptions 1, 2, 3, 5, 7, and let* $\{u_n\}$ *be a sequence of* G*-convex functions in* X *such that for every open convex set* $\omega \subset\subset X$*, the following holds:*

$$\sup_n ||u_n||_{W^{1,1}(\omega)} < +\infty.$$

Then there exists a function u^* which is G-convex in X, a measurable subset A of X, and a subsequence again labeled $\{u_n\}$ such that
1. $\{u_n\}$ converges to u^* uniformly on compact subsets of X;
2. $\{\nabla u_n\}$ converges to ∇u^* pointwise in A and $dim_H(X \setminus A) \leq M - 1$.

Proof. In this proof, we will show that the sequence of G-convex functions is convergent by applying results from Lemma 3.4.2, then prove that the limit function is also G-convex. Assume $\{u_n\}$ is a sequence of G-convex functions in X such that for every open convex set $\omega \subset\subset X$, the following holds:

$$\sup_n ||u_n||_{W^{1,1}(\omega)} < +\infty.$$

Step 1: By Assumption 5, there exists $k > 0$, such that for any $(x, x') \in X^2$, $y \in cl(Y)$ and $z \in cl(Z)$, one has $||D_xG(x, y, z) - D_xG(x', y, z)|| \leq k||x - x'||$. Denote $G_\lambda(x, y, z) := G(x, y, z) + \lambda||x||^2$, where $\lambda \geq \frac{1}{2}\,\mathrm{Lip}(D_xG)$, with $\mathrm{Lip}(D_xG) := \sup\limits_{\{(x,x',y,z)\in X\times X\times cl(Y)\times cl(Z):\ x\neq x'\}} \frac{||D_xG(x,y,z)-D_xG(x',y,z)||}{||x-x'||}$.

Then, for any $(x, x') \in X^2$, by Cauchy–Schwarz inequality, one has

$$\begin{aligned}
&\langle D_xG_\lambda(x, y, z) - D_xG_\lambda(x', y, z), x - x'\rangle \\
&= \langle D_xG(x, y, z) - D_xG(x', y, z), x - x'\rangle + 2\lambda||x - x'||^2 && \text{(By definition of } G_\lambda(x, y, z))\\
&\geq -||D_xG(x, y, z) - D_xG(x', y, z)||||x - x'|| + 2\lambda||x - x'||^2 && \text{(By Cauchy–Schwarz inequality)}\\
&\geq [2\lambda - \mathrm{Lip}(D_xG)]||x - x'||^2 && \text{(By definition of } \mathrm{Lip}(D_xG))\\
&\geq 0.
\end{aligned}$$

Thus, $G_\lambda(\cdot, y, z)$ is a convex function on X, for any fixed $(y, z) \in cl(Y) \times cl(Z)$.

Step 2: Since u_n is G-convex, by Lemma 2.2.3, we know

$$u_n(x) = \max_{x'\in X, y\in\partial^G u_n(x')} G(x, y, H(x', y, u_n(x'))).$$

Define $v_n(x) := u_n(x) + \lambda||x||^2$. Then

$$\begin{aligned}
v_n(x) &= \max_{x'\in X, y\in\partial^G u_n(x')} G(x, y, H(x', y, u_n(x'))) + \lambda||x||^2\\
&= \max_{x'\in X, y\in\partial^G u_n(x')} (G(x, y, H(x', y, u_n(x'))) + \lambda||x||^2)\\
&= \max_{x'\in X, y\in\partial^G u_n(x')} G_\lambda(x, y, H(x', y, u_n(x'))).
\end{aligned}$$

Since $G_\lambda(\cdot, y, H(x', y, u_n(x')))$ is convex for each (x', y), we have $v_n(x)$, as supremum of convex functions, is also convex, for all $n \in \mathbf{N}$.

Step 3: Since $v_n := u_n + \lambda||x||^2$ and $\sup\limits_n ||u_n||_{W^{1,1}(\omega)} < +\infty$, one has $\sup\limits_n ||v_n||_{W^{1,1}(\omega)} < +\infty$, for any $\omega \subset\subset X$. Hence $\{v_n\}$ satisfies all the assumptions of Lemma 3.4.2. So, by conclusion of Lemma 3.4.2, there exists a convex function v^* in X and a measurable set $A \subset X$, such that $dim(X \setminus A) \leq M - 1$ and up to a subsequence, $\{v_n\}$ converges to v^* uniformly on compact subset of X and (∇v_n) converges to ∇v^* pointwise in A.

Let $u^*(x) := v^*(x) - \lambda||x||^2$, then (u_n) converges to u^* uniformly on compact subset of X and (∇u_n)

converges to ∇u^* pointwise in A.

Step 4: Finally, let us prove that u^* is G-convex.

Define $\Gamma(x) := \cap_{i\geq 1}\overline{\cup_{n\geq i}\partial^G u_n(x)}$, for all $x \in X$.

Step 4.1. Claim: For any $x' \in X$, we have $\Gamma(x') \neq \emptyset$.

Proof of this Claim:

Step 4.1.1. Let us first show for any $\omega \subset\subset X$, $\sup_n ||u_n||_{L^\infty(\bar{\omega})} < +\infty$.

If not, then there exits a sequence $\{x_n\}_{n=1}^\infty \subset \bar{\omega}$, such that $\limsup_n |u_n(x_n)| = +\infty$.

Since $\bar{\omega}$ is compact, there exists $\bar{x} \in \bar{\omega}$, such that, up to a subsequence, $x_n \to \bar{x}$. Again up to a subsequence, we may assume that $u_n(x_n) \to +\infty$.

Since $\bar{x} \in \bar{\omega} \subset\subset X$, there exists $\delta > 0$, such that $\bar{x} + \delta x_{-1} \in X$, where $x_{-1} := (\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}, \cdots, \frac{1}{\sqrt{M}})$ is a unit vector in $\mathbf{R}^M$ with each coordinate equal to $\frac{1}{\sqrt{M}}$. For any $x > \bar{x} + \delta x_{-1}$, there exists n_0, such that for any $n > n_0$, we have $x > x_n$. By Proposition 3.3.6, u_n are nondecreasing, and thus

$$\int_{\{x\in X, x>\bar{x}+\delta x_{-1}\}} u_n(x)dx \geq m\{x \in X, x > \bar{x} + \delta x_{-1}\}u_n(x_n) \to +\infty, \tag{3.4.1}$$

where $m\{x \in X, x > \bar{x} + \delta x_{-1}\}$ denotes Lebesgue measure of $\{x \in X, x > \bar{x} + \delta x_{-1}\}$, which is positive.

Therefore, we have $||u_n||_{W^{1,1}(\omega')} \geq ||u_n||_{L^1(\omega')} \geq \int_{\omega'} u_n(x)dx \to +\infty$. This implies $\sup_n ||u_n||_{W^{1,1}(\omega')} = +\infty$.

On the other hand, denote $\omega' := \{x \in X|\ x > \bar{x} + \delta x_{-1}\}$, then $\omega' = X \cap \{x \in \mathbf{R}^M|\ x > \bar{x} + \delta x_{-1}\}$. Since both X and $\{x \in \mathbf{R}^M|\ x > \bar{x} + \delta x_{-1}\}$ are open and convex, we have ω' is also open and convex. Therefore, by assumption, we have $\sup_n ||u_n||_{W^{1,1}(\omega')} < +\infty$.

This is a contradiction, and thus for any $\omega \subset\subset X$, we have $\sup_n ||u_n||_{L^\infty(\bar{\omega})} < +\infty$.

Step 4.1.2. For any fixed $x' \in X$, there exists an open set $\omega \subset\subset X$ and $\delta > 0$, such that $x' \in \omega$ and $\omega + \delta\overline{B(0,1)} \subset\subset X$.

From Step 4.1.1, we know $\sup_n ||u_n||_{L^\infty(\omega+\delta\overline{B(0,1)})} < +\infty$. So there exists $R > 0$, such that for all $n \in \mathbf{N}$, we have $|u_n(x)| \leq R$, for all $x \in \omega + \delta\overline{B(0,1)}$. Since u_n are G-convex functions, by Proposition 3.3.7, there exists $T = T(\omega, \delta, R) > 0$, independent of n, such that $||y|| \leq T$, for any $y \in \partial^G u_n(x')$ and any $n \in \mathbf{N}$. Thus, there exists a sequence $\{y_n\}$, such that $y_n \in \partial^G u_n(x')$ and $||y_n|| \leq T$, for all $n \in \mathbf{N}$.

By compactness theorem for sequence $\{y_n\}$, there exists y', such that, up to a subsequence, $y_n \to y'$. Thus, we have $y' \in \overline{\cup_{n\geq i}\partial^G u_n(x')}$, for all $i \in \mathbf{N}$. It implies $y' \in \cap_{i\geq 1}\overline{\cup_{n\geq i}\partial^G u_n(x')} = \Gamma(x')$.

Therefore $\Gamma(x') \neq \emptyset$, for all $x' \in X$.

Step 4.2. Now for any fixed $x \in X$, and any $y \in \Gamma(x)$, by Cantor's diagonal argument, there exists $\{y_{n_k}\}_{k=1}^\infty$, such that $y_{n_k} \in \partial^G u_{n_k}(x)$ and $\lim_{k\to\infty} y_{n_k} = y$. For any $k \in \mathbf{N}$, by definition of G-subdifferentiability, $u_{n_k}(x') \geq G(x', y_{n_k}, H(x, y_{n_k}, u_{n_k}(x)))$, for any $x' \in X$. Take limit $k \to \infty$ at both sides, we get $u^*(x') \geq G(x', y, H(x, y, u^*(x)))$, for any $x' \in X$. Here we use the fact that both functions G and H are continuous by Assumption 1 and Proposition 3.3.5. Then by definition of G-subdifferentiability, the above inequality implies $y \in \partial^G u^*(x)$.

So $\partial^G u^*(x) \neq \emptyset$, for any $x \in X$, which means u^* is G-subdifferentiable everywhere. By Lemma 2.2.3, u^* is G-convex. □

Lastly, we show the proof of the main theorem.

Proof of the Existence Theorem. **Step 1:** Define $\Phi_u : x \longmapsto argmin_{\partial^G u(x)}\{-\pi(x,\cdot,H(x,\cdot,u(x)))\}$, then by Proposition 3.3.7, for any compact set $\omega \subset X$, one has $\cup_{x\in\omega}\partial^G u(x)$ is nonempty and compact, $\Phi_u(x)$ is nonempty and compact for all $x \in \omega$, and $\cup_{x\in\omega}\{(x,y)|y \in \Phi_u(x)\}$ is a Borel set. By the measurable selection theorem (cf. [8, Theorem 1.2, Chapter VIII]), there exists a measurable mapping $y : \omega \to Y$ such that for almost all x, $y(x) \in \Phi_u(x)$. Let $\{\omega_n\}_{n=1}^{\infty}$ denote a sequence of compact sets such that $\omega_1 \subset \omega_2 \subset ... \subset \omega_n \subset ... \subset X$ with $\cup_n \omega_n = X$. On each ω_n, there exists a measurable selection map $y^n : \omega_n \to Y$. Define $\bar{y} : X \to Y$, such that $\bar{y} = y^1$ on ω_1 and $\bar{y} = y^n$ on $\omega_n \setminus \omega_{n-1}$ for $n \geq 2$. Then $\bar{y}$ is a measurable selection of Φ_u, i.e., $\bar{y}$ is measurable and $\bar{y}(x) \in \Phi_u(x)$ for almost every x.

Let $\{(u_n, y_n)\}$ be a maximizing sequence of (P_2), where maps $u_n : X \to \mathbf{R}$ and $y_n : X \to cl(Y)$, for all $n \in \mathbf{N}$. Without loss of generality, we may assume that for all n, $y_n(\cdot)$ is measurable and $y_n(x) \in \Phi_{u_n}(x)$, for each $x \in X$. Starting from $\{(u_n, y_n)\}$, we would find an value-product pair (u^*, y^*) satisfying all the constraints in (3.3.1), and show that it is actually a maximizer.

Step 2: From Assumption 4, there exist $\alpha \geq 1$, $a_1, a_2 > 0$ and $b \in \mathbf{R}$, such that for each $x \in X$ and $n \in \mathbf{N}$,

$$\begin{aligned} a_1||y_n(x)||_\alpha^\alpha \leq & -\pi(x, y_n(x), H(x, y_n(x), u_n(x))) - a_2 u_n(x) + b \\ \leq & -\pi(x, y_n(x), H(x, y_n(x), u_n(x))) - a_2 u_\emptyset(x) + b, \end{aligned}$$

where the second inequality comes from $u_n \geq u_\emptyset$. Together with Assumption 8, this implies $\{y_n\}$ is bounded in $L^\alpha(X)$.

By participation constraint and Assumption 4, we know

$$u_\emptyset(x) \leq u_n(x) = G(x, y_n(x), H(x, y_n(x), u_n(x))) \leq \frac{1}{a_2}(b - \pi(x, y_n(x), H(x, y_n(x), u_n(x)))).$$

Together with Assumption 8, we know $\{u_n\}$ is bounded in $L^1(X)$.

By G-subdifferentiability, $Du_n(x) = D_x G(x, y_n(x), H(x, y_n(x), u_n(x)))$. By Assumption 6, we have $||Du_n||_1 \leq c||y_n||_\beta^\beta + d \leq c(N + ||y_n||_\alpha^\alpha) + d$. The last inequality holds because $\beta \in (0, \alpha]$. Because X is bounded and $\{y_n\}$ is bounded in $L^\alpha(X)$, we know $\{Du_n\}$ is bounded in $L^1(X)$.

Since both $\{u_n\}$ and $\{Du_n\}$ are bounded in $L^1(X)$, one has $\{u_n\}$ is bounded in $W^{1,1}(X)$. By Proposition 3.4.3, there exists a G-convex function u^* on X, such that, up to a subsequence, $\{u_n\}$ converges to u^* in L^1 and uniformly on compact subset of X, and ∇u_n converges to ∇u^* almost everywhere.

Step 3: Denote $y^*(x)$ as a measurable selection of Φ_{u^*}. Let us show (u^*, y^*) is a maximizer of the principal's program (P_2).

Step 3.1: By Assumption 4, for all x, $y_n(x)$ and $u_n(x)$, one has

$$\begin{aligned} & -\pi(x, y_n(x), H(x, y_n(x), u_n(x))) \\ \geq & \, a_2 G(x, y_n(x), H(x, y_n(x), u_n(x))) - b \\ = & \, a_2 u_n(x) - b \\ \geq & \, a_2 u_\emptyset(x) - b. \end{aligned}$$

By Assumption 8, $u_\emptyset$ is measurable, thus one can apply Fatou's Lemma and get

$$\begin{aligned} \sup \tilde{\Pi}(u,y) &= \limsup_n \tilde{\Pi}(u_n, y_n) \\ &= -\liminf_n \int_X -\pi(x, y_n(x), H(x, y_n(x), u_n(x)))\ d\mu(x) \\ &\leq -\int_X \liminf_n -\pi(x, y_n(x), H(x, y_n(x), u_n(x)))\ d\mu(x). \end{aligned} \tag{3.4.2}$$

Let $\gamma(x) := \liminf_n -\pi(x, y_n(x), H(x, y_n(x), u_n(x)))$. For each $x \in X$, by extracting a subsequence of $\{y_n\}$, which is denoted as $\{y_{n_x}\}$, we assume $\gamma(x) = \lim_{n_x} -\pi(x, y_{n_x}(x), H(x, y_{n_x}(x), u_{n_x}(x)))$.

Step 3.2: For any fixed $x \in X$, since u_{n_x} are G-convex functions and $\{u_{n_x}\}$ is bounded in $L^1(X)$, by Proposition 3.3.6, it is also bounded in $L^\infty_{loc}(X)$. Then by Proposition 3.3.7, $\{y_{n_x}\}$ is also bounded in $L^\infty_{loc}(X)$. Thus there exists a subsequence of $\{y_{n_x}(x)\}$, again denoted as $\{y_{n_x}(x)\}$, that converges. Denote $\tilde{y}$ a mapping on X such that $y_{n_x}(x) \to \tilde{y}(x)$.

Since π and H are continuous, we have $\gamma(x) = -\pi(x, \tilde{y}(x), H(x, \tilde{y}(x), u^*(x)))$.

For each fixed $x \in X$, since u_{n_x} are G-convex and $y_{n_x}(x) \in \partial^G u_{n_x}(x)$, for any $x' \in X$, we have

$$u_{n_x}(x') \geq G(x', y_{n_x}(x), H(x, y_{n_x}(x), u_{n_x}(x))).$$

Take limit $n_x \to +\infty$ at both sides, we get $u^*(x') \geq G(x', \tilde{y}(x), H(x, \tilde{y}(x), u^*(x)))$, for any $x' \in X$. By definition of G-subdifferentiability, we have $\tilde{y}(x) \in \partial^G u^*(x)$.

Step 3.3: By definition of y^*, one has

$$-\pi(x, y^*(x), H(x, y^*(x), u^*(x))) \leq -\pi(x, \tilde{y}(x), H(x, \tilde{y}(x), u^*(x))) = \gamma(x).$$

So, together with (3.4.2), we know

$$\sup \tilde{\Pi}(u,y) \leq -\int_X \gamma(x) d\mu(x) \leq -\int_X -\pi(x, y^*(x), H(x, y^*(x), u^*(x))) d\mu(x) = \tilde{\Pi}(u^*, y^*). \tag{3.4.3}$$

Since $\{u_n\}$ converges to u^*, and $u_n(x) \geq u_\emptyset(x)$ for all $n \in \mathbf{N}$ and $x \in X$, we have $u^*(x) \geq u_\emptyset(x)$ for all $x \in X$. In addition, because u^* is G-convex and $y^*(x) \in \partial^G u^*(x)$, we know (u^*, y^*) satisfies all the constraints in (3.3.1). Together with (3.4.3), we proved (u^*, y^*) is a solution of the principal's program. $\square$

Chapter 4

Existence: bounded product spaces

4.1 Introduction

In this chapter, we will first state the hypotheses that will be needed for this and most of the following chapters. The purpose of Section 4.2 is to fix terminology for the main results of the following chapters.

In Section 4.3, we will reformulate the principal's program in the language of G-convexity and G-subdifferentiability, state and prove the existence theorem, where the product space is bounded.

4.2 Hypotheses

For notational convenience, we adopt the following technical hypotheses, inspired by those of Trudinger [44] and Figalli-Kim-McCann [11].

The following hypotheses will be relevant: (G1)-(G3) represent partial analogs of the twist, domain convexity, and non-negative cross-curvature hypotheses from the quasilinear setting [11] [20]; (G4) encodes a form of the desirability of money to each agent, while (G5) quantifies the assertion that the maximum price $\bar{z}$ is high enough that no agent prefers paying it for any product y to the outside option.

(G0) $G \in C^1(cl(X \times Y \times Z))$, where $X \subset \mathbf{R}^m, Y \subset \mathbf{R}^n$ are open and bounded and $Z = (\underline{z}, \bar{z})$ with $-\infty < \underline{z} < \bar{z} \leq +\infty$.

(G1) For each $x \in X$, the map $(y, z) \in cl(Y \times Z) \longmapsto (G_x, G)(x, y, z)$ is a homeomorphism onto its range;

(G2) its range $(cl(Y \times Z))_x := (G_x, G)(x, cl(Y \times Z)) \subset \mathbf{R}^{m+1}$ is convex.

For each $x_0 \in X$ and $(y_0, z_0), (y_1, z_1) \in cl(Y \times Z)$, define $(y_t, z_t) \in cl(Y \times Z)$ such that the following equation holds:

$$(G_x, G)(x_0, y_t, z_t) = (1-t)(G_x, G)(x_0, y_0, z_0) + t(G_x, G)(x_0, y_1, z_1), \quad \text{for each } t \in [0, 1]. \tag{4.2.1}$$

By (G1) and (G2), (y_t, z_t) is uniquely determined by (4.2.1). We call $t \in [0,1] \longmapsto (x_0, y_t, z_t)$ the G-segment connecting (x_0, y_0, z_0) and (x_0, y_1, z_1).

(G3) For each $x, x_0 \in X$, assume $t \in [0,1] \longmapsto G(x, y_t, z_t)$ is convex along all G-segments (4.2.1).

(G4) For each $(x, y, z) \in X \times cl(Y) \times cl(Z)$, assume $G_z(x, y, z) < 0$.

(G5) $\pi \in C^0(cl(X \times Y \times Z))$ and $u_\emptyset(x) := G(x, y_\emptyset, z_\emptyset)$ for some fixed $(y_\emptyset, z_\emptyset) \in cl(Y \times Z)$ satisfying

$$G(x, y, \bar{z}) := \lim_{z \to \bar{z}} G(x, y, z) \leq G(x, y_\emptyset, z_\emptyset) \text{ for all } (x, y) \in X \times cl(Y).$$

When $\bar{z} = +\infty$ assume this inequality is strict, and moreover that z sufficiently large implies

$$G(x, y, z) < G(x, y_\emptyset, z_\emptyset) \text{ for all } (x, y) \in X \times cl(Y).$$

For each $u \in \mathbf{R}$, (G4) allows us to define $H(x, y, u) := z$ if $G(x, y, z) = u$, i.e. $H(x, y, \cdot) = G^{-1}(x, y, \cdot)$.

4.3 Reformulation of the principal's program, existence theorem

In this section, we reformulate the principal's program using u as a proxy for the prices v controlled by the principal, thus generalizing Carlier's approach [5] to the non-quasilinear setting. Moreover, the agent's indirect utility u and product selling price v are G-dual to each other in the sense of [44].

We now show each G-convex function defined in Definition 2.2.1 can be achieved by some price menu v, and conversely each price menu yields a G-convex indirect utility [44]. We require either (G5) or (4.3.1), which asserts all agents are repelled by the maximum price and insensitive to which contract they receive at that price.

Proposition 4.3.1 (Duality between prices and indirect utilities)**.** *Assume* (G0) *and* (G4)*.* (a) *If*

$$G(x, y, \bar{z}) := \lim_{z \to \bar{z}} G(x, y, z) = \inf_{(\tilde{y}, \tilde{z}) \in cl(Y \times Z)} G(x, \tilde{y}, \tilde{z}), \quad \text{for all } (x, y) \in X \times cl(Y), \tag{4.3.1}$$

then a function $u \in C^0(X)$ is G-convex if and only if there exist a lower semicontinuous $v : cl(Y) \longrightarrow cl(Z)$ such that $u(x) = \max_{y \in cl(Y)} G(x, y, v(y))$. (b) *If instead of* (4.3.1) *we assume* (G5)*, then $u_\emptyset \leq u \in C^0(X)$ is G-convex if and only if there exists a lower semicontinuous function $v : cl(Y) \longrightarrow cl(Z)$ with $v(y_\emptyset) \leq z_\emptyset$ such that $u(x) = \max_{y \in cl(Y)} G(x, y, v(y))$.*

Proof. 1. Suppose u is G-convex. Then for any agent type $x_0 \in X$, there exists a product and price $(y_0, z_0) \in cl(Y \times Z)$, such that $u(x_0) = G(x_0, y_0, z_0)$ and $u(x) \geq G(x, y_0, z_0)$, for all $x \in X$.

Let $A := \cup_{x \in X} \partial^G u(x)$ denote the corresponding set of products. For $y_0 \in A$, define $v(y_0) = z_0$, where $z_0 \in cl(Z)$ and $x_0 \in X$ satisfy $u(x_0) = G(x_0, y_0, z_0)$ and $u(x) \geq G(x, y_0, z_0)$ for all $x \in X$. We shall shortly show this makes $v : A \longrightarrow cl(Z)$ (i) well-defined and (ii) lower semicontinuous. Taking (i) for granted, our construction yields

$$u(x) = \max_{y \in A} G(x, y, v(y)) \qquad \forall x \in X. \tag{4.3.2}$$

(i) Now for $y_0 \in A$, suppose there exist $(x_0, z_0), (x_1, z_1) \in X \times cl(Z)$ with $z_0 \neq z_1$, such that $u(x_i) = G(x_i, y_0, z_i)$ and $u(x) \geq G(x, y_0, z_i)$ for all $x \in X$ and $i = 0, 1$. Without loss of generality, assume $z_0 < z_1$. By (G4), we know $u(x_1) = G(x_1, y_0, z_1) < G(x_1, y_0, z_0)$, contradicting $u(x) \geq G(x, y_0, z_0)$, for all $x \in X$. Having shown $v : A \longrightarrow cl(Z)$ is well-defined, we now show it is lower semicontinuous.

(ii) Suppose $\{y_k\} \subset A$ converges to $y_0 \in A$ and $z_\infty := \lim_{k\to\infty} v(y_k) = \liminf_{y \to y_0} v(y)$. We need to show $v(y_0) \leq z_\infty$. Letting $z_k := v(y_k)$ for each k, there exists $x_k \in X$ such that

$$u(x) \geq G(x, y_k, z_k) \qquad \forall x \in X \text{ and } k = 0, 1, 2, \ldots, \tag{4.3.3}$$

with equality holding at $x = x_k$. In case (b) we deduce $z_\infty < \infty$ from

$$G(x_k, y_k, z_k) = u(x_k) \geq G(x_k, y_\emptyset, z_\emptyset)$$

and (G5). Taking $k \to \infty$, (G0) (or (4.3.1) in case (a) when $z_\infty = +\infty$) implies

$$u(x) \geq G(x, y_0, z_\infty) \qquad \forall x \in X. \tag{4.3.4}$$

Applying (G4) to $G(x_0, y_0, z_0) = u(x_0) \geq G(x_0, y_0, z_\infty)$ yields the desired semicontinuity: $z_0 \leq z_\infty$.

(iii) We extend v from A to $cl(Y)$ by taking its lower semicontinuous hull; this does not change the values of v on A, but satisfies $v(y_0) := \bar{z}$ on $y_0 \notin cl(A)$. We now show this choice of price menu v yields (1.1.1). Recall for each $x \in X$, there exists $(y_0, z_0) \in cl(Y \times Z)$ such that

$$u(x) = G(x, y_0, z_0) \geq (u_\emptyset(x) := G(x, y_\emptyset, z_\emptyset) \geq) \sup_{y \in cl(Y) \setminus cl(A)} G(x, y, v(y)),$$

in view of (4.3.1) (or (G5)), and the fact that $v(y) = \bar{z}$ for each y outside $cl(A)$. Thus to establish (1.1.1), we need only show that (4.3.2) remains true when the domain of the maximum is enlarged from A to $cl(A)$. Since we have chosen the *largest* lower semicontinuous extension of v outside of A, each $y_0 \in cl(A) \setminus A$ is approximated by a sequence $\{y_k\} \subset A$ for which $z_k := v(y_k)$ converges to $z_\infty := v(y_0)$. As before, (4.3.3) holds and implies (4.3.4), showing (4.3.2) indeed remains true when the domain of the maximum is enlarged from A to $cl(A)$, and establishing (1.1.1). Finally, if $v(y_\emptyset) > z_\emptyset$ in case (b) then (G4) yields $u(x) \geq u_\emptyset(x) > G(x, y_\emptyset, v(y_\emptyset))$, and we may redefine $v(y_\emptyset) := z_\emptyset$ without violating either (1.1.1) or the lower semicontinuity of v.

2. Conversely, suppose there exist a lower semicontinuous function $v : cl(Y) \longrightarrow cl(Z)$, such that $u(x) = \max_{y \in cl(Y)} G(x, y, v(y))$. Then for any $x_0 \in X$, there exists $y_0 \in cl(Y)$, such that $u(x_0) = G(x_0, y_0, v(y_0))$. Let $z_0 := v(y_0)$, then $u(x_0) = G(x_0, y_0, z_0)$, and for all $x \in X$, $u(x) \geq G(x, y_0, z_0)$. By definition, u is G-convex. If $v(y_\emptyset) \leq z_\emptyset$ then $u(\cdot) \geq G(\cdot, y_\emptyset, v(y_\emptyset)) \geq u_\emptyset(\cdot)$ by (1.1.1) and (G4). □

Remark 4.3.2 (Optimal agent strategies). Assume (G0) and (G4). When $\bar{z} < \infty$, lower semicontinuity of $v : cl(Y) \longrightarrow cl(Z)$ is enough to ensure the maximum (1.1.1) is attained. However, when $\bar{z} = +\infty$ we can reach the same conclusion either by assuming the limit (4.3.1) converges uniformly with respect to $y \in cl(Y)$, or else by assuming $v(y_\emptyset) \le z_\emptyset$ and (G5).

Proof. For any fixed $x \in X$ let $u(x) = \sup_{y \in cl(Y)} G(x, y, v(y))$. We will show that the maximum is attained. Since $cl(Y)$ is compact, suppose $\{y_k\} \subset cl(Y)$ converges to $y_0 \in cl(Y)$, $z_\infty := \limsup_{k\to\infty} v(y_k)$ and $u(x) = \lim_{k\to\infty} G(x, y_k, v(y_k))$. By extracting subsequence of $\{y_k\}$ and relabelling, without loss of generality, assume $\lim_{k\to\infty} v(y_k) = z_\infty$.

1. If $z_\infty < \bar{z}$ then lower semicontinuity of v yields $v(y_0) \le z_\infty < +\infty$. By (G4), one has

$$\begin{aligned} G(x, y_0, v(y_0)) \ge G(x, y_0, z_\infty) &= \lim_{k\to\infty} G(x, y_k, v(y_k)) \\ &= u(x) = \sup_{y \in cl(Y)} G(x, y, v(y)). \end{aligned} \tag{4.3.5}$$

Therefore, the maximum is attained by y_0.

2. If $z_\infty = \bar{z}$ then $\lim_{k\to\infty} v(y_k) = \bar{z} = +\infty$.

2.1. By assuming the limit (4.3.1) converges uniformly with respect to $y \in cl(Y)$, we have

$$\begin{aligned} \inf_{(\tilde{y},\tilde{z}) \in cl(Y\times Z)} G(x, \tilde{y}, \tilde{z}) = G(x, y_0, \bar{z}) &= \lim_{k\to\infty} G(x, y_k, v(y_k)) \\ &= u(x) = \sup_{y \in cl(Y)} G(x, y, v(y)). \end{aligned}$$

In this case, the maximum is attained by y_0.

2.2. By assuming (G5), for sufficient large k, we have $G(x, y_k, v(y_k)) < G(x, y_\emptyset, z_\emptyset)$. Taking $k \to \infty$, by $v(y_\emptyset) \le z_\emptyset$ and (G4), one has

$$\begin{aligned} \sup_{y \in cl(Y)} G(x, y, v(y)) = u(x) &= \lim_{k\to\infty} G(x, y_k, v(y_k)) \\ &\le G(x, y_\emptyset, z_\emptyset) \le G(x, y_\emptyset, v(y_\emptyset)). \end{aligned}$$

Thus, the maximum is attained by $y_\emptyset$. □

From the definition of G-convexity, we know if u is a G-convex function, for any $x \in X$ where u happens to be differentiable, denoted $x \in \mathrm{dom}Du$, there exists $y \in cl(Y)$ and $z \in cl(Z)$ such that

$$u(x) = G(x, y, z), \quad Du(x) = D_x G(x, y, z). \tag{4.3.6}$$

Conversely, when (4.3.6) holds, one can identify $(y, z) \in cl(Y \times Z)$ in terms of $u(x)$ and $Du(x)$, according to Condition (G1). We denote it as

$$\bar{y}_G(x, u(x), Du(x)) := (y_G, z_G)(x, u(x), Du(x)),$$

and drop the subscript G when it is clear from the context. Under our hypotheses, $\bar{y}_G$ is a continuous function on the relevant domain of definition.[1] It will often prove convenient to augment the types x

[1] Namely $(id_X, G, G_x)(\{(x, y, z) \in cl(X \times Y \times Z) \mid G(x, y, z) \ge G(x, y_\emptyset, z_\emptyset)\})$.

and y with an extra real variable; here and later we use the notation $\bar{x} \in \mathbf{R}^{m+1}$ and $\bar{y} \in \mathbf{R}^{n+1}$ to signify this augmentation. Besides, the set $X \setminus \mathrm{dom} Du$ has Lebesgue measure zero, which will be shown in the proof of Theorem 4.3.3.

The following proposition not only reformulates the principal's problem but manifests the existence of maximizer(s). Besides Chapter 3 which relaxes relative compactness of the domain, for other existence results guaranteeing this supremum is attained in the non-quasilinear setting, see Nöldeke-Samuelson [32] who require mere continuity of the direct utility G.

Theorem 4.3.3 (Reformulating the principal's program using the agents' indirect utilities). *Assume hypotheses* (G0)-(G1) *and* (G4)-(G5), $\bar{z} < +\infty$ *and* $\mu \ll \mathcal{L}^m$. *Setting*

$$\tilde{\Pi}(u, y) = \int_X \pi(x, y(x), H(x, y(x), u(x)))d\mu(x),$$

the principal's problem (P_0) *is equivalent to*

$$(P_3) \begin{cases} \max \tilde{\Pi}(u, y) \\ \text{among } G\text{-convex } u(x) \geq u_\emptyset(x) \text{ with } y(x) \in \partial^G u(x) \text{ for all } x \in X. \end{cases}$$

This maximum is attained. Moreover, u *determines* $y(x)$ *uniquely for a.e.* $x \in X$.

Proof. 1. Proposition 4.3.1 encodes a bijective correspondence between lower semicontinuous price menus $v : cl(Y) \longrightarrow cl(Z)$ with $v(y_\emptyset) \leq z_\emptyset$ and G-convex indirect utilities $u \geq u_\emptyset$; it also shows (1.1.1) is attained. Fix a G-convex $u \geq u_\emptyset$ and the corresponding price menu v. For each $x \in X$ let $y(x)$ denote the point achieving the maximum (1.1.1), so that $u(x) = G(x, y(x), z(x))$ with $z(x) := v(y(x)) = H(x, y(x), u(x))$ and $\Pi(v, y) = \tilde{\Pi}(u, y)$. From (1.1.1) we see

$$G(\cdot, y(\cdot), v \circ y(\cdot)) = u(\cdot) \geq G(\cdot, y(x), H(x, y(x), u(x))), \tag{4.3.7}$$

so that $y(x) \in \partial^G u(x)$. Apart from the measurability established below, Proposition 2.2.4 asserts incentive compatibility of $(y, v \circ y)$, while $u \geq u_\emptyset$ shows individual rationality, so $(P_3) \leq (P_0)$.

2. The reverse inequality begins with a lower semicontinuous price menu $v : cl(Y) \longrightarrow cl(Z)$ with $v(y_\emptyset) \leq z_\emptyset$ and an incentive compatible, individually rational map $(y, v \circ y)$ on X. Proposition 2.2.4 then asserts G-convexity of $u(\cdot) := G(\cdot, y(\cdot), v(y(\cdot)))$ and that $y(x) \in \partial^G u(x)$ for each $x \in X$. Choosing $\cdot = x$ in the corresponding inequality (4.3.7) produces equality, whence (G4) implies $v(y(x)) = H(x, y(x), u(x))$ and $\Pi(v, y) = \tilde{\Pi}(u, y)$. Since $u \geq u_\emptyset$ follows from individual rationality, we have established equivalence of (P_3) to (P_0). Let us now argue the supremum (P_3) is attained.

3. Let us first show $\pi(x, y(x), H(x, y(x), u(x)))$ is measurable on X for all G-convex u and $y(x) \in \partial^G u(x)$.

By (G0), we know G is Lipschitz, i.e., there exists $L > 0$, such that $|G(x_1, y_1, z_1) - G(x_2, y_2, z_2)| < L||(x_1 - x_2, y_1 - y_2, z_1 - z_2)||$, for all (x_1, y_1, z_1), $(x_2, y_2, z_2) \in cl(X \times Y \times Z)$. Since u is G-convex, for any $x_1, x_2 \in X$, there exist $(y_1, z_1), (y_2, z_2) \in cl(Y \times Z)$, such that $u(x_i) = G(x_i, y_i, z_i)$, for $i = 1, 2$.

Therefore,

$$u(x_1) - u(x_2) \geq G(x_1, y_2, z_2) - G(x_2, y_2, z_2) > -L||x_1 - x_2||;$$
$$u(x_1) - u(x_2) \leq G(x_1, y_1, z_1) - G(x_2, y_1, z_1) < L||x_1 - x_2||.$$

That is to say, u is also Lipschitz with Lipschitz constant L. By Rademacher's theorem and $\mu \ll \mathcal{L}^m$, we have $\mu(X \setminus \mathrm{dom}Du) = \mathcal{L}^m(X \setminus \mathrm{dom}Du) = 0$. Moreover, since u is continuous, $\frac{\partial u(x)}{\partial x_j} = \lim_{h\to 0} \frac{u(x+he_j)-u(x)}{h}$ is measurable on $\mathrm{dom}Du$, for $j = 1, 2, ..., m$, where $e_j = (0, ...0, 1, 0, ..., 0)$ is the unit vector in $\mathbf{R}^m$ with j-th coordinate nonzero. Thus, Du is also Borel on $\mathrm{dom}Du$.

Since $y(x) \in \partial^G u(x)$, for all $x \in \mathrm{dom}Du$, we have

$$\begin{aligned} u(x) &= G(x, y(x), H(x, y(x), u(x))), \\ Du(x) &= D_x G(x, y(x), H(x, y(x), u(x))). \end{aligned} \tag{4.3.8}$$

By (G1), there exists a continuous function y_G, such that

$$y(x) = y_G(x, u(x), Du(x)).$$

Thus $y(x)$ is Borel on $\mathrm{dom}Du$, which implies $\pi(x, y(x), H(x, y(x), u(x)))$ is measurable on X, given $\pi \in C^0(cl(X \times Y \times Z))$ and $\mu \ll \mathcal{L}^m$. Here we use the fact that H is also continuous since G is continuous and strictly decreasing with respect to its third variable.

4. To show the supremum is attained, let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of G-convex functions, $u_k(x) \geq u_\emptyset(x)$ and $y_k(x) \in \partial^G u_k(x)$ for any $x \in X$ and $k \in \mathbb{N}$, such that $\lim_{k\to\infty} \tilde{\Pi}(u_k, y_k) = \sup \tilde{\Pi}(u, y)$, among all feasible (u, y). Below we construct a feasible pair (u_∞, y_∞) attaining the maximum.

4.1. Claim: There exists $M > 0$, such that $|u(x)| < M$, for any G-convex u and any $x \in X$. Thus $\{u_k\}_{k\in\mathbb{N}}$ is uniformly bounded.

Proof: Since u is G-convex, for any $x \in X$, there exists $(y, z) \in cl(Y \times Z)$, such that $u(x) = G(x, y, z)$. Notice that G is bounded, since G is continuous on a compact set. Thus, there exists $M > 0$, such that $|u(x)| = |G(x, y, z)| < M$ is also bounded.

4.2. From part 1, we know $\{u_k\}_{k\in\mathbb{N}}$ are uniformly Lipschitz with Lipschitz constant L, thus $\{u_k\}_{k\in\mathbb{N}}$ are uniformly equicontinuous.

4.3. By Arzelà-Ascoli theorem, there exists a subsequence of $\{u_k\}_{k\in\mathbb{N}}$, again denoted as $\{u_k\}_{k\in\mathbb{N}}$, and $u_\infty : X \longrightarrow \mathbf{R}$ such that $\{u_k\}_{k\in\mathbb{N}}$ converges uniformly to u_∞ on X.

4.4. Claim: u_∞ is also Lipschitz.

Proof: For any $\varepsilon > 0$, any $x_1, x_2 \in X$, since $\{u_k\}_{k\in\mathbb{N}}$ converges to u_∞ uniformly, there exist $K > 0$, such that for any $k > K$, we have $|u_k(x_i) - u_\infty(x_i)| < \varepsilon$, for $i = 1, 2$. Therefore,

$$\begin{aligned} & |u_\infty(x_1) - u_\infty(x_2)| \\ \leq\; & |u_k(x_1) - u_\infty(x_1)| + |u_k(x_2) - u_\infty(x_2)| + |u_k(x_1) - u_k(x_2)| \\ <\; & 2\varepsilon + L||x_1 - x_2||. \end{aligned}$$

Since the above inequality is true for all $\varepsilon > 0$, thus u_∞ is also Lipschitz.

4.5. For any $x \in X$, since $u_k(x) \geq u_\emptyset(x)$ and $\lim_{k\to\infty} u_k(x) = u_\infty(x)$, we have $u_\infty(x) \geq u_\emptyset(x)$. Therefore, u_∞ satisfies the participation constraint.

4.6. For any fixed $x \in X$, since $\{y_k(x)\}_{k\in\mathbb{N}} \subset cl(Y)$ which is compact, there exists a subsequence $\{y_{k_l}(x)\}_{l\in\mathbb{N}}$ which converges. Define $y_\infty(x) := \lim_{l\to\infty} y_{k_l}(x) \in cl(Y)$. For each $l \in \mathbb{N}$, because $y_{k_l}(x) \in \partial^G u_{k_l}(x)$, by definition, we have $u_{k_l}(x_0) \geq G(x_0, y_{k_l}(x), H(x, y_{k_l}(x), u_{k_l}(x)))$, for any $x_0 \in X$. This implies, for all $x_0 \in X$, we have

$$\begin{aligned} u_\infty(x_0) = \lim_{l\to\infty} u_{k_l}(x_0) &\geq \lim_{l\to\infty} G(x_0, y_{k_l}(x), H(x, y_{k_l}(x), u_{k_l}(x))) \\ &\geq G(x_0, y_\infty(x), H(x, y_\infty(x), u_\infty(x))). \end{aligned}$$

Thus, $y_\infty(x) \in \partial^G u_\infty(x)$.

Therefore, $\partial^G u_\infty(x) \neq \emptyset$, for any $x \in X$. By Lemma 2.2.3, this implies u_∞ is G-convex.

At this point, we have found a feasible pair (u_∞, y_∞), satisfying all the constraints in (P_3).

4.7. Claim: For any $x \in \mathrm{dom}Du_\infty$, the sequence $\{y_k(x)\}_{k\in\mathbb{N}} \subset cl(Y)$ converges to $y_\infty(x)$.

Proof: Since u_∞ is Lipschitz, by Rademacher's theorem, u_∞ is differentiable almost everywhere in X, i.e. $\mu(X \setminus \mathrm{dom}Du_\infty) = \mathcal{L}^m(X \setminus \mathrm{dom}Du_\infty) = 0$.

For any $x \in \mathrm{dom}Du_\infty$ and any $\tilde{y} \in \partial^G u_\infty(x)$, we have

$$\tilde{y}(x) = y_G(x, u_\infty(x), Du_\infty(x)),$$

according to equation (4.3.8) and hypothesis (G1). This implies $\partial^G u_\infty(x)$ is a singleton for each $x \in \mathrm{dom}Du_\infty$, i.e. $\partial^G u_\infty(x) = \{y_\infty(x)\}$.

For any $x \in \mathrm{dom}Du_\infty$, by similar argument to that above in part 4.6, we can show that any (other) accumulation points of $\{y_k(x)\}_{k\in\mathbb{N}}$ are elements in the set $\partial^G u_\infty(x) = \{y_\infty(x)\}$, i.e. the sequence $\{y_k(x)\}_{k\in\mathbb{N}}$ converges to $y_\infty(x)$.

4.8. Finally, since $\mu \ll \mathcal{L}^m$, by Fatou's lemma, we have

$$\tilde{\Pi}(u_\infty, y_\infty) = \int_X \pi(x, y_\infty(x), H(x, y_\infty(x), u_\infty(x)))d\mu(x)$$

$$\begin{aligned} &= \int_X \limsup_{k\to\infty} \pi(x, y_k(x), H(x, y_k(x), u_k(x)))d\mu(x) \\ &\geq \limsup_{k\to\infty} \int_X \pi(x, y_k(x), H(x, y_k(x), u_k(x)))d\mu(x) \\ &= \lim_{k\to\infty} \tilde{\Pi}(u_k, y_k) \\ &= \sup \tilde{\Pi}(u, y), \end{aligned}$$

among all feasible (u, y). Thus, the supremum is attained. □

Remark 4.3.4 (More singular measures)*.* If $G \in C^2$ (uniformly in $z \in Z$) the same conclusions extend to μ which need not be absolutely continuous with respect to the Lebesgue measure, provide μ vanishes on all hypersurfaces parameterized locally as a difference of convex functions [11] [13], essentially because G-convexity then implies semiconvexity of u. On the other hand, apart from its final sentence, the proposition extends to all probability measures μ if G is merely continuous, according to Nöldeke-Samuelson [32]. Our argument is simpler than theirs on one point however: Borel measurability of $y(x)$ on $\mathrm{dom} Du$ follows automatically from $(G0) - (G1)$; in the absence of these extra hypotheses, they are required to make a measurable selection from among each agent's preferred products to define $y(x)$.

Remark 4.3.5 (Tie-breaking rules for singular measures). When an agent x finds more than one product which maximize his utility, in order to reduce the ambiguity, it is convenient to assume the principal has satisfactory persuasion to convince the agent to choose one of those products which maximize the principal's profit. According to equation (4.3.6) and condition $(G1)$, this scenario would occur only for $x \in X \setminus \mathrm{dom} Du$, which has Lebesgue measure zero. Thus, this convention has no effect for absolutely continuous measures, but can be used as in Figalli-Kim-McCann [11] to extend our result to singular measures.

Chapter 5

Convexity

5.1 Introduction

In this chapter, we will show concavity and uniqueness results of the principal's problem, under the settings in Section 4.2.

In Section 5.2, we will first rewrite the principal's problem as (5.2.1), then state the equivalent condition to convexity of the functional domain $\mathcal{U}_\emptyset$. Then we will show a variety of necessary and sufficient conditions for concavity (and convexity) of the principal's problem, and the resulting uniqueness of her optimal strategy.

In Section 5.3, we assume the monopolist's utility does not depend on the agent's private information, which in certain circumstances allows us to provide a necessary and sufficient condition for the concavity of her profit functional.

5.2 Concavity and convexity results

The advantage of the reformulation from Section 4.3 is to make the principal's objective $\mathbf{\Pi}$ depend on a scalar function u instead of a vector field y. By (G1), the optimal choice $y(x)$ of Lebesgue almost every agent $x \in X$ is uniquely determined by u. Recall that $\bar{y}_G(x, u(x), Du(x))$ is the unique solution (y, z) of the system (4.3.6), for any $x \in \mathrm{dom}Du$. Then the principal's problem (P_3) can be rewritten as maximizing a functional depending only on the agents' indirect utility u:

$$(P_4) \qquad \max_{\substack{u \geq u_\emptyset \\ u \text{ is } G\text{-convex}}} \mathbf{\Pi}(u) := \max_{\substack{u \geq u_\emptyset \\ u \text{ is } G\text{-convex}}} \int_X \pi(x, \bar{y}_G(x, u(x), Du(x))) d\mu(x). \tag{5.2.1}$$

Define $\mathcal{U} := \{u : X \longrightarrow \mathbf{R} \mid u \text{ is } G\text{-convex}\}$ and $\mathcal{U}_\emptyset := \{u \in \mathcal{U} \mid u \geq u_\emptyset\}$. Then the problem becomes to maximize $\mathbf{\Pi}$ on $\mathcal{U}_\emptyset$. In this section, we give conditions under which the function space $\mathcal{U}_\emptyset$ is convex and the functional $\mathbf{\Pi}$ is concave, often strictly. Uniqueness and stability of the principal's maximizing strategy follow from strict concavity as in [11]. We also provide conditions under which $\mathbf{\Pi}$ is convex. In this situation, the maximizers of $\mathbf{\Pi}$ may not be unique but are attained at extreme points of $\mathcal{U}_\emptyset$. (Recall that $u \in \mathcal{U}$ is called *extreme* if u does not lie at the midpoint of any segment in $\mathcal{U}$.)

Theorem 5.2.1 (G-convex functions form a convex set)**.** *If $G : cl(X\times Y\times Z) \longrightarrow \mathbf{R}$ satisfies* (G0)-(G2)*, then* (G3) *becomes necessary and sufficient for the convexity of the set $\mathcal{U}$.*

Proof. Assuming (G0)-(G2), for any $u_0, u_1 \in \mathcal{U}$, define $u_t(x) := (1-t)u_0(x) + tu_1(x)$, $t \in (0,1)$. We want to show u_t is G-convex as well, for each $t \in (0,1)$.

For any fixed $x_0 \in X$, since u_0, u_1 are G-convex, there exist (y_0, z_0), $(y_1, z_1) \in cl(Y \times Z)$, such that $u_0(x_0) = G(x_0, y_0, z_0)$, $u_1(x_0) = G(x_0, y_1, z_1)$, $u_0(x) \geq G(x, y_0, z_0)$ and $u_1(x) \geq G(x, y_1, z_1)$, for all $x \in X$.

Denote (x_0, y_t, z_t) the G-segment connecting (x_0, y_0, z_0) and (x_0, y_1, z_1). Then $u_t(x_0) = (1-t)u_0(x_0) + tu_1(x_0) = (1-t)G(x_0, y_0, z_0) + tG(x_0, y_1, z_1) = G(x_0, y_t, z_t)$, where the last equality comes from (4.2.1).

In order to prove u_t is G-convex, it remains to show $u_t(x) \geq G(x, y_t, z_t)$, for all $x \in X$.

By (G3), $G(x, y_t, z_t)$ is convex in t, i.e., $G(x, y_t, z_t) \leq (1-t)G(x, y_0, z_0) + tG(x, y_1, z_1)$. So, $u_t(x) = (1-t)u_0(x) + tu_1(x) \geq (1-t)G(x, y_0, z_0) + tG(x, y_1, z_1) \geq G(x, y_t, z_t)$, for each $x \in X$. By definition, u_t is G-convex, i.e., $u_t \in \mathcal{U}$, for all $t \in (0,1)$. Thus, $\mathcal{U}$ is convex.

Conversely, assume $\mathcal{U}$ is convex. For any fixed $x_0 \in X$, $(y_t, z_t) \in cl(Y \times Z)$ with (x_0, y_t, z_t) being a G-segment, we would like to show $G(x, y_t, z_t) \leq (1-t)G(x, y_0, z_0) + tG(x, y_1, z_1)$, for any $x \in X$.

Define $u_i(x) := G(x, y_i, z_i)$, for $i = 0, 1$. Then by definition of G-convexity, $u_0, u_1 \in \mathcal{U}$. Denote $u_t := (1-t)u_0 + tu_1$, for all $t \in (0,1)$. Since $\mathcal{U}$ is a convex set, u_t is also G-convex. For this x_0 and each $t \in (0,1)$, there exists $(\tilde{y}_t, \tilde{z}_t) \in cl(Y \times Z)$, such that $u_t(x) \geq G(x, \tilde{y}_t, \tilde{z}_t)$, for all $x \in X$, and equality holds at x_0. Thus, $Du_t(x_0) = D_xG(x_0, \tilde{y}_t, \tilde{z}_t)$.

Since (x_0, y_t, z_t) is a G-segment, from (4.2.1), we know $D_xG(x_0, y_t, z_t) = (1-t)D_xG(x_0, y_0, z_0) + tD_xG(x_0, y_1, z_1) = (1-t)Du_0(x_0) + tDu_1(x_0) = Du_t(x_0)$. Thus, by (G1), $(\tilde{y}_t, \tilde{z}_t) = (y_t, z_t)$, for each $t \in (0,1)$. Therefore, $(1-t)G(x, y_0, z_0) + tG(x, y_1, z_1) = u_t \geq G(x, \tilde{y}_t, \tilde{z}_t) = G(x, y_t, z_t)$, for all $x \in X$, i.e., $G(x, y_t, z_t)$ is convex in t along any G-segment (x_0, y_t, z_t). ☐

The following theorem provides a necessary and sufficient condition for the functional $\mathbf{\Pi}(u)$ to be concave. It reveals the relationship between linear interpolations on the function space $\mathcal{U}$ and G-segments on the underlying type space $cl(Y \times Z)$.

Theorem 5.2.2 (Concavity of the principal's objective)**.** *If G and $\pi : cl(X \times Y \times Z) \longrightarrow \mathbf{R}$ satisfy* (G0)-(G5)*, the following statements are equivalent:*

(i) $t \in [0,1] \longmapsto \pi(x, y_t, z_t)$ *is concave along all G-segments (x, y_t, z_t);*

(ii) $\mathbf{\Pi}(u)$ *is concave in $\mathcal{U}$ for all $\mu \ll \mathcal{L}^m$.*

Proof. *(i)* $\Rightarrow$ *(ii)*. For any $u_0, u_1 \in \mathcal{U}$, $t \in (0,1)$, define $u_t = (1-t)u_0 + tu_1$. We want to prove $\mathbf{\Pi}(u_t) \geq (1-t)\mathbf{\Pi}(u_0) + t\mathbf{\Pi}(u_1)$, for any $\mu \ll \mathcal{L}^m$.

Equations (4.3.6) implies that there exist $y_0, y_1 : \mathrm{dom}Du \longrightarrow cl(Y)$ and $z_0, z_1 : \mathrm{dom}Du \longrightarrow cl(Z)$ such that

$$\begin{aligned} (G_x, G)(x, y_0(x), z_0(x)) &= (Du_0, u_0)(x), \\ (G_x, G)(x, y_1(x), z_1(x)) &= (Du_1, u_1)(x). \end{aligned} \tag{5.2.2}$$

For each $x \in \mathrm{dom}Du$, $(y_0(x), z_0(x)), (y_1(x), z_1(x)) \in cl(Y \times Z)$, let $t \in [0,1] \longmapsto (x, y_t(x), z_t(x))$ be the G-segment connecting $(x, y_0(x), z_0(x))$ and $(x, y_1(x), z_1(x))$. Combining (5.2.2) and (4.2.1), we have

$$(G_x, G)(x, y_t(x), z_t(x)) = (Du_t, u_t)(x). \tag{5.2.3}$$

Thus, by concavity of π on G-segments, for every $t \in [0,1]$,

$$\begin{aligned}
\mathbf{\Pi}(u_t) &= \int_X \pi(x, y_t(x), z_t(x)) d\mu(x) \\
&\geq \int_X (1-t)\pi(x, y_0(x), z_0(x)) + t\pi(x, y_1(x), z_1(x)) d\mu(x) \\
&= (1-t)\mathbf{\Pi}(u_0) + t\mathbf{\Pi}(u_1).
\end{aligned}$$

Thus, $\mathbf{\Pi}$ is concave in $\mathcal{U}$.

$(ii) \Rightarrow (i)$. To derive a contradiction, assume (i) fails. Then there exists a G-segment $(x_0, y_t(x_0), z_t(x_0))$ and $t_0 \in (0,1)$ such that

$$\pi(x_0, y_{t_0}(x_0), z_{t_0}(x_0)) < (1-t_0)\pi(x_0, y_0(x_0), z_0(x_0)) + t_0\pi(x_0, y_1(x_0), z_1(x_0)).$$

Let $u_0(x) := G(x, y_0(x_0), z_0(x_0))$, $u_1(x) := G(x, y_1(x_0), z_1(x_0))$ and $u_{t_0} = (1-t_0)u_0 + t_0 u_1$. Then $u_0, u_1, u_{t_0} \in \mathcal{U}$. From (5.2.2) we know, $y_i(x) \equiv y_i(x_0)$, $z_i(x) \equiv z_i(x_0)$, for $i = 0, 1$. Let $t \in [0,1] \longmapsto (x, y_t(x), z_t(x))$ be the G-segment connecting $(x, y_0(x), z_0(x))$ and $(x, y_1(x), z_1(x))$. And combining (4.3.6) and (4.2.1), we have

$$\begin{aligned}
(G_x, G)(x, y_0(x_0), z_0(x_0)) &= (Du_0, u_0)(x), \\
(G_x, G)(x, y_1(x_0), z_1(x_0)) &= (Du_1, u_1)(x). \\
(G_x, G)(x, y_{t_0}(x), z_{t_0}(x)) &= (Du_{t_0}, u_{t_0})(x).
\end{aligned}$$

Since π, y_{t_0} and z_{t_0} are continuous, there exists $\varepsilon > 0$, such that for all $x \in B_\varepsilon(x_0)$, one has

$$\pi(x, y_{t_0}(x), z_{t_0}(x)) < (1-t_0)\pi(x, y_0(x_0), z_0(x_0)) + t_0\pi(x, y_1(x_0), z_1(x_0)).$$

Here we use $B_\varepsilon(x_0)$ denote the open ball in $\mathbf{R}^m$ centered at x_0 with radius ε. Take $d\mu = d\mathcal{L}^m \mid_{B_\varepsilon(x_0)} / \mathcal{L}^m(B_\varepsilon(x_0))$ to be uniform measure on $B_\varepsilon(x_0)$. Thus,

$$\begin{aligned}
\mathbf{\Pi}(u_{t_0}) &= \int_X \pi(x, y_{t_0}(x), z_{t_0}(x)) d\mu(x) \\
&< \int_X (1-t_0)\pi(x, y_0(x_0), z_0(x_0)) + t_0\pi(x, y_1(x_0), z_1(x_0)) d\mu(x) \\
&= (1-t_0)\mathbf{\Pi}(u_0) + t_0\mathbf{\Pi}(u_1).
\end{aligned}$$

This contradicts the concavity of $\mathbf{\Pi}$. □

A similar proof shows the following result. Corollary 5.2.3 implies that the concavity of the principal's profit is equivalent to the concavity of principal's utility along qualified G-segments. Moreover, Theorem 5.2.1 and Corollary 5.2.3 together imply that the principal's profit $\mathbf{\Pi}$ is a concave functional on a convex space, under assumptions (G0)-(G5), $\mu \ll \mathcal{L}^m$, and $(i)'$ below.

Corollary 5.2.3. *If G and π satisfy* (G0)-(G5)*, the following are equivalent:*

$(i)'$ *$t \in [0,1] \longmapsto \pi(x, y_t(x), z_t(x))$ is concave along all G-segments $(x, y_t(x), z_t(x))$ whose endpoints satisfy $\min\{G(x, y_0(x), z_0(x)), G(x, y_1(x),\ z_1(x))\} \geq u_\emptyset(x)$;*

$(ii)'$ *$\mathbf{\Pi}(u)$ is concave in $\mathcal{U}_\emptyset$ for all $\mu \ll \mathcal{L}^m$.*

To obtain uniqueness and stability of optimizers requires a stronger form of convexity. Recall that a function f defined on a convex subset of a normed space is said to be strictly convex if $f((1-t)x+ty) > (1-t)f(x) + tf(y)$ whenever $0 < t < 1$ and $x \neq y$. It is said to be (2-)uniformly concave, if there exists $\lambda > 0$, such that for any x, y in the domain of f and $t \in [0,1]$, the following inequality holds.

$$f((1-t)x + ty) - (1-t)f(x) - tf(y) \geq t(1-t)\lambda||x-y||^2.$$

For such strengthenings, it is necessary to view indirect utilities $u \in \mathcal{U}$ as equivalence classes of functions which differ only on sets of μ measure zero. More precisely, it is natural to adopt the Sobolev norm

$$\|u\|^2_{W^{1,2}(X,d\mu)} := \int_X (|u|^2 + |Du|^2) d\mu(x)$$

on $\mathcal{U}$ and $\mathcal{U}_\emptyset$. We then have the following results:

Corollary 5.2.4. *Let π and G satisfy* (G0)-(G5)*. If*

(iii) *$t \in [0,1] \longmapsto \pi(x, y_t, z_t)$ is strictly concave along all G-segments (x, y_t, z_t), then*

(iv) *$\mathbf{\Pi}(u)$ is strictly concave in $\mathcal{U} \subset W^{1,2}(X, d\mu)$ for all $\mu \ll \mathcal{L}^m$. If*

$(iii)'$ *$t \in [0,1] \longmapsto \pi(x, y_t(x), z_t(x))$ is strictly concave along all G-segments $(x, y_t(x), z_t(x))$ whose endpoints satisfy $\min\{G(x, y_0(x), z_0(x)),\ G(x,\ y_1(x), z_1(x))\} \geq u_\emptyset(x)$, then*

$(iv)'$ *$\mathbf{\Pi}(u)$ is strictly concave in $\mathcal{U}_\emptyset \subset W^{1,2}(X, d\mu)$ for all $\mu \ll \mathcal{L}^m$.*

Besides, Theorem 5.2.1 and Corollary 5.2.4 together imply strict concavity of principal's profit on a convex space, which guarantees a unique solution to the monopolist's problem.

Define $\bar{G}(\bar{x}, \bar{y}) = \bar{G}(x, x_0, y, z) := x_0 G(x, y, z)$, where $\bar{x} = (x, x_0)$, $\bar{y} = (y, z)$ and $x_0 \in X_0$, where $X_0 \subset (-\infty, 0)$ is an open bounded interval containing -1. Hereafter, in this chapter and Section 8.2 only, we use x_0 to denote a number in X_0. For further applications, we need the following non-degeneracy assumption.

(G6) $G \in C^2(cl(X \times Y \times Z))$, and $D_{\bar{x},\bar{y}}(\bar{G})(x, -1, y, z)$ has full rank, for each $(x, y, z) \in cl(X \times Y \times Z)$.

Since (G1) implies $m \geq n$, full rank means $D_{\bar{x},\bar{y}}(\bar{G})(x, -1, y, z)$ has rank $n+1$.

Theorem 5.2.5 (Uniform concavity of the principal's objective)**.** *Assume $G \in C^2(cl(X \times Y \times Z))$ satisfies* (G0)-(G6)*. In case $\bar{z} = +\infty$, assume the homeomorphisms of* (G1) *are uniformly bi-Lipschitz. Then the following statements are equivalent:*

(*v*) *Uniform concavity of* π *along* G*-segments, i.e., there exists* $\lambda > 0$*, for any* G*-segment* (x, y_t, z_t)*, and any* $t \in [0,1]$*,*

$$\begin{aligned} &\pi(x, y_t(x), z_t(x)) - (1-t)\pi(x, y_0(x), z_0(x)) - t\pi(x, y_1(x), z_1(x)) \\ \geq &t(1-t)\lambda||(y_1(x) - y_0(x), z_1(x) - z_0(x))||^2_{\mathbf{R}^{n+1}} \end{aligned} \tag{5.2.4}$$

(*vi*) $\mathbf{\Pi}(u)$ *is uniformly concave in* $\mathcal{U} \subset W^{1,2}(X, d\mu)$*, uniformly for all* $\mu \ll \mathcal{L}^m$*.*

Proof. $(v) \Rightarrow (vi)$. With the same notation as last proof, we want to prove there exists $\tilde{\lambda} > 0$, such that $\mathbf{\Pi}(u_t) - (1-t)\mathbf{\Pi}(u_0) - t\mathbf{\Pi}(u_1) \geq t(1-t)\tilde{\lambda}||u_1 - u_0||^2_{W^{1,2}(X,d\mu)}$, for any $\mu \ll \mathcal{L}^m$, $u_0, u_1 \in \mathcal{U}$ and $t \in (0,1)$.

Similar to the last proof, we have (5.2.2) and (5.2.3). Denote $\mathrm{Lip}(G_x, G)$ the uniform Lipschitz constant of the map $(x, y, z) \in X \times Y \times Z \longmapsto (G_x, G)(x, y, z)$.

Thus by uniform concavity of π on G-segments, there exists $\lambda > 0$, such that for every $t \in [0,1]$,

$$\begin{aligned} &\mathbf{\Pi}(u_t) - (1-t)\mathbf{\Pi}(u_0) - t\mathbf{\Pi}(u_1) \\ =&\int_X \pi(x, y_t(x), z_t(x)) - (1-t)\pi(x, y_0(x), z_0(x)) - t\pi(x, y_1(x), z_1(x))d\mu(x) \\ \geq &\int_X t(1-t)\lambda||(y_1(x) - y_0(x), z_1(x) - z_0(x))||^2_{\mathbf{R}^{n+1}}d\mu(x) \\ \geq &\int_X t(1-t)\lambda||(Du_1(x) - Du_0(x), u_1(x) - u_0(x))||^2_{\mathbf{R}^{n+1}}/\mathrm{Lip}^2(G_x, G)d\mu(x) \\ =& \, t(1-t)\frac{\lambda}{\mathrm{Lip}^2(G_x, G)}||u_1 - u_0||_{W^{1,2}(X,d\mu)}. \end{aligned}$$

Thus, $\mathbf{\Pi}$ is uniformly concave in $\mathcal{U}$, with $\tilde{\lambda} = \frac{\lambda}{\mathrm{Lip}^2(G_x,G)} > 0$.

$(vi) \Rightarrow (v)$. To derive a contradiction, assume (v) fails. Then for any $\lambda > 0$, there exists a G-segment $(x^0, y_t(x^0), z_t(x^0))$, and some $\tau \in (0,1)$, such that $\pi(x^0, y_\tau(x^0), z_\tau(x^0)) - (1-\tau)\pi(x^0, y_0(x^0), z_0(x^0)) - \tau\pi(x^0, y_1(x^0), z_1(x^0)) < \tau(1-\tau)\lambda||(y_1(x^0) - y_0(x^0), z_1(x^0) - z_0(x^0))||^2_{\mathbf{R}^{n+1}}$.

Take $u_0(x) := G(x, y_0(x^0), z_0(x^0))$, $u_1(x) := G(x, y_1(x^0), z_1(x^0))$ and for $t \in (0,1)$, assign $u_t := (1-t)u_0 + tu_1$. Then $u_t \in \mathcal{U}$, for $t \in [0,1]$. From (5.2.2) we know, $y_i(x) \equiv y_i(x^0)$, $z_i(x) \equiv z_i(x^0)$, for $i = 0, 1$. Let $t \in [0,1] \longmapsto (x, y_t(x), z_t(x))$ be the G-segment connecting $(x, y_0(x), z_0(x))$ and $(x, y_1(x), z_1(x))$. And combining (4.3.6) and (4.2.1), we have

$$\begin{aligned} (G_x, G)(x, y_0(x^0), z_0(x^0)) &= (Du_0, u_0)(x), \\ (G_x, G)(x, y_1(x^0), z_1(x^0)) &= (Du_1, u_1)(x). \\ (G_x, G)(x, y_t(x), z_t(x)) &= (Du_t, u_t)(x). \end{aligned}$$

Since π, y_τ and z_τ are continuous, there exists $\varepsilon > 0$, such that for all $x \in B_\varepsilon(x^0)$,

$$\begin{aligned} &\pi(x, y_\tau(x), z_\tau(x)) - (1-\tau)\pi(x, y_0(x^0), z_0(x^0)) - \tau\pi(x, y_1(x^0), z_1(x^0)) \\ &< \tau(1-\tau)\lambda||(y_1(x^0) - y_0(x^0), z_1(x^0) - z_0(x^0))||^2_{\mathbf{R}^{n+1}}. \end{aligned}$$

Here we use $B_\varepsilon(x^0)$ denote the open ball in $\mathbf{R}^m$ centered at x^0 with radius ε. Take $d\mu = d\mathcal{L}^m \mid_{B_\varepsilon(x^0)} /\mathcal{L}^m(B_\varepsilon(x^0))$ to be uniform measure on $B_\varepsilon(x^0)$. By (G6), the map $\bar{y}_G : (x, p, q) \longmapsto (y, z)$, which solves equation (4.3.6), is uniformly Lipschitz on $X \times \mathbf{R} \times \mathbf{R}^m$. Denote $\mathrm{Lip}(\bar{y}_G)$ its Lipschitz constant.

Thus for such τ, u_0, u_1 and μ, we have

$$\begin{aligned}
&\mathbf{\Pi}(u_\tau) - (1-\tau)\mathbf{\Pi}(u_0) - \tau\mathbf{\Pi}(u_1) \\
&= \int_X \pi(x, y_\tau(x), z_\tau(x)) - (1-\tau)\pi(x, y_0(x^0), z_0(x^0)) - \tau\pi(x, y_1(x^0), z_1(x^0))d\mu(x) \\
&< \int_X \tau(1-\tau)\lambda||(y_1 - y_0, z_1 - z_0)||^2_{\mathbf{R}^{n+1}} d\mu(x) \\
&\leq \tau(1-\tau)\lambda \mathrm{Lip}^2(\bar{y}_G)||u_1 - u_0||^2_{W^{1,2}(X,d\mu)}.
\end{aligned}$$

This contradicts the uniform concavity of $\mathbf{\Pi}$. □

A similar argument implies the following equivalence. Theorem 5.2.1 and Corollary 5.2.6 together imply that the principal's profit $\mathbf{\Pi}$ is a uniformly concave functional on a convex space, under assumptions (G0)-(G6), $\mu \ll \mathcal{L}^m$, and $(v)'$.

Corollary 5.2.6. *Under the same assumptions as in Theorem 5.2.5, the following are equivalent:*

$(v)'$ *Uniform concavity of π (in the sense of equation (5.2.4)) along G-segments $(x, y_t(x), z_t(x))$ whose endpoints satisfy $\min\{G(x, y_0(x), z_0(x)), G(x,\ y_1(x), z_1(x))\} \geq u_\emptyset(x)$;*

$(vi)'$ *$\mathbf{\Pi}(u)$ is uniformly concave in $\mathcal{U}_\emptyset \subset W^{1,2}(X, d\mu)$ uniformly for all $\mu \ll \mathcal{L}^m$.*

The preceding concavity results also have convexity analogs. Unlike strict concavity, strict convexity does not imply uniqueness of the principal's profit-maximizing strategy, though it suggests it should only be attained at extreme points of the strategy space $\mathcal{U}$, where extreme point needs to be interpreted appropriately.

Remark 5.2.7 (Convexity of principal's objective). If π and G satisfy (G0)-(G5), the equivalences $(i) \Leftrightarrow (ii)$ and $(i)' \Leftrightarrow (ii)'$ and implications $(iii) \Rightarrow (iv)$ and $(iii)' \Rightarrow (iv)'$ remain true when all occurences of concavity are replaced by convexity. Similarly, the equivalences $(v) \Leftrightarrow (vi)$ and $(v)' \Leftrightarrow (vi)'$ remain true when both occurences of uniform concavity are replaced by uniform convexity in Theorem 5.2.5.

Assuming (G6), we denote $(\bar{G}_{\bar{x},\bar{y}})^{-1}$ the left inverse of $D_{\bar{x},\bar{y}}(\bar{G})(x, x_0, y, z)$. Starting from now, for subscripts, we use $i, k, j, l, \alpha, \beta$ denoting integers from either $\{1, ..., m\}$ or $\{1, ..., n\}$, and $\bar{i}, \bar{k}, \bar{j}, \bar{l}$ denoting augmented indices from $\{1, ..., m+1\}$ or $\{1, ..., n+1\}$. For instance, $\pi_{i,}$ denotes first order derivative with respect to x only, $\pi_{,\bar{k}\bar{j}}$ represents Hessian matrix with respect to $\bar{y}$ only, and $\bar{G}_{\bar{i},\bar{k}\bar{j}}$ denotes a third order derivative tensor which can be viewed as taking $\bar{x}$-derivative of $\bar{G}_{,\bar{k}\bar{j}}$.

The following remark reformulates concavity of π on G-segments using non-positive definiteness of a matrix. This equivalent form provides a simple method to verify the concavity condition stated in Theorem 5.2.2. We will apply this matrix form to establish Corollary 5.3.2 and Example 8.2.1 – 8.2.3.

Lemma 5.2.8 (Characterizing concavity of principal's profit in the smooth case)**.** *When $G \in C^3(cl(X \times Y \times Z))$ satisfies* (G0)-(G6) *and $\pi \in C^2(cl(X \times Y \times Z))$, then differentiating π along an arbitrary G-segment $t \in [0,1] \longrightarrow (x, y_t, z_t)$ yields*

$$\frac{d^2}{dt^2}\pi(x, y_t, z_t) = (\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}}\bar{G}^{\bar{i},\bar{l}}\bar{G}_{\bar{i},\bar{k}\bar{j}})\dot{\bar{y}}^{\bar{k}}\dot{\bar{y}}^{\bar{j}} \tag{5.2.5}$$

where $\bar{G}^{\bar{i},\bar{l}}$ denotes the left inverse of the matrix $\bar{G}_{\bar{i},\bar{k}}$ and $\dot{\bar{y}}^{\bar{k}} = (\frac{d}{dt})\bar{y}_t^{\bar{k}}$. Thus (i) in Theorem 5.2.2 is equivalent to non-positive definiteness of the quadratic form $\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}}\bar{G}^{\bar{i},\bar{l}}\bar{G}_{\bar{i},\bar{k}\bar{j}}$ on $T_{\bar{y}}(Y \times Z) = \mathbf{R}^{n+1}$, for each $(x, \bar{y}) \in X \times Y \times Z$. Similarly, Theorem 5.2.5 (v) is equivalent to uniform negative definiteness of the same form.

Proof. For any G-segments (x, y_t, z_t) satisfying equation (5.2.3) and $\pi \in C^2(cl(X \times Y \times Z))$, $t \in [0, 1] \longmapsto \pi(x, y_t, z_t)$ is concave [uniformly concave] if and only if $\frac{d^2}{dt^2}\pi(x, y_t, z_t) \leq 0$ $[\leq -\lambda||(\dot{y}_t, \dot{z}_t)||^2_{\mathbf{R}^{n+1}} < 0]$, for all $t \in [0, 1]$.

On the one hand, since $\frac{d}{dt}\pi(x, y_t, z_t) = \pi_{,\bar{k}}\dot{\bar{y}}^{\bar{k}}$, taking another derivative with respect to t gives

$$\frac{d^2}{dt^2}\pi(x, y_t, z_t) = \pi_{,\bar{k}\bar{j}}\dot{\bar{y}}^{\bar{k}}\dot{\bar{y}}^{\bar{j}} + \pi_{,\bar{l}}\ddot{\bar{y}}^{\bar{l}}. \tag{5.2.6}$$

On the other hand, taking second derivative with respect to t at both sides of equation (5.2.3), which is equivalent to $\bar{G}_{\bar{i},}(x, x_0, y_t(x), z_t(x)) = (x_0 Du_t, u_t)(x)$, for some fixed $x_0 \in X_0$, implies

$$\bar{G}_{\bar{i},\bar{k}\bar{j}}\dot{\bar{y}}^{\bar{k}}\dot{\bar{y}}^{\bar{j}} + \bar{G}_{\bar{i},\bar{k}}\ddot{\bar{y}}^{\bar{k}} = 0 \tag{5.2.7}$$

Combining equations (5.2.6) with (5.2.7) yields (5.2.5). For $x \in X$, there is a G-segment with any given tangent direction through $\bar{y} = (y, z) \in Y \times Z$. Thus, the non-positivity of $\frac{d^2}{dt^2}\pi(x, y_t, z_t)$ along all G-segments (x, y_t, z_t) is equivalent to non-positive definiteness of the matrix $(\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}}\bar{G}^{\bar{i},\bar{l}}\bar{G}_{\bar{i},\bar{k}\bar{j}})$ on $T_{\bar{y}}(Y \times Z) = \mathbf{R}^{n+1}$.

In addition, the uniform concavity of $\pi(x, y_t, z_t)$ along all G-segments (x, y_t, z_t) is equivalent to uniform negative definiteness of $(\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}}\bar{G}^{\bar{i},\bar{l}}\bar{G}_{\bar{i},\bar{k}\bar{j}})$ on $\mathbf{R}^{n+1}$. □

5.3 Concavity of principal's objective when her utility does not depend directly on agents' private types: A sharper, more local result

In this section, we reveal a necessary and sufficient condition for the concavity of principal's maximization problem, not for some specific examples in Chapter 8, but for many other private-value circumstances, where principal's utility only directly depends on the products sold and their selling prices, but not the buyer's type.

Before we state the results, we need the following definition, which is a generalized Legendre transform (see Moreau [28], Kutateladze-Rubinov [19], Elster-Nehse [9], Balder [2], Dolecki-Kurcyusz [7], Gangbo-McCann[12], Singer[41], Rubinov[40, 39], and Martínez-Legaz [22] for more references).

Definition 5.3.1 ($\bar{G}$-concavity, $\bar{G}^*$-concavity)**.** A function $\phi : cl(X \times X_0) \longrightarrow \mathbf{R}$ is called $\bar{G}$-concave if $\phi = (\phi^{\bar{G}^*})^{\bar{G}}$ and a function $\psi : cl(Y \times Z) \longrightarrow \mathbf{R}$ is called $\bar{G}^*$-concave if $\psi = (\psi^{\bar{G}})^{\bar{G}^*}$, where

$$\begin{aligned} \psi^{\bar{G}}(\bar{x}) &= \min_{\bar{y} \in cl(Y \times Z)} \bar{G}(\bar{x}, \bar{y}) - \psi(\bar{y}), \\ \text{and } \phi^{\bar{G}^*}(\bar{y}) &= \min_{\bar{x} \in cl(X \times X_0)} \bar{G}(\bar{x}, \bar{y}) - \phi(\bar{x}). \end{aligned} \tag{5.3.1}$$

We say ψ is strictly $\bar{G}^*$-concave, if in addition $\psi^{\bar{G}} \in C^1(X \times X_0)$.

Note that, apart from an overall sign and the extra variables, Definition 5.3.1 coincides with a quasilinear version $G(\bar{x}, \bar{y}, z) = \bar{G}(\bar{x}, \bar{y}) - z$ of Definition 2.2.1.

The following corollary characterizes the concavity of principal's profit when her utility, on the one hand, is not influenced by the agents' identity, and, on the other hand, has adequate generality to encompass a tangled nonlinear relationship between products and selling prices. It generalizes the convexity result in Figalli-Kim-McCann [11], where $G(x, y, z) = b(x, y) - z$ and $\pi(x, y, z) = z - a(y)$.

Corollary 5.3.2 (Concavity of principal's objective with her utility not depending on agents' types)**.** *If $G \in C^3(cl(X \times Y \times Z))$ satisfies* (G0)-(G6), *$\pi \in C^2(cl(Y \times Z))$ is $\bar{G}^*$-concave and $\mu \ll \mathcal{L}^m$, then $\mathbf{\Pi}$ is concave.*

Proof. According to Lemma 5.2.8, for concavity, we only need to show non-positive definiteness of $(\pi_{\bar{k}\bar{j}} - \pi_{\bar{l}} \bar{G}^{\bar{i},\bar{l}} \bar{G}_{\bar{i},\bar{k}\bar{j}})$ on $\mathbf{R}^{n+1}$, i.e., for any $\bar{x} = (x, x_0) \in X \times X_0$, $\bar{y} \in Y \times Z$ and $\xi \in \mathbf{R}^{n+1}$, $\big(\pi_{\bar{k}\bar{j}}(\bar{y}) - \pi_{\bar{l}}(\bar{y}) \bar{G}^{\bar{i},\bar{l}}(\bar{x}, \bar{y}) \bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\big) \xi^{\bar{k}} \xi^{\bar{j}} \leq 0$.

For any fixed $\bar{x} = (x, x_0) \in X \times X_0$, $\bar{y} \in Y \times Z$, $\xi \in \mathbf{R}^{n+1}$, there exist $\delta > 0$ and $t \in (-\delta, \delta) \longmapsto \bar{y}_t \in Y \times Z$, such that $\bar{y}_t|_{t=0} = \bar{y}$, $\dot{\bar{y}}|_{t=0} = \xi$ and $\frac{d^2}{dt^2} \bar{G}_{\bar{i},}(\bar{x}, \bar{y}_t) = 0$. Thus,

$$0 = \frac{d^2}{dt^2}\bigg|_{t=0} \bar{G}_{\bar{i},}(\bar{x}, \bar{y}_t) = \bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y}) \xi^{\bar{k}} \xi^{\bar{j}} + \bar{G}_{\bar{i},\bar{k}}(\bar{x}, \bar{y}) (\ddot{\bar{y}}_t)^{\bar{k}}\bigg|_{t=0}. \tag{5.3.2}$$

Since π is $\bar{G}^*$-concave, we have $\pi(\bar{y}) = \min_{\tilde{x} \in cl(X \times X_0)} \bar{G}(\tilde{x}, \bar{y}) - \phi(\tilde{x})$, for some $\bar{G}$-concave function ϕ. Since $cl(X \times X_0)$ is compact, for this $\bar{y}$, there exists $\bar{x}^* = (x^*, {x_0}^*) \in cl(X \times X_0)$, such that $\pi_{\bar{l}}(\bar{y}) = \bar{G}_{,\bar{l}}(\bar{x}^*, \bar{y})$ for each $\bar{l} = 1, 2, ..., n+1$ and $\pi_{\bar{k}\bar{j}}(\bar{y}) \xi^{\bar{k}} \xi^{\bar{j}} \leq \bar{G}_{,\bar{k}\bar{j}}(\bar{x}^*, \bar{y}) \xi^{\bar{k}} \xi^{\bar{j}}$ for each $\xi \in \mathbf{R}^{n+1}$. Combined with (5.3.2) this yields

$$\begin{aligned}
&\big(\pi_{\bar{k}\bar{j}}(\bar{y}) - \pi_{\bar{l}}(\bar{y}) \bar{G}^{\bar{i},\bar{l}}(\bar{x}, \bar{y}) \bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\big) \xi^{\bar{k}} \xi^{\bar{j}} \\
&\leq \big(\bar{G}_{,\bar{k}\bar{j}}(\bar{x}^*, \bar{y}) - \bar{G}_{,\bar{l}}(\bar{x}^*, \bar{y}) \bar{G}^{\bar{i},\bar{l}}(\bar{x}, \bar{y}) \bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\big) \xi^{\bar{k}} \xi^{\bar{j}} \\
&= \bar{G}_{,\bar{k}\bar{j}}(\bar{x}^*, \bar{y}) \xi^{\bar{k}} \xi^{\bar{j}} + \bar{G}_{,\bar{l}}(\bar{x}^*, \bar{y}) \cdot (\ddot{\bar{y}}_t)^{\bar{l}}\big|_{t=0} \\
&= \frac{d^2}{dt^2}\bigg|_{t=0} \bar{G}(\bar{x}^*, \bar{y}_t) \\
&= {x_0}^* \cdot \frac{d^2}{dt^2}\bigg|_{t=0} G(x^*, \bar{y}_t) \\
&\leq 0.
\end{aligned}$$

The last inequality comes from ${x_0}^* \leq 0$ and (G3). □

The following proposition shows a version of a necessary and sufficient condition to the concavity in corollary 5.3.2.

Proposition 5.3.3 (Concavity of principal's objective when her payoff is independent of agents' types)**.** *Suppose $G \in C^3(cl(X \times Y \times Z))$ satisfies* (G0)-(G6), *$\pi \in C^2(cl(Y \times Z))$, and assume there exists a set $J \subset cl(X)$ such that for each $\bar{y} \in Y \times Z$, $0 \in (\pi_{\bar{y}} + G_{\bar{y}})(cl(J), \bar{y})$. Then the following statements are equivalent:*

(i) local $\bar{G}^*$-concavity of π*: i.e.* $\pi_{\bar{y}\bar{y}}(\bar{y}) + G_{\bar{y}\bar{y}}(x, \bar{y})$ *is non-positive definite whenever* $(x, \bar{y}) \in cl(J) \times Y \times Z$ *satisfies* $\pi_{\bar{y}}(\bar{y}) + G_{\bar{y}}(x, \bar{y}) = 0$*;*

(ii) $\mathbf{\Pi}$ *is concave on* $\mathcal{U}$ *for all* $\mu \ll \mathcal{L}^m$*.*

Remark 5.3.4. The sufficient condition, i.e., existence of $J \subset cl(X)$ (such that for each $\bar{y} \in Y \times Z$, $0 \in (\pi_{\bar{y}} + G_{\bar{y}})(cl(J), \bar{y})$), make the statement more general than taking some specific subset of $cl(X)$ instead. In particular, if $J = cl(X)$, this condition is equivalent to: for each $\bar{y} \in Y \times Z$, there exists $x \in cl(X)$, such that $(\pi_{\bar{y}} + G_{\bar{y}})(x, \bar{y}) = 0$. One of its economic interpretations is that for each product-price type, there exists a customer type, such that his marginal disutility, the gradient with respect to product type (e.g., quality, quantity, etc.) and price type, coincides with the marginal utility of the monopolist.

Proof of Proposition 5.3.3. *(i)* $\Rightarrow$ *(ii)*. Similar to the proof of Corollary 5.3.2, we only need to show non-positive definiteness of $(\pi_{\bar{k}\bar{j}} - \pi_{\bar{l}}\bar{G}^{\bar{i},\bar{l}}\bar{G}_{\bar{i},\bar{k}\bar{j}})$, i.e., for any $\bar{x} = (x, x_0) \in X \times X_0$, $\bar{y} \in Y \times Z$ and $\xi \in \mathbf{R}^{n+1}$, $\big(\pi_{\bar{k}\bar{j}}(\bar{y}) - \pi_{\bar{l}}(\bar{y})\bar{G}^{\bar{i},\bar{l}}(\bar{x}, \bar{y})\bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\big)\xi^{\bar{k}}\xi^{\bar{j}} \leq 0$.

For any fixed $\bar{x} = (x, x_0) \in X \times X_0$, $\bar{y} \in Y \times Z$, $\xi \in \mathbf{R}^{n+1}$, there exist $\delta > 0$ and a curve $t \in (-\delta, \delta) \longmapsto \bar{y}_t \in Y \times Z$, such that $\bar{y}_t|_{t=0} = \bar{y}$, $\dot{\bar{y}}_t|_{t=0} = \xi$ and $\frac{d^2}{dt^2}\bar{G}_{\bar{i},}(\bar{x}, \bar{y}_t) = 0$. Thus,

$$0 = \frac{d^2}{dt^2}\Big|_{t=0} \bar{G}_{\bar{i},}(\bar{x}, \bar{y}_t) = \bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\xi^{\bar{k}}\xi^{\bar{j}} + \bar{G}_{\bar{i},\bar{k}}(\bar{x}, \bar{y}) \cdot (\ddot{\bar{y}}_t)^{\bar{k}}\big|_{t=0} \tag{5.3.3}$$

For this $\bar{y}$, since $0 \in (\pi_{\bar{y}} + G_{\bar{y}})(cl(J), \bar{y})$, there exists $x^* \in cl(J)$, such that $(\pi_{\bar{y}} + G_{\bar{y}})(x^*, \bar{y}) = 0$. By property *(i)*, one has $(\pi_{\bar{y}\bar{y}}(\bar{y}) + G_{\bar{y}\bar{y}}(x^*, \bar{y}))\xi^{\bar{k}}\xi^{\bar{j}} \leq 0$. Let $\bar{x}^* = (x^*, -1)$, then $\pi_{\bar{l}}(\bar{y}) = \bar{G}_{,\bar{l}}(\bar{x}^*, \bar{y})$ and $\pi_{\bar{k}\bar{j}}(\bar{y})\xi^{\bar{k}}\xi^{\bar{j}} \leq \bar{G}_{,\bar{k}\bar{j}}(\bar{x}^*, \bar{y})\xi^{\bar{k}}\xi^{\bar{j}}$, for each $\bar{l} = 1, 2, ..., n+1$. Thus, combining (5.3.3) and (G3), we have

$$\begin{aligned}
&\big(\pi_{\bar{k}\bar{j}}(\bar{y}) - \pi_{\bar{l}}(\bar{y})\bar{G}^{\bar{i},\bar{l}}(\bar{x}, \bar{y})\bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\big)\xi^{\bar{k}}\xi^{\bar{j}} \\
&\leq \big(\bar{G}_{,\bar{k}\bar{j}}(\bar{x}^*, \bar{y}) - \bar{G}_{,\bar{l}}(\bar{x}^*, \bar{y})\bar{G}^{\bar{i},\bar{l}}(\bar{x}, \bar{y})\bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\big)\xi^{\bar{k}}\xi^{\bar{j}} \\
&= \bar{G}_{,\bar{k}\bar{j}}(\bar{x}^*, \bar{y})\xi^{\bar{k}}\xi^{\bar{j}} + \bar{G}_{,\bar{l}}(\bar{x}^*, \bar{y}) \cdot (\ddot{\bar{y}}_t)^{\bar{l}}\big|_{t=0} \\
&= \frac{d^2}{dt^2}\Big|_{t=0} \bar{G}(\bar{x}^*, \bar{y}_t) \\
&= -\frac{d^2}{dt^2}\Big|_{t=0} G(x^*, \bar{y}_t) \\
&\leq 0.
\end{aligned} \tag{5.3.4}$$

(ii) $\Rightarrow$ *(i)*. For any $(x, \bar{y}) \in cl(J) \times Y \times Z$, satisfying $\pi_{\bar{y}}(\bar{y}) + G_{\bar{y}}(x, \bar{y}) = 0$, we would like to show $(\pi_{\bar{k}\bar{j}}(\bar{y}) + G_{,\bar{k}\bar{j}}(x, \bar{y}))\xi^{\bar{k}}\xi^{\bar{j}} \leq 0$, for any $\xi \in \mathbf{R}^{n+1}$. Let $\bar{x} = (x, -1)$, there exist $\delta > 0$ and a curve $t \in (-\delta, \delta) \longmapsto \bar{y}_t \in Y \times Z$, such that $\bar{y}_t|_{t=0} = \bar{y}$, $\dot{\bar{y}}_t|_{t=0} = \xi$ and $\frac{d^2}{dt^2}\bar{G}_{\bar{i},}(\bar{x}, \bar{y}_t) = 0$. Thus, equation (5.3.3) holds.

Since $\mathbf{\Pi}$ is concave, by Theorem 5.2.2 and Lemma 5.2.8 as well as equation (5.3.3), we have

$$\begin{aligned}
0 \geq& \big(\pi_{\bar{k}\bar{j}}(\bar{y}) - \pi_{\bar{l}}(\bar{y})\bar{G}^{\bar{i},\bar{l}}(\bar{x}, \bar{y})\bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\big)\xi^{\bar{k}}\xi^{\bar{j}} \\
=& \big(\pi_{\bar{k}\bar{j}}(\bar{y}) - \bar{G}_{,\bar{k}\bar{j}}(\bar{x}, \bar{y}) + \bar{G}_{,\bar{k}\bar{j}}(\bar{x}, \bar{y}) - \bar{G}_{,\bar{l}}(\bar{x}, \bar{y})\bar{G}^{\bar{i},\bar{l}}(\bar{x}, \bar{y})\bar{G}_{\bar{i},\bar{k}\bar{j}}(\bar{x}, \bar{y})\big)\xi^{\bar{k}}\xi^{\bar{j}} \\
=& \big(\pi_{\bar{k}\bar{j}}(\bar{y}) - \bar{G}_{,\bar{k}\bar{j}}(\bar{x}, \bar{y})\big)\xi^{\bar{k}}\xi^{\bar{j}} + \frac{d^2}{dt^2}\Big|_{t=0} \bar{G}(\bar{x}, \bar{y}_t)
\end{aligned}$$

$$=\big(\pi_{\bar{k}\bar{j}}(\bar{y}) - \bar{G}_{,\bar{k}\bar{j}}(\bar{x},\bar{y})\big)\xi^{\bar{k}}\xi^{\bar{j}}$$
$$=(\pi_{\bar{k}\bar{j}}(\bar{y}) + G_{,\bar{k}\bar{j}}(x,\bar{y}))\xi^{\bar{k}}\xi^{\bar{j}},$$

which completes the proof. □

The following remark provides an equivalent condition for the uniform concavity of principal's maximization problem. Its proof is very similar to that of the above proposition.

Remark 5.3.5. In addition to the hypotheses of Proposition 5.3.3, when $\bar{z} = +\infty$ assume the homeomorphisms of (G1) are uniformly bi-Lipschitz. Then the following statements are equivalent:

(i) $\pi_{\bar{y}\bar{y}}(\bar{y}) + G_{\bar{y}\bar{y}}(x,\bar{y})$ is uniformly negative definite for all $(x,\bar{y}) \in cl(J) \times Y \times Z$ such that $\pi_{\bar{y}}(\bar{y}) + G_{\bar{y}}(x,\bar{y}) = 0$;

(ii) $\mathbf{\Pi}$ is uniformly concave on $\mathcal{U} \subset W^{1,2}(X, d\mu)$, uniformly for all $\mu \ll \mathcal{L}^m$.

When $m = n$, $G(x,y,z) = b(x,y) - z \in C^3(cl(X \times Y \times Z))$ satisfies (G0)-(G8), and $\pi(y,z) = z - a(y) \in C^2(cl(Y \times Z))$, then Corollary 5.3.2 shows b^*-convexity of a is a sufficient condition for concavity of $\mathbf{\Pi}$ for all $\mu \ll \mathcal{L}^m$. One may wonder under what hypotheses it would become a necessary condition as well. From Theorem A.1 in [18], under the same assumptions as above, the manufacturing cost a is b^*-convex if and only if it satisfies the following local b^*-convexity hypothesis: $D^2a(y) \geq D^2_{yy}b(x,y)$ whenever $Da(y) = D_y b(x,y)$.

Combined with Proposition 5.3.3, we have the following corollary.

Corollary 5.3.6. *Adopting the terminology of Figalli-Kim-McCann [11], i.e. (B0)-(B3), $G(x,y,z) = b(x,y) - z \in C^3(cl(X \times Y \times Z)$ and $\pi(x,y,z) = z - a(y) \in C^2(cl(Y \times Z)$, then $a(y)$ is b^*-convex if and only if $\mathbf{\Pi}$ is concave on $\mathcal{U}$ and for every $y \in Y$, there exists $x \in cl(X)$ such that $Da(y) = D_y b(x,y)$.*

Proof. Assume a is b^*-convex, by definition, there exists a function $a^* : cl(X) \to \mathbf{R}$, such that for any $y \in Y$, $a(y) = \max_{x\in cl(X)} b(x,y) - a^*(x)$. Therefore, for any $y^0 \in Y$, there exists $x^0 \in cl(X)$, such that $a(y) \geq b(x^0,y) - a^*(x^0)$ for all $y \in Y$, with equality holds at $y = y^0$. This implies, $Da(y^0) = D_y b(x^0,y^0)$. Taking $J = cl(X)$ and applying Proposition 5.3.3, we have concavity of $\mathbf{\Pi}$, since local b^*-convexity of a is automatically satisfied by a b^*-convex function a.

On the other hand, assuming $\mathbf{\Pi}$ is concave on $\mathcal{U}$ and for every $y \in Y$, there exists $x \in cl(X)$ such that $Da(y) = D_y b(x,y)$, Proposition 5.3.3 implies local b^*-convexity of a. Together with Theorem A.1 in [18], we know a is b^*-convex. □

Chapter 6

Analytic representation of condition (G3)

6.1 A fourth-order differential re-expression of condition (G3)

In our convexity argument, hypothesis (G3) plays a crucial role. In this chapter, we localize this hypothesis using differential calculus. Inspired by and strongly connected with Trudinger's theory of generalized prescribed Jacobian equations, this form is analogous to the non-negative cross-curvature condition (B3) of Figalli-Kim-McCann [11], a fourth order condition in the spirit of the Ma-Trudinger-Wang [21]. For another formulation, see e.g. [15].

Apart from the assumptions of Section 4.2, we shall need the non-degeneracy condition assumed in Section 5.2:

(G6) $G \in C^2(cl(X \times Y \times Z))$, and $D_{\bar{x},\bar{y}}(\bar{G})(x, -1, y, z)$ has full rank, for each $(x, y, z) \in cl(X \times Y \times Z)$.

For this and the next chapter only, we assume the dimensions of spaces X and Y are equal, i.e. $m = n$, so that the matrix mentioned in (G6) is square. We shall also need to extend the twist and convex range hypotheses (G1) and (G2) to the function H in place of G. This is equivalent to assuming:

(G7) For each $(y, z) \in cl(Y \times Z)$ the map $x \in X \longmapsto \frac{G_y}{G_z}(\cdot, y, z)$ is one-to-one;

(G8) its range $X_{(y,z)} := \frac{G_y}{G_z}(X, y, z) \subset \mathbf{R}^n$ is convex.

We can now state:

Theorem 6.1.1. *Assume* (G0)-(G2) *and* (G4)-(G8). If, in addition, $G \in C^4(cl(X \times Y \times Z))$, then the following statements are equivalent:

(i) (G3).

(ii) For any given $x_0, x_1 \in X$, any curve $(y_t, z_t) \in cl(Y \times Z)$ connecting (y_0, z_0) and (y_1, z_1), we have

$$\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\bigg|_{t=t_0} \leq 0,$$

whenever $s \in [0,1] \longmapsto \frac{G_y}{G_z}(x_s, y_{t_0}, z_{t_0})$ forms an affinely parametrized line segment for some $t_0 \in [0,1]$.

(iii) For any given curve $x_s \in X$ connecting x_0 and x_1, any $(y_0, z_0), (y_1, z_1) \in cl(Y \times Z)$, we have

$$\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\bigg|_{s=s_0} \leq 0,$$

whenever $t \in [0,1] \longmapsto (G_x, G)(x_{s_0}, y_t, z_t)$ forms an affinely parametrized line segment for some $s_0 \in [0,1]$.

The proof of this theorem and its embellishments are represented in the following section:

6.2 Proof and variations on Theorem 6.1.1

Proof of Theorem 6.1.1. (i) $\Rightarrow$ (ii). Suppose for some $t_0 \in [0,1]$, $s \in [0,1] \longmapsto \frac{G_y}{G_z}(x_s, y_{t_0}, z_{t_0})$ forms an affinely parametrized line segment.

For any fixed $s_0 \in [0,1]$, consider $x_{s_0} \in X$, there is a G-segment $(x_{s_0}, y_t^{s_0}, z_t^{s_0})$ passing through $(x_{s_0}, y_{t_0}, z_{t_0})$ at $t = t_0$ with the same tangent vector as (x_{s_0}, y_t, z_t) at $t = t_0$, i.e., there exists another curve $(y_t^{s_0}, z_t^{s_0}) \in cl(Y \times Z)$, such that $(y_t^{s_0}, z_t^{s_0}) \mid_{t=t_0} = (y_t, z_t) \mid_{t=t_0}$, $(\dot{y_t}^{s_0}, \dot{z_t}^{s_0}) \mid_{t=t_0} \| (\dot{y}_t, \dot{z}_t) \mid_{t=t_0}$, and $(G_x, G)(x_{s_0}, y_t^{s_0}, z_t^{s_0}) = (1-t)(G_x, G)(x_{s_0}, y_0^{s_0}, z_0^{s_0}) + t(G_x, G)(x_{s_0}, y_1^{s_0}, z_1^{s_0})$.

Computing the fourth mixed derivative yields

$$\begin{aligned}
&\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\\
=\ & \frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z}\right)\frac{\partial^2}{\partial t^2}G + 2\frac{\partial}{\partial s}\left(\frac{1}{G_z}\right)\frac{\partial^3}{\partial s \partial t^2}G + \frac{1}{G_z}\frac{\partial^4}{\partial s^2 \partial t^2}G\\
=\ & [-(G_z)^{-2}G_{i,z}\ddot{x_s}^i - (G_z)^{-2}G_{ij,z}\dot{x_s}^i\dot{x_s}^j + 2(G_z)^{-3}G_{i,z}G_{j,z}\dot{x_s}^i\dot{x_s}^j] \cdot [G_{,k}\ddot{y_t}^k + G_z\ddot{z}_t + G_{,kl}\dot{y_t}^k\dot{y_t}^l\\
&\quad + 2G_{,kz}\dot{y_t}^k\dot{z}_t + G_{zz}(\dot{z}_t)^2]\\
& + 2[-(G_z)^{-2}G_{i,z}\dot{x_s}^i] \cdot [G_{j,k}\dot{x_s}^j\ddot{y_t}^k + G_{j,z}\dot{x_s}^j\ddot{z}_t + G_{j,kl}\dot{x_s}^j\dot{y_t}^k\dot{y_t}^l + 2G_{j,kz}\dot{x_s}^j\dot{y_t}^k\dot{z}_t + G_{j,zz}\dot{x_s}^j(\dot{z}_t)^2]\\
& + (G_z)^{-1}[G_{i,k}\ddot{x_s}^i\ddot{y_t}^k + G_{ij,k}\dot{x_s}^i\dot{x_s}^j\ddot{y_t}^k + G_{i,z}\ddot{x_s}^i\ddot{z}_t + G_{ij,z}\dot{x_s}^i\dot{x_s}^j\ddot{z}_t + G_{i,kl}\ddot{x_s}^i\dot{y_t}^k\dot{y_t}^l + G_{ij,kl}\dot{x_s}^i\dot{x_s}^j\dot{y_t}^k\dot{y_t}^l\\
&\quad + 2G_{i,kz}\ddot{x_s}^i\dot{y_t}^k\dot{z}_t + 2G_{ij,kz}\dot{x_s}^i\dot{x_s}^j\dot{y_t}^k\dot{z}_t + G_{i,zz}\ddot{x_s}^i(\dot{z}_t)^2 + G_{ij,zz}\dot{x_s}^i\dot{x_s}^j(\dot{z}_t)^2]\\
=\ & [((G_z)^{-1}G_{i,k} - (G_z)^{-2}G_{i,z}G_{,k})\ddot{x_s}^i + ((G_z)^{-1}G_{ij,k} - (G_z)^{-2}G_{,k}G_{ij,z} - 2(G_z)^{-2}G_{i,z}G_{j,k}\\
&\quad + 2(G_z)^{-3}G_{,k}G_{i,z}G_{j,z})\dot{x_s}^i\dot{x_s}^j]\ddot{y_t}^k\\
& + [(G_z)^{-1}G_{i,kl} - (G_z)^{-2}G_{i,z}G_{,kl}]\ddot{x_s}^i\dot{y_t}^k\dot{y_t}^l\\
& + [(G_z)^{-1}G_{ij,kl} - (G_z)^{-2}G_{ij,z}G_{,kl} + 2(G_z)^{-3}G_{i,z}G_{j,z}G_{,kl} - 2(G_z)^{-2}G_{i,z}G_{j,kl}]\dot{x_s}^i\dot{x_s}^j\dot{y_t}^k\dot{y_t}^l\\
& + [2(G_z)^{-1}G_{i,kz} - 2(G_z)^{-2}G_{i,z}G_{,kz}]\ddot{x_s}^i\dot{y_t}^k\dot{z}_t\\
& + [2(G_z)^{-1}G_{ij,kz} - 2(G_z)^{-2}G_{ij,z}G_{,kz} + 4(G_z)^{-3}G_{i,z}G_{j,z}G_{,kz} - 4(G_z)^{-2}G_{i,z}G_{j,kz}]\dot{x_s}^i\dot{x_s}^j\dot{y_t}^k\dot{z}_t\\
& + [(G_z)^{-1}G_{i,zz} - (G_z)^{-2}G_{i,z}G_{zz}]\ddot{x_s}^i(\dot{z}_t)^2\\
& + [(G_z)^{-1}G_{ij,zz} - (G_z)^{-2}G_{ij,z}G_{zz} + 2(G_z)^{-3}G_{i,z}G_{j,z}G_{zz} - 2(G_z)^{-2}G_{i,z}G_{j,zz}]\dot{x_s}^i\dot{x_s}^j(\dot{z}_t)^2.
\end{aligned}$$

The coefficient of $\ddot{y_t}^k$ vanishes when this expression is evaluated at $t = t_0$, due to the assumption that $s \in [0,1] \longmapsto \frac{G_y}{G_z}(x_s, y_{t_0}, z_{t_0})$ forms an affinely parametrized line segment, which implies

$$\begin{aligned} 0 = &\ \frac{\partial^2}{\partial s^2}\frac{G_y}{G_z}(x_s, y_{t_0}, z_{t_0}) \\ = &\ [((G_z)^{-1}G_{i,y} - (G_z)^{-2}G_{i,z}G_{,y})\ddot{x}^i + ((G_z)^{-1}G_{ij,y} - (G_z)^{-2}G_{,y}G_{ij,z} \\ &\quad - 2(G_z)^{-2}G_{i,z}G_{j,y} + 2(G_z)^{-3}G_{,y}G_{i,z}G_{j,z})\dot{x}^i\dot{x}^j] \end{aligned}$$

for all $s \in [0,1]$.

Since $(\dot{y_t}^{s_0}, \dot{z_t}^{s_0})|_{t=t_0} \parallel (\dot{y}_t, \dot{z}_t)|_{t=t_0}$, there exists some constant $C_1 > 0$, such that $(\dot{y}_t, \dot{z}_t)|_{t=t_0} = C_1(\dot{y_t}^{s_0}, \dot{z_t}^{s_0})|_{t=t_0}$. Moreover, since $(y_t, z_t)\ |_{t=t_0} = (y_t^{s_0}, z_t^{s_0})\ |_{t=t_0}$, we have

$$\begin{aligned} &\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\Bigg|_{t=t_0} \\ = &\ C_1^2\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t^{s_0}, z_t^{s_0})}\frac{\partial^2}{\partial t^2}G(x_s, y_t^{s_0}, z_t^{s_0})\right)\Bigg|_{t=t_0}. \end{aligned}$$

Denote $g(s) := \frac{\partial^2}{\partial t^2}|_{t=t_0}G(x_s, y_t^{s_0}, z_t^{s_0})$, for $s \in [0,1]$. Since $(x_{s_0}, y_t^{s_0}, z_t^{s_0})$ is a G-segment, by (G3), we have $g(s) \geq 0$, for all $s \in [0,1]$. By definition of $(y_t^{s_0}, z_t^{s_0})$, it is clear that $g(s_0) = 0$. The first- and second-order conditions for an interior minimum then give $g'(s_0) = 0 \leq g''(s_0)$; (in fact $g'(s_0) = 0$ also follows directly from the definition of a G-segment).

By the assumption (G4), we have $G_z < 0$, thus,

$$\begin{aligned} &\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\Bigg|_{(s,t)=(s_0,t_0)} \\ = &\ C_1^2\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t^{s_0}, z_t^{s_0})}\frac{\partial^2}{\partial t^2}G(x_s, y_t^{s_0}, z_t^{s_0})\right)\Bigg|_{(s,t)=(s_0,t_0)} \\ = &\ C_1^2\frac{\partial^2}{\partial s^2}\Bigg|_{(s,t)=(s_0,t_0)}\left(\frac{1}{G_z}(x_s, y_t^{s_0}, z_t^{s_0})\right)g(s_0) \\ &\ + 2C_1^2\frac{\partial}{\partial s}\Bigg|_{(s,t)=(s_0,t_0)}\left(\frac{1}{G_z(x_s, y_t^{s_0}, z_t^{s_0})}\right)g'(s_0) + \frac{C_1^2}{G_z(x_s, y_t^{s_0}, z_t^{s_0})}g''(s_0) \\ \leq &\ 0. \end{aligned}$$

(ii) $\Rightarrow$ (i). For any fixed $x_0 \in X$ and G-segment (x_0, y_t, z_t), we need to show $\frac{\partial^2}{\partial t^2}G(x_1, y_t, z_t) \geq 0$, for all $t \in [0,1]$ and $x_1 \in X$.

For any fixed $t_0 \in [0,1]$ and $x_1 \in X$, define x_s as the solution $\hat{x}$ to the equation

$$\frac{G_y}{G_z}(\hat{x}, y_{t_0}, z_{t_0}) = (1-s)\frac{G_y}{G_z}(x_0, y_{t_0}, z_{t_0}) + s\frac{G_y}{G_z}(x_1, y_{t_0}, z_{t_0}). \tag{6.2.1}$$

By (G7) and (G8), x_s is uniquely determined for each $s \in (0,1)$. In addition, x_0 and x_1 satisfy the above equation for $s = 0$ and $s = 1$, respectively.

Define $g(s) := \frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\Big|_{t=t_0}$ for $s \in [0,1]$.

Then $g(0) = 0 = g'(0)$ from the two conditions defining a G-segment.

In our setting, $s \in [0,1] \longmapsto \frac{G_y}{G_z}(x_s, y_{t_0}, z_{t_0})$ forms an affinely parametrized line segment, thus $0 \geq \frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s,y_t,z_t)}\frac{\partial^2}{\partial t^2}G(x_s,y_t,z_t)\right)\Big|_{t=t_0} = g''(s)$ for all $s \in [0,1]$ by hypothesis (ii).

Hence g is concave in $[0,1]$, and $g(0) = 0$ is a critical point, thus $g(1) \leq 0$. Since $G_z < 0$ this implies $\frac{\partial^2}{\partial t^2}\Big|_{t=t_0} G(x_1, y_t, z_t) \geq 0$ for any $t_0 \in [0,1]$ and $x_1 \in X$, as desired.

(i) $\Rightarrow$ (iii). For any fixed $s_0 \in [0,1]$, suppose $t \in [0,1] \longmapsto (G_x, G)(x_{s_0}, y_t,\ z_t)$ forms an affinely parametrized line segment. For any fixed $t_0 \in [0,1]$, define $g(s) := \left(\frac{1}{G_z(x_s,y_t,z_t)}\frac{\partial^2}{\partial t^2}G(x_s,y_t,z_t)\right)\Big|_{t=t_0}$, for all $s \in [0,1]$. By (G3)-(G4), we know $g(s) \leq 0$, for all $s \in [0,1]$. By the definition of (y_t, z_t), we have $g(s_0) = g'(s_0) = 0$. Thus $g''(s_0) \leq 0$.

(iii) $\Rightarrow$ (i). For any fixed $x_0 \in X$, suppose (x_0, y_t^0, z_t^0) is a G-segment, then we need to show $\frac{\partial^2}{\partial t^2}G(x_1, y_t^0, z_t^0) \geq 0$, for all $t \in [0,1], x_1 \in X$.

For any fixed $t_0 \in [0,1]$, $x_1 \in X$, define x_s as the solution $\hat{x}$ of equation

$$\frac{G_y}{G_z}(\hat{x}, y_{t_0}^0, z_{t_0}^0) = (1-s)\frac{G_y}{G_z}(x_0, y_{t_0}^0, z_{t_0}^0) + s\frac{G_y}{G_z}(x_1, y_{t_0}^0, z_{t_0}^0).$$

By (G7) and (G8), x_s is uniquely determined for each $s \in (0,1)$. In addition, x_0 and x_1 satisfy the above equation for $s = 0$ and $s = 1$ respectively.

Define $g(s) := \frac{1}{G_z(x_s,y_t^0,z_t^0)}\frac{\partial^2}{\partial t^2}G(x_s, y_t^0, z_t^0)\Big|_{t=t_0}$, for $s \in [0,1]$.

Then $g(0) = g'(0) = 0$ by the two conditions defining a G-segment.

For any fixed $s_0 \in [0,1]$, there is a G-segment $(x_{s_0}, y_t^{s_0}, z_t^{s_0})$ passing through $(x_{s_0}, y_{t_0}^0, z_{t_0}^0)$ at $t = t_0$ with the same tangent vector as (x_{s_0}, y_t^0, z_t^0) at $t = t_0$, i.e., there exists another curve $(y_t^{s_0}, z_t^{s_0}) \in cl(Y \times Z)$ and some constant $C_2 > 0$, such that $(y_t^{s_0}, z_t^{s_0})\mid_{t=t_0} = (y_t^0, z_t^0)\mid_{t=t_0}$, $(\dot{y}_t{}^{s_0}, \dot{z}_t{}^{s_0})\mid_{t=t_0} = \frac{1}{C_2}(\dot{y}_t^0, \dot{z}_t^0)\mid_{t=t_0}$, and $(G_x, G)(x_{s_0}, y_t^{s_0}, z_t^{s_0}) = (1-t)(G_x, G)(x_{s_0}, y_0^{s_0}, z_0^{s_0}) + t(G_x, G)(x_{s_0}, y_1^{s_0}, z_1^{s_0})$.

Computing the mixed fourth derivative yields

$$\begin{aligned}
&\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t^0, z_t^0)}\frac{\partial^2}{\partial t^2}G(x_s, y_t^0, z_t^0)\right)\Bigg|_{(s,t)=(s_0,t_0)} \\
=\ & C_2^2\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t^{s_0}, z_t^{s_0})}\frac{\partial^2}{\partial t^2}G(x_s, y_t^{s_0}, z_t^{s_0})\right)\Bigg|_{(s,t)=(s_0,t_0)},
\end{aligned}$$

where the equality is derived from the condition that $s \in [0,1] \longmapsto \frac{G_y}{G_z}(x_s, y_{t_0},\ z_{t_0})$ forms an affinely parametrized line segment, $(y_t^{s_0}, z_t^{s_0})\mid_{t=t_0} = (y_t, z_t)\mid_{t=t_0}$ and $(\dot{y}_t^0, \dot{z}_t^0)\mid_{t=t_0} = C_2(\dot{y}_t{}^{s_0}, \dot{z}_t{}^{s_0})\mid_{t=t_0}$. Moreover, the latter expression is non-positive by assumption (iii).

Thus $g''(s_0) \leq 0$ for all $s_0 \in [0,1]$. Since g is concave in $[0,1]$, and $g(0) = 0$ is a critical point, we have $g(1) \leq 0$. Thus $G_z < 0$ implies $\frac{\partial^2}{\partial t^2}\Big|_{t=t_0} G(x_1, y_t^0, z_t^0) \geq 0$ for all $t_0 \in [0,1]$ and $x_1 \in X$, as desired. $\square$

For strictly concavity of the profit functional, one might need a strict version of hypothesis (G3):

$(G3)_s$ For each $x, x_0 \in X$ and $x \neq x_0$, assume $t \in [0,1] \longmapsto G(x, y_t, z_t)$ is strictly convex along all G-segments (x_0, y_t, z_t) defined in (4.2.1).

Remark 6.2.1. Strict inequality in (ii) [or (iii)] implies $(G3)_s$ but the reverse is not necessarily true, i.e. $(G3)_s$ is intermediate in strength between (G3) and strict inequality version of (ii) [or (iii)]. Besides, strict inequality versions of (ii) and (iii) are equivalent, and denoted by $(G3)_u$.

Note inequality (6.2.2) below and its strict and uniform versions $(\mathrm{G3})_s$ and $(\mathrm{G3})_u$ precisely generalize of the analogous hypotheses $(B3)$, $(B3)_s$ and $(B3)_u$ from the quasilinear case in [11].

Proof. We only show strict inequality of (ii) implies that of (iii) here since the other direction is similar.

For any fixed $s_0 \in [0,1]$, suppose $t \in [0,1] \longmapsto (G_x, G)(x_{s_0}, y_t, z_t)$ forms an affinely parametrized line segment. For any fixed $t_0 \in [0,1]$, define $x_s^{t_0}$ as a solution to the equation $\frac{G_y}{G_z}(x_s^{t_0}, y_{t_0}, z_{t_0}) = (1-s)\frac{G_y}{G_z}(x_0^{t_0}, y_{t_0}, z_{t_0}) + s\frac{G_y}{G_z}(x_1^{t_0}, y_{t_0}, z_{t_0})$, with initial conditions $x_s^{t_0}|_{s=s_0} = x_{s_0}$ and $\dot{x}_s^{t_0}|_{s=s_0} = C_1 \cdot \dot{x}_s|_{s=s_0}$, for some constant $C_1 > 0$. Thus, by strict inequality of (ii), we have

$$\begin{aligned}
0 > \ & \frac{\partial^2}{\partial s^2}\Big(\frac{1}{G_z(x_s^{t_0}, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s^{t_0}, y_t, z_t)\Big)\Big|_{(s,t)=(s_0,t_0)} \\
= \ & \frac{\partial^2}{\partial s^2}\Big(\frac{1}{G_z(x_s^{t_0}, y_t, z_t)}\Big)\frac{\partial^2}{\partial t^2}G(x_s^{t_0}, y_t, z_t)\Big|_{(s,t)=(s_0,t_0)} \\
& + \frac{\partial}{\partial s}\Big(\frac{1}{G_z(x_s^{t_0}, y_t, z_t)}\Big)\frac{\partial^3}{\partial s \partial t^2}G(x_s^{t_0}, y_t, z_t)\Big|_{(s,t)=(s_0,t_0)} \\
& + \frac{1}{G_z(x_s^{t_0}, y_t, z_t)}\frac{\partial^4}{\partial s^2 \partial t^2}G(x_s^{t_0}, y_t, z_t)\Big|_{(s,t)=(s_0,t_0)} \\
= \ & -\frac{G_{x,z}(x_s^{t_0}, y_t, z_t)}{G_z^2(x_s^{t_0}, y_t, z_t)}\frac{\partial^2}{\partial t^2}G_x(x_s^{t_0}, y_t, z_t)(\dot{x}_s^{t_0})^2\Big|_{(s,t)=(s_0,t_0)} \\
& + \frac{1}{G_z(x_s^{t_0}, y_t, z_t)}\frac{\partial^2}{\partial t^2}G_{xx}(x_s^{t_0}, y_t, z_t)(\dot{x}_s^{t_0})^2\Big|_{(s,t)=(s_0,t_0)} \\
= \ & C_1^2\Big[-\frac{G_{x,z}(x_s, y_t, z_t)}{G_z^2(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G_x(x_s, y_t, z_t)(\dot{x}_s)^2\Big|_{(s,t)=(s_0,t_0)} \\
& + \frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G_{xx}(x_s, y_t, z_t)(\dot{x}_s)^2\Big|_{(s,t)=(s_0,t_0)}\Big] \\
= \ & C_1^2\frac{\partial^2}{\partial s^2}\Big(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\Big)\Big|_{(s,t)=(s_0,t_0)}.
\end{aligned}$$

Here we use the initial condition $x_s^{t_0}|_{s=s_0} = x_{s_0}$ and $\dot{x}_s^{t_0}|_{s=s_0} = C_1\dot{x}_s|_{s=s_0}$. Besides, since (x_{s_0}, y_t, z_t) forms a G-segment, therefore we have

$$\frac{\partial^2}{\partial t^2}G(x_s^{t_0}, y_t, z_t)\Big|_{(s,t)=(s_0,t_0)} = \frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\Big|_{(s,t)=(s_0,t_0)} = 0,$$

$$\text{and} \quad \frac{\partial^2}{\partial t^2}G_x(x_s^{t_0}, y_t, z_t)\Big|_{(s,t)=(s_0,t_0)} = \frac{\partial^2}{\partial t^2}G_x(x_s, y_t, z_t)\Big|_{(s,t)=(s_0,t_0)} = 0.$$

From the above inequality and $C_1 > 0$, one has

$$\frac{\partial^2}{\partial s^2}\Big(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\Big)\Big|_{(s,t)=(s_0,t_0)} < 0,$$

whenever $\dot{x_s}|_{s=s_0}$ and $(\dot{y_t}, \dot{z_t})|_{t=t_0}$ are nonzero. Since this inequality holds for each fixed $t_0 \in [0,1]$, the strict version of (iii) is proved. □

Combining (ii) and (iii), one can conclude they are also equivalent to the following statement (iv).

Corollary 6.2.2. *Assuming* (G0)-(G2), (G4)-(G8) and $G \in C^4(cl(X \times Y \times Z))$, then (G3) is equivalent to the following statement:

(iv) For any given curve $x_s \in X$ connecting x_0 and x_1, and any curve $(y_t, z_t) \in cl(Y \times Z)$ connecting (y_0, z_0) and (y_1, z_1), we have

$$\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\Bigg|_{(s,t)=(s_0,t_0)} \leq 0, \tag{6.2.2}$$

whenever either of the two curves $t \in [0,1] \longmapsto (G_x, G)(x_{s_0}, y_t, z_t)$ and $s \in [0,1] \longmapsto \frac{G_y}{G_z}(x_s, y_{t_0}, z_{t_0})$ forms an affinely parametrized line segment.

Chapter 7

Geometric re-expression of condition (G3)

7.1 Introduction

As part of optimal transport theory, Ma-Trudinger-Wang [21] in 2005 gave sufficient conditions on a transportation cost to guarantee smoothness of the optimal transportation map, while Loeper [20] showed these conditions are also necessary. In 2010, Kim-McCann [18] expressed them via non-negativity of the sectional curvature of certain null-planes in a novel but natural pseudo-Riemannian geometry which was induced by the cost function on some product space.

In Chapter 6, we show that (G3) is, in fact, a fourth order condition in the spirit of the Ma-Trudinger-Wang condition. Inspired by Kim-McCann [18], we will show in this chapter a geometric representation of condition (G3), which is non-negativity of the sectional curvature in a pseudo-Riemannian geometry induced by the utility $\bar{G}$, up to the additional variable, on the product space $X \times \mathbf{R} \times Y \times \mathbf{R}$.

7.2 Context

In this section, we will define the pseudo-Riemannian metric g and calculate the Christoffel symbols Γ and the curvature tensor R.

Let $\bar{G}(x, w, y, z) = wG(x, y, z)$, $\bar{x} = (x, w)$, $\bar{y} = (y, z)$, and $\delta(\bar{x}, \bar{y}, \bar{x}_0, \bar{y}_0) = -\bar{G}(\bar{x}, \bar{y}) - \bar{G}(\bar{x}_0, \bar{y}_0) + \bar{G}(\bar{x}, \bar{y}_0) + \bar{G}(\bar{x}_0, \bar{y})$.

For some fixed $(\bar{x}_0, \bar{y}_0)$, one can view the δ defined above as a function of variable $\Xi = (\bar{x}, \bar{y})$ on the space $M = X \times \mathbf{R} \times Y \times \mathbf{R}$, with the following first order derivatives.

$$\nabla_i \delta(\bar{x}, \bar{y}, \bar{x}_0, \bar{y}_0) = \begin{cases} -\bar{G}_{i,}(\bar{x}, \bar{y}) + \bar{G}_{i,}(\bar{x}, \bar{y}_0), & i \leq n+1 \\ -\bar{G}_{,\bar{i}}(\bar{x}, \bar{y}) + \bar{G}_{,\bar{i}}(\bar{x}_0, \bar{y}), & i > n+1 \end{cases} \tag{7.2.1}$$

Here we adopt this notation $\bar{i} := i - (n+1)$ for $n+1 < i \leq 2(n+1)$. As a notation convention, we use a comma to separate subscripts of $\bar{G}$, which correspond to the derivatives with respect to the variables $\bar{x}$ in spaces $X \times \mathbf{R}$ (before comma) and $\bar{y}$ in $Y \times \mathbf{R}$ (after comma), respectively.

If $G \in C^2$, for each fixed $(\bar{x}_0, \bar{y}_0)$, since the first derivative of δ vanishes at $(\bar{x}, \bar{y}) = (\bar{x}_0, \bar{y}_0)$, the second order derivatives of δ at $\Xi_0 = (\bar{x}_0, \bar{y}_0)$ is well-defined and can be written as follows.

$$\nabla_{ij}\delta(\bar{x}, \bar{y}, \bar{x}_0, \bar{y}_0)\big|_{(\bar{x},\bar{y})=(\bar{x}_0,\bar{y}_0)} = \begin{cases} -\bar{G}_{i,\bar{j}}(\bar{x}_0, \bar{y}_0), & i \le n+1 < j \\ -\bar{G}_{j,\bar{i}}(\bar{x}_0, \bar{y}_0), & i > n+1 \ge j \\ 0, & \text{otherwise} \end{cases} \tag{7.2.2}$$

Let $T_{\Xi_0}M$ denote the tangent space to M at Ξ_0. Define the pseudo-Riemannian metric $g_{\Xi_0} : T_{\Xi_0}M \times T_{\Xi_0}M \to \mathbf{R}$ at Ξ_0 on M to be the above $2(n+1) \times 2(n+1)$ symmetric matrix.

One can calculate the Christoffel symbols using the following formula with Einstein summation convention where g^{ml} is the inverse matrix of g:

$$\Gamma_{ij}^m = \frac{1}{2} g^{ml} \left(\frac{\partial g_{il}}{\partial \Xi_j} + \frac{\partial g_{jl}}{\partial \Xi_i} - \frac{\partial g_{ij}}{\partial \Xi_l} \right). \tag{7.2.3}$$

Here is a calculation of each component:

$$g_{il;j} := \frac{\partial g_{il}}{\partial \Xi_j} = \begin{cases} -\bar{G}_{i,\bar{j}\bar{l}}, & i \le n+1 < j, l; \\ -\bar{G}_{ij,\bar{l}}, & i, j \le n+1 < l; \\ -\bar{G}_{l,\bar{i}\bar{j}}, & l \le n+1 < i, j; \\ -\bar{G}_{jl,\bar{i}}, & j, l \le n+1 < i; \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.4}$$

Similarly, one has

$$g_{lj;i} := \frac{\partial g_{lj}}{\partial \Xi_i} = \begin{cases} -\bar{G}_{j,\bar{i}\bar{l}}, & j \le n+1 < i, l; \\ -\bar{G}_{ij,\bar{l}}, & i, j \le n+1 < l; \\ -\bar{G}_{l,\bar{i}\bar{j}}, & l \le n+1 < i, j; \\ -\bar{G}_{il,\bar{j}}, & i, l \le n+1 < j; \\ 0 & \text{otherwise;} \end{cases} \tag{7.2.5}$$

$$g_{ij;l} := \frac{\partial g_{ij}}{\partial \Xi_l} = \begin{cases} -\bar{G}_{i,\bar{j}\bar{l}}, & i \le n+1 < j, l; \\ -\bar{G}_{il,\bar{j}}, & i, l \le n+1 < j; \\ -\bar{G}_{j,\bar{i}\bar{l}}, & j \le n+1 < i, l; \\ -\bar{G}_{jl,\bar{i}}, & j, l \le n+1 < i; \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.6}$$

Therefore, putting together these three terms, one has

$$\frac{\partial g_{il}}{\partial \Xi_j} + \frac{\partial g_{lj}}{\partial \Xi_i} - \frac{\partial g_{ij}}{\partial \Xi_l} = \begin{cases} -2\bar{G}_{ij,\bar{l}}, & i, j \le n+1 < l; \\ -2\bar{G}_{l,\bar{i}\bar{j}}, & i, j > n+1 \ge l; \\ 0, & \text{otherwise.} \end{cases} \tag{7.2.7}$$

Denote $\bar{G}^{m,\bar{l}}$ as the inverse matrix of $\bar{G}_{k,\bar{l}}$. Since $g^{ml}g_{lk} = g_k^m$, one has

$$g^{ml} = \begin{cases} -\bar{G}^{m,\bar{l}}, & m \le n+1 < l; \\ -\bar{G}^{l,\bar{m}}, & l \le n+1 < m; \\ 0, & \text{otherwise}. \end{cases} \tag{7.2.8}$$

Therefore, the Christoffel symbols could be represented as follows.

$$\begin{aligned} \Gamma_{ij}^m &= \frac{1}{2} g^{ml}(g_{il;j} + g_{lj;i} - g_{ij;l}) \\ &= \begin{cases} \bar{G}^{m,\bar{l}} \bar{G}_{ij,\bar{l}}, & i,j,m \le n+1 < l; \\ \bar{G}^{l,\bar{m}} \bar{G}_{l,\bar{i}\bar{j}}, & i,j,m > n+1 \ge l; \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{7.2.9}$$

Then one can calculate the pseudo-Riemannian curvature tensor R.

$$R_{ijkl} = g_{im} R_{jkl}^m \tag{7.2.10}$$

$$= -g_{im}\left[\frac{\partial}{\partial \Xi_l}\Gamma_{jk}^m - \frac{\partial}{\partial \Xi_k}\Gamma_{jl}^m + \Gamma_{l\alpha}^m \Gamma_{jk}^\alpha - \Gamma_{k\alpha}^m \Gamma_{jl}^\alpha\right] \tag{7.2.11}$$

$$= \begin{cases} -\bar{G}_{il,\alpha}\bar{G}^{\beta,\alpha}\bar{G}_{\beta,\bar{j}\bar{k}} + \bar{G}_{il,\bar{j}\bar{k}}, & i,l \le n+1 < k,j; \\ \bar{G}_{ik,\alpha}\bar{G}^{\beta,\alpha}\bar{G}_{\beta,\bar{j}\bar{l}} - \bar{G}_{ik,\bar{j}\bar{l}}, & i,k \le n+1 < l,j; \\ -\bar{G}_{jk,\alpha}\bar{G}^{\beta,\alpha}\bar{G}_{\beta,\bar{i}\bar{l}} + \bar{G}_{jk,\bar{i}\bar{l}}, & j,k \le n+1 < l,i; \\ \bar{G}_{jl,\alpha}\bar{G}^{\beta,\alpha}\bar{G}_{\beta,\bar{i}\bar{k}} - \bar{G}_{jl,\bar{i}\bar{k}}, & j,l \le n+1 < k,i; \\ 0, & \text{otherwise}. \end{cases} \tag{7.2.12}$$

7.3 G-segments are geodesics

Definition 7.3.1 (G-segment with the notion of additional variable)**.** For each $\bar{x}_0 = (x_0, w_0) \in X \times \mathbf{R}$, $\bar{y}_0, \bar{y}_1 \in cl(Y \times Z)$ with $w_0 \ne 0$, define $\bar{y}_t \in cl(Y \times Z)$ such that the following equation

$$D_{\bar{x}}\bar{G}(\bar{x}_0, \bar{y}_t) = (1-t)D_{\bar{x}}\bar{G}(\bar{x}_0, \bar{y}_0) + tD_{\bar{x}}\bar{G}(\bar{x}_0, \bar{y}_1) \tag{7.3.1}$$

holds for each $t \in [0,1]$. By conditions (G1) and (G2), $\bar{y}_t$ is uniquely determined by (7.3.1). We call $t \in [0,1] \longmapsto (\bar{x}_0, \bar{y}_t)$ the G-segment connecting $(\bar{x}_0, \bar{y}_0)$ and $(\bar{x}_0, \bar{y}_1)$ on M.

For any continuous, piecewise continuously differentiable curves $\gamma : [0,1] \longrightarrow M$, let $E(\cdot)$ denote the energy functional:

$$E(\gamma) = \frac{1}{2}\int_0^1 g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt. \tag{7.3.2}$$

Then the Euler-Lagrange equations of motion for the functional E are given by

$$\frac{d^2\Xi^m}{dt^2} + \Gamma_{ij}^m \frac{d\Xi^i}{dt}\frac{d\Xi^j}{dt} = 0, \tag{7.3.3}$$

where Γ_{ij}^m is the Christoffel symbol define in (7.2.3). The above equality (7.3.3) is the so-called geodesic equation.

Proposition 7.3.2 (G-segments are geodesics)**.** *Assume* (G1) *and* (G2). *For any G-segment* $t \in [0,1] \longmapsto (\bar{x}_0, \bar{y}_t)$*, defined in Definition 7.3.1, connecting* $(\bar{x}_0, \bar{y}_0)$ *and* $(\bar{x}_0, \bar{y}_1)$ *on* M*, it satisfies the geodesic equation* (7.3.3)*.*

Proof. Since G-segment $t \in [0,1] \longmapsto (\bar{x}_0, \bar{y}_t)$ satisfies (7.3.1), the following equations hold:

$$0 = \partial_t^2 D_{\bar{x}} \bar{G}(\bar{x}_0, \bar{y}_t) \tag{7.3.4}$$

$$= D_{\bar{x}^k \bar{y}^m} \bar{G} \cdot \ddot{\bar{y}}_t^m + D_{\bar{x}^k \bar{y}^i \bar{y}^j} \bar{G} \cdot \dot{\bar{y}}_t^i \, \dot{\bar{y}}_t^j. \tag{7.3.5}$$

This implies

$$\ddot{\bar{y}}_t^m + [(D_{\bar{x}^k \bar{y}^m} \bar{G})^{-1}]^{k,m} \cdot D_{\bar{x}^k \bar{y}^i \bar{y}^j} \bar{G} \cdot \dot{\bar{y}}_t^i \, \dot{\bar{y}}_t^j = 0. \tag{7.3.6}$$

Rewrite the above equation in terms of variable Ξ, then one has

$$\ddot{\Xi}_t^m + \bar{G}^{k,\bar{m}} \cdot \bar{G}_{k,\bar{i}\bar{j}} \cdot \dot{\Xi}_t^i \, \dot{\Xi}_t^j = 0, \text{ where } k \leq n+1 < i, j, m. \tag{7.3.7}$$

Combining with (7.2.9), this implies the geodesic equation (7.3.3). □

7.4 Condition (G3) is a non-negative sectional curvature condition

Recall that in concavity arguments in Chapter 5, condition (G3) plays the most important role. In Section 6.1, we introduced the fourth-order differential re-expression of condition (G3). One may also wonder what the geometric meaning of the (G3) condition is. In this section, we are going to show the geometric re-expression of condition (G3).

(G3) For any $\bar{x} = (x, w)$ with $w > 0$, assume $\partial_t^2 \bar{G}(\bar{x}, \bar{y}_t) \geq 0$, whenever there exists $\bar{x}_0 = (x_0, 1)$ such that $\partial_t^2 D_{\bar{x}} \bar{G}(\bar{x}_0, \bar{y}_t) = 0$.

For the pseudo-Riemannian metric tensor g defined in (7.2.2) and any two tangent vectors $P, Q \in T_{\Xi_0} M$, define the unnormalized sectional curvature at the point $\Xi_0 \in M$ as

$$\sec_{\Xi_0}^{(M,g)} P \wedge Q := R_{ijkl}(\Xi_0) \cdot P^i \cdot P^l \cdot Q^j \cdot Q^k, \tag{7.4.1}$$

where R is the curvature tensor shown in (7.2.12).

The following theorem describes equivalent expressions of the (G3) condition. Part (i) and (iii) are taken from Theorem 6.1.1. While keeping the ordering number of statements from Chapter 6 (Theorem 6.1.1 and Corollary 6.2.2), part (v) is a variation of (iii) by rewriting (G3) condition with the notion of the additional variable, and part (vi) is the non-negative sectional curvature condition on the manifold M defined in Section 7.2.

Theorem 7.4.1. *Assume* (G0)-(G2) *and* (G4)-(G8). If, in addition, $G \in C^4(cl(X \times Y \times Z))$, then the following statements are equivalent:

(i) (G3).

(iii) For any given curve $x_s \in X$ connecting x_0 and x_1, any $(y_0, z_0), (y_1, z_1) \in cl(Y \times Z)$, we have

$$\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\Bigg|_{s=s_0} \leq 0, \tag{7.4.2}$$

whenever $t \in [0,1] \longmapsto (G_x, G)(x_{s_0}, y_t, z_t)$ forms an affinely parametrized line segment for some $s_0 \in [0,1]$.

(v) Let $s \in [0,1] \longmapsto w_s \in (0,\infty)$, for any given curve $x_s \in X$ connecting x_0 and x_1, any $\bar{y}_0, \bar{y}_1 \in cl(Y \times Z)$, we have

$$\frac{\partial^4}{\partial s^2 \partial t^2}\Bigg|_{s=s_0} \bar{G}(\bar{x}_s, \bar{y}_t) \geq 0, \tag{7.4.3}$$

whenever $t \in [0,1] \longmapsto (\bar{x}_{s_0}, \bar{y}_t)$ forms a G-segment for some $s_0 \in [0,1]$.

(vi) For any vectors $P = p \oplus 0$, $Q = 0 \oplus q \in \mathbf{R}^{2n+2}$, where $p, q, 0 \in \mathbf{R}^{n+1}$ and $\oplus$ represents the direct sum, and $\Xi_\emptyset = (x_\emptyset, w_\emptyset, \bar{y}_\emptyset) \in M$ with $w_\emptyset \in (0,\infty)$, the sectional curvature satisfies

$$\sec^{(M,g)}_{\Xi_\emptyset} P \wedge Q \geq 0. \tag{7.4.4}$$

Proof. (i) and (iii) are equivalent from Theorem 6.1.1. Only need to show the equivalences of (iii) and (v), (v) and (vi).

(iii) $\Rightarrow$ (v). Let x_0, x_1 be any two points on X, (y_0, z_0), (y_1, z_1) be any two points on $cl(Y \times Z)$, w_s be any curve on $(0,\infty)$, $x_s \in X$ be any curve connecting x_0 and x_1, and $(y_t, z_t) \in cl(Y \times Z)$ be any curve connecting (y_0, z_0) and (y_1, z_1). Suppose there exists $s_0 \in [0,1]$, such that $t \in [0,1] \longmapsto (\bar{x}_{s_0}, \bar{y}_t)$ forms a G-segment. Then from (7.3.4), we know $t \in [0,1] \longmapsto (G_x, G)(x_{s_0}, y_t, z_t)$ forms an affinely parametrized line segment, i.e.,

$$0 = \frac{\partial^2}{\partial t^2}G_x(x_{s_0}, y_t, z_t), \tag{7.4.5}$$

$$0 = \frac{\partial^2}{\partial t^2}G(x_{s_0}, y_t, z_t). \tag{7.4.6}$$

Therefore, one has

$$0 \geq \frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)}\frac{\partial^2}{\partial t^2}G(x_s, y_t, z_t)\right)\Bigg|_{s=s_0} \tag{7.4.7}$$

$$= \frac{1}{G_z(x_{s_0}, y_t, z_t)}\frac{\partial^4}{\partial s^2 \partial t^2}\Bigg|_{s=s_0} G(x_s, y_t, z_t) + \frac{\partial^2}{\partial s^2}\Bigg|_{s=s_0}\frac{1}{G_z}(x_s, y_t, z_t) \cdot \frac{\partial^2}{\partial t^2}G(x_{s_0}, y_t, z_t) \tag{7.4.8}$$

$$+ 2\frac{\partial}{\partial s}\Bigg|_{s=s_0}\frac{1}{G_z(x_s, y_t, z_t)} \cdot \frac{\partial^2}{\partial t^2}G_x(x_{s_0}, y_t, z_t) \cdot \dot{x}_s\Bigg|_{s=s_0} \tag{7.4.9}$$

$$= \frac{1}{G_z(x_{s_0}, y_t, z_t)}\frac{\partial^4}{\partial s^2 \partial t^2}\Bigg|_{s=s_0} G(x_s, y_t, z_t). \tag{7.4.10}$$

Notice that $G_z < 0$ because of condition (G4). Thus, the above inequality is equivalent to

$$\left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{s=s_0} G(x_s, y_t, z_t) \geq 0. \tag{7.4.11}$$

On the other hand, for the same curves described above, since $\bar{G}(\bar{x}_s, \bar{y}_t) = \bar{G}(x_s, w_s, y_t, z_t) = w_s G(x_s, y_t, z_t)$, applying (7.4.5) and (7.4.6), one has

$$\frac{\partial}{\partial s}\bar{G}(\bar{x}_s, \bar{y}_t) = \dot{w}_s G(x_s, y_t, z_t) + w_s \frac{\partial}{\partial s} G(x_s, y_t, z_t) \tag{7.4.12}$$

$$\frac{\partial^2}{\partial s^2}\bar{G}(\bar{x}_s, \bar{y}_t) = \ddot{w}_s G(x_s, y_t, z_t) + 2\dot{w}_s \frac{\partial}{\partial s} G(x_s, y_t, z_t) + w_s \frac{\partial^2}{\partial s^2} G(x_s, y_t, z_t) \tag{7.4.13}$$

$$\left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{s=s_0} \bar{G}(\bar{x}_s, \bar{y}_t) = \ddot{w}_{s_0} \frac{\partial^2}{\partial t^2} G(x_{s_0}, y_t, z_t) + 2\dot{w}_{s_0} \dot{x}_{s_0} \frac{\partial^2}{\partial t^2} G_x(x_{s_0}, y_t, z_t) + w_{s_0} \left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{s=s_0} G(x_s, y_t, z_t) \tag{7.4.14}$$

$$= w_{s_0} \left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{s=s_0} G(x_s, y_t, z_t). \tag{7.4.15}$$

Since w_{s_0} is positive, the equation (7.4.11) is equivalent to

$$\left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{s=s_0} \bar{G}(\bar{x}_s, \bar{y}_t) \geq 0. \tag{7.4.16}$$

(v) $\Rightarrow$ (iii). Let x_0, x_1 be any two points on X, (y_0, z_0), (y_1, z_1) be any two points on $cl(Y \times Z)$, $x_s \in X$ be any curve connecting x_0 and x_1, and $(y_t, z_t) \in cl(Y \times Z)$ be any curve connecting (y_0, z_0) and (y_1, z_1), satisfying that $t \in [0,1] \longmapsto (G_x, G)(x_{s_0}, y_t, z_t)$ forms an affinely parametrized line segment. Let $w_s \equiv 1$, for all $s =\in [0,1]$. Then, by definition, $t \in [0,1] \longmapsto (\bar{x}_{s_0}, \bar{y}_t)$ forms a G-segment. From (v), one has

$$\left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{s=s_0} \bar{G}(\bar{x}_s, \bar{y}_t) \geq 0. \tag{7.4.17}$$

By the similar computations as in the first part and $G_z < 0$ because of condition (G4), one has

$$\left.\frac{\partial^2}{\partial s^2}\left(\frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t)\right)\right|_{s=s_0} = \frac{1}{G_z(x_{s_0}, y_t, z_t)} \left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{s=s_0} G(x_s, y_t, z_t) \tag{7.4.18}$$

$$= \frac{1}{G_z(x_{s_0}, y_t, z_t)} \left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{s=s_0} \bar{G}(\bar{x}_s, \bar{y}_t) \leq 0. \tag{7.4.19}$$

(v) $\Rightarrow$ (vi). Let $\bar{x}_s$ be any curve on $X \times (0, \infty)$ with $\bar{x}_s|_{s=0} = (x_\emptyset, w_\emptyset)$ and $\dot{\bar{x}}_s|_{s=0} = p$, $\bar{y}_t$ be any curve on $cl(Y \times Z)$, satisfying $\bar{y}_t|_{t=0} = \bar{y}_\emptyset$, $\dot{\bar{y}}_t|_{t=0} = q$ and the following equation

$$\partial_t^2 D_{\bar{x}} \bar{G}(\bar{x}_0, \bar{y}_t) = 0. \tag{7.4.20}$$

That is, the curve $t \in [0,1] \longmapsto (\bar{x}_0, \bar{y}_t)$ forms a G-segment. Thus by (v) we know

$$\frac{\partial^4}{\partial s^2 \partial t^2}\bigg|_{s=0} \bar{G}(\bar{x}_s, \bar{y}_t) \geq 0. \tag{7.4.21}$$

On the other hand, from (7.4.20) we know

$$\bar{G}_{i,j}(\bar{x}_0, \bar{y}_t) \cdot \ddot{\bar{y}}_t^j + \bar{G}_{i,kl}(\bar{x}_0, \bar{y}_t) \cdot \dot{\bar{y}}_t^k \cdot \dot{\bar{y}}_t^l = 0. \tag{7.4.22}$$

This implies

$$\ddot{\bar{y}}_t^j = -\bar{G}^{i,j}(\bar{x}_0, \bar{y}_t) \cdot \bar{G}_{i,kl}(\bar{x}_0, \bar{y}_t) \cdot \dot{\bar{y}}_t^k \cdot \dot{\bar{y}}_t^l. \tag{7.4.23}$$

Thus

$$\begin{aligned}
& \frac{\partial^4}{\partial s^2 \partial t^2} \bar{G}(\bar{x}_s, \bar{y}_t)\bigg|_{s=0} && (7.4.24)\\
&= \frac{\partial^2}{\partial t^2}[\bar{G}_{i,}(\bar{x}_s, \bar{y}_t) \cdot \ddot{\bar{x}}_s^i + \bar{G}_{il,}(\bar{x}_s, \bar{y}_t) \cdot \dot{\bar{x}}_s^i \cdot \dot{\bar{x}}_s^l]\bigg|_{s=0} && (7.4.25)\\
&= \frac{\partial^2}{\partial t^2} \bar{G}_{il,}(\bar{x}_0, \bar{y}_t) \cdot \dot{\bar{x}}_0^i \cdot \dot{\bar{x}}_0^l && (7.4.26)\\
&= \bar{G}_{il,j}(\bar{x}_0, \bar{y}_t) \cdot \dot{\bar{x}}_0^i \cdot \dot{\bar{x}}_0^l \cdot \ddot{\bar{y}}_t^j + \bar{G}_{il,jk}(\bar{x}_0, \bar{y}_t) \cdot \dot{\bar{x}}_0^i \cdot \dot{\bar{x}}_0^l \cdot \dot{\bar{y}}_t^j \cdot \dot{\bar{y}}_t^k && (7.4.27)\\
&= [-\bar{G}_{il,\alpha}(\bar{x}_0, \bar{y}_t) \cdot \bar{G}^{\beta,\alpha}(\bar{x}_0, \bar{y}_t) \cdot \bar{G}_{\beta,jk}(\bar{x}_0, \bar{y}_t) + \bar{G}_{il,jk}(\bar{x}_0, \bar{y}_t)] \cdot \dot{\bar{x}}_0^i \cdot \dot{\bar{x}}_0^l \cdot \dot{\bar{y}}_t^j \cdot \dot{\bar{y}}_t^k, && (7.4.28)
\end{aligned}$$

where $i, l, j, k, \alpha, \beta = 1, 2, ..., n+1$.

Denote $\Xi_t := (\bar{x}_0, \bar{y}_t)$. Since $P = \dot{\bar{x}}_0 \oplus 0$, one can rewrite (7.4.28) as

$$\begin{aligned}
& \sum_{i,l=1}^{n+1} \sum_{j,k=n+2}^{2n+2} \sum_{\alpha,\beta=1}^{n+1} [-\bar{G}_{il,\alpha}(\Xi_t) \cdot \bar{G}^{\beta,\alpha}(\Xi_t) \cdot \bar{G}_{\beta,\bar{j}\bar{k}}(\Xi_t) + \bar{G}_{il,\bar{j}\bar{k}}(\Xi_t)] \cdot P^i \cdot P^l \cdot \dot{\Xi}_t^j \cdot \dot{\Xi}_t^k && (7.4.29)\\
&= \sum_{i,l=1}^{n+1} \sum_{j,k=n+2}^{2n+2} R_{ijkl}(\Xi_t) \cdot P^i \cdot P^l \cdot \dot{\Xi}_t^j \cdot \dot{\Xi}_t^k && (7.4.30)\\
&= \sum_{i,l,j,k=1}^{2n+2} R_{ijkl}(\Xi_t) \cdot P^i \cdot P^l \cdot \dot{\Xi}_t^j \cdot \dot{\Xi}_t^k && (7.4.31)\\
&= \sec_{\Xi_t}^{(M,g)} P \wedge \dot{\Xi}_t. && (7.4.32)
\end{aligned}$$

Therefore, equation (7.4.21) implies

$$\sec_{\Xi_t}^{(M,g)} P \wedge \dot{\Xi}_t \geq 0. \tag{7.4.33}$$

In particular, for $t = 0$, since $Q = \dot{\Xi}_0 \in \mathbf{R}^{2n+2}$,

$$\sec_{\Xi_0}^{(M,g)} P \wedge Q \geq 0. \tag{7.4.34}$$

(vi) $\Rightarrow$ (v). Let x_0, x_1 be any two points on X, (y_0, z_0), (y_1, z_1) be any two points on $cl(Y \times Z)$, w_s be any curve on $(0, \infty)$, $x_s \in X$ be any curve connecting x_0 and x_1, and $(y_t, z_t) \in cl(Y \times Z)$ be any curve connecting (y_0, z_0) and (y_1, z_1). Suppose there exists $s_0 \in [0, 1]$, such that $t \in [0, 1] \longmapsto (\bar{x}_{s_0}, \bar{y}_t)$ forms a G-segment. Then from (7.3.4), we know

$$\bar{G}_{i,j}(\bar{x}_{s_0}, \bar{y}_t) \cdot \ddot{\bar{y}}_t^j + \bar{G}_{i,kl}(\bar{x}_{s_0}, \bar{y}_t) \cdot \dot{\bar{y}}_t^k \cdot \dot{\bar{y}}_t^l = 0. \tag{7.4.35}$$

$$\ddot{\bar{y}}_t^j = -\bar{G}^{i,j}(\bar{x}_{s_0}, \bar{y}_t) \cdot \bar{G}_{i,kl}(\bar{x}_{s_0}, \bar{y}_t) \cdot \dot{\bar{y}}_t^k \cdot \dot{\bar{y}}_t^l. \tag{7.4.36}$$

Denote $\Xi_t := (\bar{x}_{s_0}, \bar{y}_t)$. For any fixed $t_0 \in [0, 1]$, let $P = \dot{\bar{x}}_{s_0} \oplus 0$, $Q = \dot{\Xi}_{t_0} = 0 \oplus \dot{\bar{y}}_{t_0} \in \mathbf{R}^{2n+2}$. Thus from part (vi),

$$\sec_{\Xi_{t_0}}^{(M,g)} P \wedge Q \geq 0. \tag{7.4.37}$$

On the other hand, with similar calculations as above, one has

$$\frac{\partial^4}{\partial s^2 \partial t^2} \bar{G}(\bar{x}_s, \bar{y}_t)\bigg|_{s=s_0, t=t_0} \tag{7.4.38}$$

$$= [-\bar{G}_{il,\alpha}(\bar{x}_{s_0}, \bar{y}_{t_0}) \cdot \bar{G}^{\beta,\alpha}(\bar{x}_{s_0}, \bar{y}_{t_0}) \cdot \bar{G}_{\beta,jk}(\bar{x}_{s_0}, \bar{y}_{t_0}) + \bar{G}_{il,jk}(\bar{x}_{s_0}, \bar{y}_{t_0})] \cdot \dot{\bar{x}}_{s_0}^i \cdot \dot{\bar{x}}_{s_0}^l \cdot \dot{\bar{y}}_{t_0}^j \cdot \dot{\bar{y}}_{t_0}^k \tag{7.4.39}$$

$$= \sec_{\Xi_{t_0}}^{(M,g)} P \wedge Q. \tag{7.4.40}$$

Therefore, (7.4.3) holds for all $t \in [0, 1]$. $\square$

Remark 7.4.2. Strict inequality versions of (v) and (vi) in Theorem 7.4.1 are equivalent to strict inequality of (iii), and thus condition $(\mathrm{G3})_u$.

Chapter 8

Examples

8.1 Several examples for the quasilinear case with explicit solutions

In 2011, Figalli-Kim-McCann [11] provided a non-negative curvature condition (B3) which is equivalent to the convexity of the domain of the objective $\mathbf{\Pi}$ or the domain of $\mathcal{L}$ defined below, under some other constraints. One may wonder the question whether this curvature condition (B3) is necessary for uniqueness of the optimizer as well.

According to Loeper [20], for $c = d^2$, where d is a Riemannian distance, (B3) is satisfied only if the Riemannian sectional curvature is non-negative. This section shows a negative answer to the above question, via uniqueness examples on the hyperbolic spaces with constant negative curvatures, where (B3) is violated.

Let $\mathcal{D}$ be a disk with a small radius $\bar{r}$ on $\mathbf{H}^n (n \geq 2)$, and consider spaces $X = Y = \mathcal{D}$, utility $G(x, y, z) = -\frac{1}{2}d_H^2(x, y) - z$, and profit $\pi(x, y, z) = z$ (i.e., $\pi(x, y, z) = z - a(y)$ with $a \equiv 0$), where

$$
\begin{aligned}
&d_H(x, y) = R\cosh^{-1}\left(\frac{x_0y_0 - x_1y_1 - \cdots - x_ny_n}{R^2}\right) = R\cosh^{-1}\left[\cosh\frac{r}{R}\cosh\frac{s}{R} - \sinh\frac{r}{R}\sinh\frac{s}{R}M\right],\\
&M = \sum_{i=1}^{n-1}\cos\theta_i\cos\varphi_i\left(\prod_{j=1}^{i-1}\sin\theta_j\sin\varphi_j\right) + \prod_{j=1}^{n-1}\sin\theta_j\sin\varphi_j,\\
&\text{here } x = (x_0, x_1, \dots, x_n),\ y = (y_0, y_1, \dots, y_n) \in \mathcal{D},\\
&x_0 = R\cosh\frac{r}{R},\ x_i = R\sinh\frac{r}{R}\cos\theta_i\Pi_{j=1}^{i-1}\sin\theta_j,\ \text{for all } i = 1, 2, ..., n-1,\ x_n = R\sinh\frac{r}{R}\Pi_{j=1}^{n-1}\sin\theta_j;\\
&y_0 = R\cosh\frac{s}{R},\ y_i = R\sinh\frac{s}{R}\cos\varphi_i\Pi_{j=1}^{i-1}\sin\varphi_j,\ \text{for all } i = 1, 2, ..., n-1,\ y_n = R\sinh\frac{s}{R}\Pi_{j=1}^{n-1}\sin\varphi_j.
\end{aligned}
\tag{8.1.1}
$$

Here $\prod$ means the product notation. In order to distinguish it from the profit functional $\mathbf{\Pi}$, equivalently, in this section, we minimize $\mathcal{L} := -\mathbf{\Pi}$. Let μ be the uniform measure on this hyperbolic disk $\mathcal{D}$. The participation constraint is $u_\emptyset(x) = -\frac{1}{2}d_H^2(x, y_\emptyset)$, where $y_\emptyset = (R, 0, 0, ..., 0) \in \mathcal{D}$ is the outside option with a fixed price $z_\emptyset = 0$.

Thus, the monopolist problem becomes

$$
\begin{aligned}
\max_{\substack{u\geq u_\emptyset\\ u\text{ is }G\text{-convex}}} \mathbf{\Pi}(u) &= \max_{\substack{u\geq u_\emptyset\\ u\text{ is }G\text{-convex}}} \int_X \pi(x,\bar{y}_G(x,u(x),Du(x)))d\mu(x) && (8.1.2)\\
&= \max_{\substack{u\geq u_\emptyset\\ u\text{ is }G\text{-convex}}} \int_X -\frac{1}{2}d_H^2(x,y_G(x,u(x),Du(x))) - u(x)d\mu(x). && (8.1.3)\\
&= -\min_{\substack{u\geq u_\emptyset\\ u\text{ is }G\text{-convex}}} \int_X \frac{1}{2}d_H^2(x,y_G(x,u(x),Du(x))) + u(x)d\mu(x). && (8.1.4)\\
(P_5) \qquad &= -\min_{\substack{u\geq u_\emptyset\\ u\text{ is }G\text{-convex}}} \mathcal{L}(u). && (8.1.5)
\end{aligned}
$$

Lemma 8.1.1. *Let $x(t)$ be any curve on $\mathcal{D}$, with $|\dot{x}(t_0)| = 1$, and y be any point on $\mathcal{D}$. Suppose $D_t d_H(x(t),y)|_{t=t_0}$ exists, then $|D_t d_H(x(t),y)|_{t=t_0}| \leq 1$.*

Proof. By the triangle inequality, we have

$$
\begin{aligned}
|D_t d_H(x(t),y)|_{t=t_0}| &= \left|\lim_{t\to t_0}\frac{d_H(x(t),y)-d_H(x(t_0),y)}{t-t_0}\right| = \lim_{t\to t_0}\left|\frac{d_H(x(t),y)-d_H(x(t_0),y)}{t-t_0}\right|\\
&\leq \lim_{t\to t_0^+}\frac{d_H(x(t),x(t_0))}{t-t_0} = |\dot{x}(t_0)| = 1.
\end{aligned}
$$

□

Corollary 8.1.2. *Since $\left|\frac{\partial x(r,\theta_1,\ldots,\theta_{n-1})}{\partial r}\right| = \lim_{s\to 0}\frac{d(x(r,\theta_1,\ldots,\theta_{n-1}),x(r+s,\theta_1,\ldots,\theta_{n-1}))}{s} = \lim_{s\to 0}\frac{s}{s} = 1$, by Lemma (8.1.1), we have $|D_r d_H(x(r,\theta_1,...,\theta_{n-1}),y)| \leq 1$, for all $x, y \in \mathcal{D}$.*

The following theorem shows a unique solution $\bar{u}$, with the explicit formula, to the principal-agent problem, on a negative curvature space $\mathcal{D}$. Its proof has three parts. In step 1, we first derive $\bar{u}$ as a local minimizer among the class of C^1 radially symmetric functions which are bounded below by the reservation utility, using the Calculus of Variations, then show it is also the unique global minimizer in this class. Then in step 2, we prove by definition $\bar{u}$ is G-convex. In step 3, we show that $\bar{u}$ is also a minimizer among all the G-convex functions. Moreover, the minimizer is unique.

Theorem 8.1.3. *The program (P_5) has a unique minimizer on $\mathcal{D}$. And*

$$
\operatorname*{argmin}_{\substack{u\geq -\frac{1}{2}r^2\\ u\text{ is radially symmetric}\\ u\in C^1(\mathcal{D})}} \mathcal{L}(u) = \operatorname*{argmin}_{\substack{u\geq -\frac{1}{2}r^2\\ u\text{ is radially symmetric}\\ u\text{ is }G\text{-convex}}} \mathcal{L}(u) = \operatorname*{argmin}_{\substack{u\geq u_\emptyset\\ u\text{ is }G\text{-convex}}} \mathcal{L}(u) =: \bar{u}(r)
$$

where

$$
\bar{u}(r) = \begin{cases} -\dfrac{1}{2}r^2, & 0\leq r\leq \tilde{r};\\[2ex] \displaystyle\int_{\tilde{r}}^r \sinh^{1-n}\left(\frac{t}{R}\right)\int_0^t \sinh^{n-1}\left(\frac{\sigma}{R}\right)d\sigma dt & \\ \displaystyle\quad -\int_0^{\tilde{r}}\sinh^{n-1}\left(\frac{\sigma}{R}\right)d\sigma\int_{\tilde{r}}^r \sinh^{1-n}\left(\frac{t}{R}\right)dt - \frac{1}{2}(\tilde{r})^2, & \tilde{r} < r\leq \bar{r}. \end{cases}
$$

Here $\tilde{r}$ satisfies

$$\int_{\bar{r}}^{\tilde{r}} \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma + \tilde{r}\sinh^{n-1}\left(\frac{\tilde{r}}{R}\right) = 0.$$

Proof. **Step 1:** Firstly, find the minimizer of $\mathcal{L}(u)$ for all u satisfying $u(r) \geq -\frac{1}{2}r^2$ and u is radially symmetric. Assume $\bar{u} \in C^2$ piecewisely on $\mathcal{D}$ is such a minimizer. For each agent x, $\bar{u}(x) = \sup\limits_y -\frac{1}{2}d_H^2(x,y) - v(y)$, one can find the optimal $y_G(x, \bar{u}(x), D\bar{u}(x))$ via

$$\begin{aligned} D_r\bar{u}(x) &= D_r\left(-\frac{1}{2}d_H^2(x, y_G(x,\bar{u}(x),D\bar{u}(x)))\right), \\ D_{\theta_i}\bar{u}(x) &= D_{\theta_i}\left(-\frac{1}{2}d_H^2(x, y_G(x,\bar{u}(x),D\bar{u}(x)))\right). \end{aligned} \tag{8.1.6}$$

From the above equations, we can see that $y_G(x,\bar{u}(x),D\bar{u}(x))$ could be uniquely determined by x and $D\bar{u}(x)$. In this section, we use $y_G(x,D\bar{u}(x))$ to denote $y_G(x,\bar{u}(x),D\bar{u}(x))$.

Since $\bar{u}$ is radially symmetric, thus $D_{\theta_i}\bar{u}(x) = 0$, for all $i = 1,2,...,n-1$, and $D_r\bar{u}(x) = \bar{u}'(r)$. From (8.1.6), one can derive $\theta_i = \varphi_i$, for all $i = 1,2,...,n-1$, and $d_H(x, y_G(x,D\bar{u}(x))) = |r-s|$, $D_r d_H(x, y_G(x,D\bar{u}(x))) = \frac{\sin\frac{r-s}{R}}{|\sin\frac{r-s}{R}|} = \text{sign}(r-s)$, for $x, y_G(x,D\bar{u}(x)) \in \mathcal{D}$ with polar coordinates introduced in (8.1.1).

Again from (8.1.6), $\bar{u}'(r) + |r-s|\cdot\text{sign}(r-s) = 0$ implies $s = r + \bar{u}'(r)$, and $(\bar{u}'(r))^2 = (s-r)^2 = d_H^2(x, y_G(x,D\bar{u}(x)))$. Notice here magnitude of $y_G(x,D\bar{u}(x))$ should be non-negative, so we have constraint $r + \bar{u}'(r) \geq 0$, which will be used later.

After calculating $y_G(x,D\bar{u}(x))$, one can compute

$$\begin{aligned} \mathcal{L}(\bar{u}) &= \int_{\mathcal{D}} \frac{1}{2}d_H^2(x, y_G(x,D\bar{u}(x))) + \bar{u}(x)d\mu(x) \\ &= \int_0^{\bar{r}}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\left[\frac{(\bar{u}'(r))^2}{2} + \bar{u}(r)\right]R^{n-1}\left(\sinh\frac{r}{R}\right)^{n-1}(\sin\theta_1)^{n-2}\cdots(\sin\theta_{n-2})d\theta_{n-1}\cdots d\theta_1 dr \\ &= C_0\int_0^{\bar{r}}\left[\frac{(\bar{u}'(r))^2}{2} + \bar{u}(r)\right]\left(\sinh\frac{r}{R}\right)^{n-1} dr. \end{aligned}$$

Here $C_0 = \int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi} R^{n-1}(\sin\theta_1)^{n-2}\cdots(\sin\theta_{n-2})d\theta_{n-1}\cdots d\theta_1$ is a positive constant.

Since $\bar{u}(r) \geq -\frac{1}{2}r^2$, let $A \in [0,\bar{r}]$, such that $A\times[0,\pi]^{n-2}\times[0,2\pi] = \left\{(r,\theta_1,...,\theta_{n-1}) \mid \bar{u}(r) = -\frac{1}{2}r^2\right\}$. Define $B = [0,\bar{r}]\setminus A$, so $\bar{u}(r) > -\frac{1}{2}r^2$ on B.

Denote $U_1 = \left\{w \in C^1(\mathcal{D}) \mid w \text{ is radially symmetric and } w = 0 \text{ on } A\times[0,\pi]^{n-2}\times[0,2\pi]\right\}$. Since $\bar{u}$ is a minimizer of $\mathcal{L}(u)$, for any $w \in U_1$, one has

$$\begin{aligned} 0 &= \left.\frac{\partial\mathcal{L}(\bar{u}+\varepsilon w)}{\partial\varepsilon}\right|_{\varepsilon=0} \\ &= C_0\int_B [\bar{u}'(r)w'(r) + w(r)]\left(\sinh\frac{r}{R}\right)^{n-1} dr \\ &= C_0\int_B w\left[\left(\sinh\frac{r}{R}\right)^{n-1} - \bar{u}''(r)\left(\sinh\frac{r}{R}\right)^{n-1} - \frac{n-1}{R}\bar{u}'(r)\left(\sinh\frac{r}{R}\right)^{n-2}\cosh\frac{r}{R}\right]dr \\ &\quad + C_0\bar{u}'(r)w(r)\left(\sinh\frac{r}{R}\right)^{n-1}\Bigg|_{\partial B}. \end{aligned}$$

By the fundamental lemma of the Calculus of Variations, and the inequality we derived from non-negativity of the magnitude of $y_G(x, D\bar{u}(x))$, we have following constraints for $\bar{u}$:

$$(ODE)\begin{cases} r+\bar{u}'(r)\geq 0, & (1)\text{ on } B;\\ \bar{u}''(r)+\frac{n-1}{R}\cdot\bar{u}'(r)(\coth\frac{r}{R})-1=0, & (2)\text{ on } B\setminus\partial B;\\ w(r)\bar{u}'(r)(\sinh\frac{r}{R})^{n-1}|_{\partial B}=0, & (3)\text{ for all } w\in U_1.\end{cases}$$

The equation (ODE)(2) implies, for all $r \in B$,

$$\bar{u}(r)=\int_0^r\left(\sinh\frac{t}{R}\right)^{1-n}\int_0^t\left(\sinh\frac{\sigma}{R}\right)^{n-1}d\sigma dt+C_1\int_0^r\left(\sinh\frac{t}{R}\right)^{1-n}dt+C_2.$$

Taking the derivatives of $\bar{u}$,

$$\bar{u}'(r)=\left(\sinh\frac{r}{R}\right)^{1-n}\left[\int_0^r\left(\sinh\frac{\sigma}{R}\right)^{n-1}d\sigma+C_1\right],$$
$$\bar{u}''(r)=1-\frac{n-1}{R}\cdot\frac{\cosh\frac{r}{R}}{\sinh^n(\frac{r}{R})}\left[\int_0^r\left(\sinh\frac{\sigma}{R}\right)^{n-1}d\sigma+C_1\right].$$

Consider the sign of C_1, there are two cases:

1. If $C_1 \geq 0$, then $\bar{u}'(r) \geq 0$, which implies $\bar{u}(r)$ is increasing on B.
2. If $C_1 < 0$, then $\bar{u}''(r) \geq 1-\frac{n-1}{R}\cdot\frac{\cosh\frac{r}{R}}{\sinh^n(\frac{r}{R})}\int_0^r(\sinh\frac{\sigma}{R})^{n-1}d\sigma$. Define

$$h_1(r):=\frac{R\sinh^n(\frac{r}{R})}{(n-1)\cosh(\frac{r}{R})}-\int_0^r\left(\sinh\frac{\sigma}{R}\right)^{n-1}d\sigma.$$

   Then $h_1(0)=0, h_1'(r)=\frac{\sinh^{n-1}(\frac{r}{R})}{(n-1)\cosh^2(\frac{r}{R})}\geq 0$, for all $r\in[0,\bar{r}]$. Thus, $h_1(r)\geq 0$, for all $r\in[0,\bar{r}]$, which implies, $u''(r)\geq 0$, i.e., $\bar{u}$ is convex on B.

In either case, A is path-connected, since one cannot join two points on $u_0(r) = -\frac{1}{2}r^2$ by either increasing or convex curve above the graph of u_0.

Assume $A = [\alpha_1, \alpha_2] \neq [0, \bar{r}]$. For the relative position of A and B, considering $A \cup B = [0, \bar{r}]$, there are three cases:

1. If $\alpha_1 > 0, \alpha_2 < \bar{r}$, assume $B = B_1 \cup B_2$ with $B_1 = [0, \alpha_1)$ and $B_2 = (\alpha_2, \bar{r}]$. Let $\bar{u}'(r) = (\sinh\frac{r}{R})^{1-n}[\int_0^r(\sinh\frac{\sigma}{R})^{n-1}d\sigma+C_{1_0}]$ on B_1 and $\bar{u}'(r)=(\sinh\frac{r}{R})^{1-n}[\int_0^r(\sinh\frac{\sigma}{R})^{n-1}d\sigma+C_{1_{\bar{r}}}]$ on B_2. Then by (ODE)(3), one has $\bar{u}'(r)(\sinh\frac{r}{R})^{n-1}|_{r=0}^{\bar{r}}=0$, i.e. $[\int_0^{\bar{r}}(\sinh\frac{\sigma}{R})^{n-1}d\sigma+C_{1_{\bar{r}}}]-C_{1_0}=0$, which implies

$$C_{1_{\bar{r}}}=C_{1_0}-\int_0^{\bar{r}}\left(\sinh\frac{\sigma}{R}\right)^{n-1}d\sigma. \tag{8.1.7}$$

   Since $\bar{u}(r) \geq u_0(r)$ on B_1 with equality holds at $r = \alpha_1$, we know $\bar{u}$ could not be increasing on B_1. Therefore, by the above discussion, we know $C_{1_0} < 0$ and $C_{1_{\bar{r}}} < 0$ by (8.1.7). Thus $\bar{u}$ is convex on B. In particular, $\bar{u}$ is convex and decreasing on B_1. Let $\tilde{u} = \bar{u}$ on $A \cup B_2$, and $\tilde{u} = u_0$ on B_1. Then for any $r \in B_1$, $\tilde{u}(r) - \bar{u}(r) \leq 0$ and $0 \geq \tilde{u}'(r) > u_0'(\alpha_1) \geq \bar{u}'(\alpha_1) \geq \bar{u}'(r)$. Thus,

$$\mathcal{L}(\tilde{u})-\mathcal{L}(\bar{u})=C_0\int_0^{\alpha_1}\left\{\left[\frac{(\tilde{u}'(r))^2}{2}-\frac{(\bar{u}'(r))^2}{2}\right]+[\tilde{u}-\bar{u}(r)]\right\}\left(\sinh\frac{r}{R}\right)^{n-1}dr<0. \tag{8.1.8}$$

Therefore, this case is reduced to the following one where $\alpha_1 = 0$.

2. If $\alpha_1 = 0, \alpha_2 \neq \bar{r}$, then by (ODE)(3), one has $\bar{u}'(r)(\sinh \frac{r}{R})^{n-1}|_{r=\bar{r}} = 0$, which implies $\bar{u}'(\bar{r}) = 0$. Thus, $C_1 = -\int_0^{\bar{r}} (\sinh \frac{\sigma}{R})^{n-1} d\sigma$. In this case, $\bar{u}$ is convex and decreasing on B.

3. If $\alpha_1 \neq 0, \alpha_2 = \bar{r}$, then by (ODE)(3), one has $\bar{u}'(r)(\sinh \frac{r}{R})^{n-1}|_{r=0} = 0$. That is,
$$\left[\int_0^r \left(\sinh \frac{\sigma}{R}\right)^{n-1} d\sigma + C_1\right]\Bigg|_{r=0} = 0,$$
which implies $C_1 = 0$. In this case, $\bar{u}$ is increasing on B. Notice that $\bar{u}(r) \geq -\frac{1}{2}r^2$ with equality holds at $r = \alpha_1$, a contradiction.

Summing up all the possible cases above, we know there exist $\alpha \in [0, \bar{r}]$, such that $A = [0, \alpha]$, $B = (\alpha, \bar{r}]$. By (ODE)(1), we have for any r on B,
$$\int_{\bar{r}}^r \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma + r \sinh^{n-1}\left(\frac{r}{R}\right) \geq 0.$$
Define
$$h_2(r) = \int_{\bar{r}}^r \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma + r \sinh^{n-1}\left(\frac{r}{R}\right).$$
Then
$$h_2'(r) = 2\sinh^{n-1}\left(\frac{r}{R}\right) + (n-1)\sinh^{n-2}\left(\frac{r}{R}\right)\left(\cosh \frac{r}{R}\right)\frac{r}{R} > 0.$$
Thus h_2 is strictly increasing. Notice that $h_2(\bar{r}) > 0$, and $h_2(0) < 0$. Thus there is a unique solution of $h_2(r) = 0$ in $[0, \bar{r}]$, denote it $\tilde{r}$. Then (ODE)(1) implies, $\alpha \geq \tilde{r}$, where $\tilde{r}$ satisfies
$$\int_{\bar{r}}^{\tilde{r}} \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma + \tilde{r} \sinh^{n-1}\left(\frac{\tilde{r}}{R}\right) = 0. \tag{8.1.9}$$
Since $\bar{u}$ is continuous, at $r = \alpha$, one have
$$-\frac{1}{2}\alpha^2 = \int_0^\alpha \left(\sinh \frac{t}{R}\right)^{1-n} \int_0^t \left(\sinh \frac{\sigma}{R}\right)^{n-1} d\sigma dt - \int_0^{\bar{r}} \left(\sinh \frac{\sigma}{R}\right)^{n-1} d\sigma \int_0^\alpha \left(\sinh \frac{t}{R}\right)^{1-n} dt + C_2.$$
This implies
$$C_2 = -\frac{1}{2}\alpha^2 - \int_0^\alpha \left(\sinh \frac{t}{R}\right)^{1-n} \int_0^t \left(\sinh \frac{\sigma}{R}\right)^{n-1} d\sigma dt + \int_0^{\bar{r}} \left(\sinh \frac{\sigma}{R}\right)^{n-1} d\sigma \int_0^\alpha \left(\sinh \frac{t}{R}\right)^{1-n} dt.$$
For any given $\alpha \in [\tilde{r}, \bar{r}]$, denote
$$\bar{u}_\alpha(r) = \begin{cases} -\dfrac{1}{2}r^2, & 0 \leq r \leq \alpha; \\ \displaystyle\int_\alpha^r (\sinh \frac{t}{R})^{1-n} \int_0^t (\sinh \frac{\sigma}{R})^{n-1} d\sigma dt & \\ \displaystyle\quad - \int_0^{\bar{r}} (\sinh \frac{\sigma}{R})^{n-1} d\sigma \int_\alpha^r (\sinh \frac{t}{R})^{1-n} dt - \frac{1}{2}\alpha^2, & \alpha < r \leq \bar{r}. \end{cases}$$

Define $h_3(\alpha) = \mathcal{L}(\bar{u}_\alpha)$ for all $\alpha \in [\tilde{r}, \bar{r}]$.

Then

$$h_3'(\alpha) = \frac{C_0}{2} \sinh^{1-n}\left(\frac{\alpha}{R}\right)\left[\alpha \sinh^{n-1}\left(\frac{\alpha}{R}\right) - \int_\alpha^{\bar{r}} \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma\right]^2 \geq 0$$

for all $\alpha \in [\tilde{r}, \bar{r}]$.

Define $h_4(\alpha) := \alpha \sinh^{n-1}(\frac{\alpha}{R}) - \int_\alpha^{\bar{r}} \sinh^{n-1}(\frac{\sigma}{R}) d\sigma$ for all $\alpha \in [\tilde{r}, \bar{r}]$.

Then

$$h_4'(\alpha) = 2\sinh^{n-1}\left(\frac{\alpha}{R}\right) + \frac{n-1}{R}\alpha \sinh^{n-2}\left(\frac{\alpha}{R}\right)\cosh\left(\frac{\alpha}{R}\right) > 0$$

on $[\tilde{r}, \bar{r}]$ and $h_4(\tilde{r}) = 0$, which implies $h_4(\alpha) > 0$, for $\alpha \in (\tilde{r}, \bar{r}]$.

Thus, $h_3'(\alpha) > 0$, for all $\alpha \in (\tilde{r}, \bar{r}]$, which implies

$$\min_{\alpha \in [\tilde{r}, \bar{r}]} \mathcal{L}(\bar{u}_\alpha) = \min_{\alpha \in [\tilde{r}, \bar{r}]} h_3(\alpha) = h_3(\tilde{r}) = \mathcal{L}(\bar{u}_{\tilde{r}}).$$

Therefore, $\bar{u}(r) = \bar{u}_{\tilde{r}}(r)$. And

$$\bar{u}(r) = \begin{cases} -\frac{1}{2}r^2 & , 0 \leq r \leq \tilde{r}; \\ \int_{\tilde{r}}^r \sinh^{1-n}\left(\frac{t}{R}\right) \int_0^t \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma dt & \\ \quad - \int_0^{\tilde{r}} \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma \int_{\tilde{r}}^r \sinh^{1-n}\left(\frac{t}{R}\right) dt - \frac{1}{2}(\tilde{r})^2 & , \tilde{r} < r \leq \bar{r}. \end{cases}$$

$$\bar{u}'(r) = \begin{cases} -r & , 0 \leq r \leq \tilde{r}; \\ \sinh^{1-n}\left(\frac{r}{R}\right) \int_{\bar{r}}^r \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma & , \tilde{r} < r \leq \bar{r}. \end{cases}$$

$$\bar{u}''(r) = \begin{cases} -1 & , 0 \leq r \leq \tilde{r}; \\ 1 - \dfrac{(n-1)\cosh\left(\frac{r}{R}\right) \int_{\bar{r}}^r \sinh^{n-1}\left(\frac{\sigma}{R}\right) d\sigma}{R \sinh^n\left(\frac{r}{R}\right)} & , \tilde{r} < r \leq \bar{r}. \end{cases}$$

Since $\partial_- \bar{u}'(\tilde{r}) = \partial_+ \bar{u}'(\tilde{r}) = -\tilde{r}$, thus $\bar{u}'(\tilde{r})$ exists, and $\bar{u}'(\tilde{r}) = -\tilde{r}$.

We now show the above $\bar{u}(r)$ is indeed the (global) minimizer, i.e.,

$$\bar{u}(r) = \underset{\substack{u \geq -\frac{1}{2}r^2 \\ u \text{ is radially symmetric} \\ u \in C^1(\mathcal{D})}}{\operatorname{argmin}} \mathcal{L}(u).$$

Denote $U_2 = \{w \in C^1(\mathcal{D}) \mid w \text{ is symmetric and } w \geq 0 \text{ on } A \times [0,\pi]^{n-2} \times [0, 2\pi]\}$. For any $w \in U_2$,

$$\begin{aligned} & \mathcal{L}(\bar{u} + w) - \mathcal{L}(\bar{u}) \\ = {} & C_0 \int_0^{\bar{r}} \left[\frac{1}{2}(\bar{u}'(r) + w'(r))^2 - \frac{1}{2}(\bar{u}'(r))^2 + w\right] \sinh^{n-1}\left(\frac{r}{R}\right) dr \end{aligned} \tag{8.1.10}$$

Drop a non-negative term with integrand $\frac{1}{2}(w'(r))^2$, plug in $\bar{u}'$ on $A \times [0,\pi]^{n-2} \times [0, 2\pi]$ and use the integrate by parts formula, one has

$$\begin{aligned}
(8.1.10) &\geq C_0 \int_0^{\bar{r}} [\bar{u}'(r)w'(r) + w] \sinh^{n-1}\left(\frac{r}{R}\right) dr. \\
&= C_0 \int_0^{\tilde{r}} [-rw'(r) + w] \sinh^{n-1}\left(\frac{r}{R}\right) dr + C_0 \int_{\tilde{r}}^{\bar{r}} [\bar{u}'(r)w'(r) + w] \sinh^{n-1}\left(\frac{r}{R}\right) dr \\
&= C_0 \int_0^{\tilde{r}} w \left[2\sinh^{n-1}\left(\frac{r}{R}\right) + \frac{n-1}{R} r \sinh^{n-2}\left(\frac{r}{R}\right)\cosh\left(\frac{r}{R}\right)\right] dr - C_0 r w(r) \sinh^{n-1}\left(\frac{r}{R}\right)\Big|_0^{\tilde{r}} \\
&\quad + C_0 \int_{\tilde{r}}^{\bar{r}} w \left[(1-\bar{u}''(r))\sinh^{n-1}\left(\frac{r}{R}\right) - \frac{n-1}{R}\bar{u}'(r)\sinh^{n-2}\left(\frac{r}{R}\right)\cosh\left(\frac{r}{R}\right)\right] dr \\
&\quad + C_0 \bar{u}'(r) w(r) \sinh^{n-1}\left(\frac{r}{R}\right)\Big|_{\tilde{r}}^{\bar{r}}
\end{aligned}$$

By equation (ODE)(2), the term

$$C_0 \int_{\tilde{r}}^{\bar{r}} w \left[(1-\bar{u}''(r))\sinh^{n-1}\left(\frac{r}{R}\right) - \frac{n-1}{R}\bar{u}'(r)\sinh^{n-2}\left(\frac{r}{R}\right)\cosh\left(\frac{r}{R}\right)\right] dr$$

vanishes. Drop the term with non-negative integrand

$$C_0 \int_0^{\tilde{r}} w \left[2\sinh^{n-1}\left(\frac{r}{R}\right) + \frac{n-1}{R} r \sinh^{n-2}\left(\frac{r}{R}\right)\cosh\left(\frac{r}{R}\right)\right] dr.$$

Thus,

$$\begin{aligned}
(8.1.10) &\geq -C_0 r w(r) \sinh^{n-1}\left(\frac{r}{R}\right)\Big|_0^{\tilde{r}} + C_0 \bar{u}'(r) w(r) \sinh^{n-1}\left(\frac{r}{R}\right)\Big|_{\tilde{r}}^{\bar{r}} \\
&= 0.
\end{aligned}$$

The last equality holds because $\bar{u}'(\tilde{r}) = -\tilde{r}$ and $\bar{u}'(\bar{r}) = 0$.

In addition, if $\mathcal{L}(\bar{u}+w) - \mathcal{L}(\bar{u}) = 0$, from above inequalities and since $w \in C^1$, we have $w'(r) = 0$ on $\mathcal{D}$, i.e., $w = C_3$, for some non-negative constant C_3. Then

$$0 = \mathcal{L}(\bar{u}+w) - \mathcal{L}(\bar{u}) \geq C_0 \int_0^{\bar{r}} [\bar{u}'(r)w'(r) + w] \sinh^{n-1}\left(\frac{r}{R}\right) dr = C_0 C_3 \int_0^{\bar{r}} \sinh^{n-1}\left(\frac{r}{R}\right) dr \geq 0.$$

Since both C_0 and $\int_0^{\bar{r}} \sinh^{n-1}(\frac{r}{R})dr$ are positive, we have $C_3 = 0$, i.e. $w \equiv 0$ on $\mathcal{D}$. Therefore, $\bar{u}$ is the unique minimizer.

Step 2: To check that $\bar{u}(x)$ is G-convex, it is equivalent to prove $\bar{u}(x)$ is b-convex in the sense of [11], or equivalently $-\bar{u}(x)$ is $(-b)$-concave in the sense of Definition 5.3.1, where $b(x,y) := -\frac{1}{2}d_H^2(x,y)$. That is, we need to show $\bar{u}(x) = -((-\bar{u})^{(-b)^*})^{(-b)}(x)$, for all $x \in \mathcal{D}$. Denote $\psi(y) = -(-\bar{u})^{(-b)^*}(y)$, and $\phi(x) = -(-\psi)^{(-b)}(x)$. Then it is equivalent to show $\bar{u}(x) = \phi(x)$, where

$$\begin{aligned}
\phi(x) &= \sup_y b(x,y) - \psi(y), \\
\text{and } \psi(y) &= \sup_x b(x,y) - \bar{u}(x).
\end{aligned}$$

By definition, we have

$$\psi(y) = \sup_x -\frac{1}{2}d_H^2(x,y) - \bar{u}(x)$$

From equation (8.1.1), we know that $|M| \le 1$ and $M = 1$ holds when $\theta_i = \varphi_i$ for each $i = 1, 2, ..., n-1$. Thus,

$$\psi(y) = \sup_r -\frac{1}{2}(r-s)^2 - \bar{u}(r)$$

For each $s \in [0, \bar{r}]$, define $h_5^s(r) := -\frac{1}{2}(r-s)^2 - \bar{u}(r)$. Then

$$(h_5^s)'(r) = \begin{cases} s & ,0 \le r \le \tilde{r}; \\ s - r - \sinh^{1-n}\left(\frac{r}{R}\right)\int_{\tilde{r}}^{r} \sinh^{n-1}\left(\frac{\sigma}{R}\right)d\sigma & ,\tilde{r} < r \le \bar{r}. \end{cases}$$

$$(h_5^s)''(r) = \begin{cases} 0 & ,0 \le r \le \tilde{r}; \\ -1 - \bar{u}''(r) < -1 & ,\tilde{r} < r \le \bar{r}. \end{cases}$$

Thus, $h_5^s(r)$ is concave on $[0, \bar{r}]$ and strictly concave on $[\tilde{r}, \bar{r}]$. Since $\bar{u}'$ is continuous, thus $(h_5^s)'(r)$ is continuous. Notice $(h_5^s)'(\tilde{r}) = s > 0$ and $(h_5^s)'(\bar{r}) < 0$. Therefore, for each $s \in [0, \bar{r}]$, $(h_5^s)'(\beta) = 0$ has exactly one solution on $[0, \bar{r}]$, which is located on $[\tilde{r}, \bar{r}]$ and takes the maximum value of h_5^s.

Let $h_6(s)$ be the unique solution of $(h_5^s)'(\beta) = 0$, for all $s \in [0, \bar{r}]$.

Then $h_6(s) \in [\tilde{r}, \bar{r}]$, and $h_6(0) = \tilde{r}$, $h_6(\bar{r}) = \bar{r}$.

Since $(h_5^s)'(s) > 0 = (h_5^s)'(h_6(s))$ and $(h_5^s)'$ is strictly decreasing on $[\tilde{r}, \bar{r}]$, we have $h_6(s) > s$.

For any $0 \le s_1 < s_2 \le \bar{r}$, $(h_5^{s_2})'(h_6(s_2)) = 0 = (h_5^{s_1})'(h_6(s_1)) < (h_5^{s_2})'(h_6(s_1))$, thus $h_6(s_1) < h_6(s_2)$, i.e., h_6 is strictly increasing. By the Implicit Function Theorem, one has $h_6 \in C^1$. Thus, $h_6' > 0$. Here, denote

$$\bar{u}(r) = \begin{cases} u_1(r) & ,0 \le r \le \tilde{r}; \\ u_2(r) & ,\tilde{r} < r \le \bar{r}. \end{cases}$$

Then $\psi(y) = \sup_r h_5^s(r) = h_5^s(h_6(s)) = -\frac{1}{2}(h_6(s) - s)^2 - u_2(h_6(s))$.

$$\begin{aligned} \phi(x) &= \sup_y -\frac{1}{2}d_H^2(x,y) - \psi(y) \\ &= \sup_s -\frac{1}{2}(r-s)^2 + \frac{1}{2}(h_6(s)-s)^2 + u_2(h_6(s)) \end{aligned}$$

For each $r \in [0, \bar{r}]$, define $h_7^r(s) := -\frac{1}{2}(r-s)^2 + \frac{1}{2}(h_6(s)-s)^2 + u_2(h_6(s))$. Then

$$\begin{aligned} (h_7^r)'(s) &= r - h_6(s), \\ (h_7^r)''(s) &= -(h_6)'(s) < 0. \end{aligned}$$

Plugging in $s = 0, \bar{r}$, we have $(h_7^r)'(0) = r - \tilde{r}$, $(h_7^r)'(\bar{r}) = r - \bar{r} \le 0$. Therefore,

1. For $r \in [0, \tilde{r}]$, $(h_7^r)'(s) < (h_7^r)'(0) \le 0$, then we have $\phi(x) = \sup_s (h_7^r)(s) = (h_7^r)(0) = -\frac{1}{2}r^2 = u_1(r)$;
2. For $r \in [\tilde{r}, \bar{r}]$, since $(h_7^r)'(0) \ge 0, (h_7^r)'(\bar{r}) \le 0$, then we have $\phi(x) = \sup_s (h_7^r)(s) = (h_7^r)((h_6^{-1})(r)) = u_2(r)$.

Thus, $\phi(x) = \bar{u}(x)$, which implies $\bar{u}$ is G-convex. So,

$$\bar{u}(r) = \underset{\substack{u \geq -\frac{1}{2}r^2 \\ u \text{ is radially symmetric} \\ u \text{ is } G\text{-convex}}}{\operatorname{argmin}} \mathcal{L}(u).$$

Step 3: We are going to show

$$\bar{u} = \underset{\substack{u \geq u_\emptyset \\ u \text{ is } G\text{-convex}}}{\operatorname{argmin}} \mathcal{L}(u).$$

Suppose u is any G-convex function (not necessarily radially symmetric), then there exists a function v, such that $u(x) = \max_y b(x, y) - v(y)$. Thus,

$$D_r u(x) = D_r b(x, y_G(x, Du(x))) = -d_H(x, y_G(x, Du(x))) \cdot D_r d_H(x, y_G(x, Du(x))),$$

where $y_G(x, Du(x)) = \operatorname{argmax}_y b(x, y) - v(y)$. By Corollary (8.1.2), we have

$$-b(x, y_G(x, Du(x))) = \frac{1}{2} d_H^2(x, y_G(x, Du(x))) = \frac{|D_r u(x)|^2}{2|D_r d_H(x, y_G(x, Du(x)))|^2} \geq \frac{|D_r u(x)|^2}{2}. \quad (8.1.11)$$

Denote $U_3 = \{w : \mathcal{D} \to \mathbf{R} \mid \bar{u} + w \text{ is } G\text{-convex, and } w \geq 0 \text{ on } A \times [0, \pi]^{n-2} \times [0, 2\pi]\}$. For any $w \in U_3$, we have

$$\begin{aligned} &\mathcal{L}(\bar{u} + w) - \mathcal{L}(\bar{u}) \\ = &\int_{\mathcal{D}} [-b(x, y_G(x, D\bar{u}(x) + Dw(x))) + b(x, y_G(x, D\bar{u}(x))) + w(x)]\, d\mu(x). \end{aligned} \quad (8.1.12)$$

Similar to (8.1.11), one has

$$-b(x, y_G(x, D\bar{u}(x) + Dw(x))) \geq \frac{1}{2}|D_r\bar{u} + D_r w|^2$$

$$\text{and} \quad b(x, y_G(x, D\bar{u}(x))) = -\frac{1}{2}|\bar{u}'(r)|^2.$$

Thus,

$$(8.1.12) \geq \int_{\mathcal{D}} \left[\frac{1}{2}|D_r\bar{u} + D_r w|^2 - \frac{1}{2}|\bar{u}'(r)|^2 + w(x)\right] d\mu(x).$$

Simplify the right hand side, drop a non-negative term with integrand $\frac{1}{2}|D_r w|^2$ and change to polar coordinates, then plug in $\bar{u}'$ on $A \times [0, \pi]^{n-2} \times [0, 2\pi]$ and $B \times [0, \pi]^{n-2} \times [0, 2\pi]$, separately. Denote

$$d\Theta = R^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} d\theta_1 d\theta_2 \cdots d\theta_{n-1}.$$

Thus

$$(8.1.12) \geq \int_{\mathcal{D}} [\bar{u}'(r) \cdot D_r w + w] R^{n-1} \sinh^{n-1}\left(\frac{r}{R}\right) \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} dr d\theta_1 d\theta_2 \cdots d\theta_{n-1}$$

$$= \int_{A\times[0,\pi]^{n-2}\times[0,2\pi]} \left[-r\cdot\sinh^{n-1}\left(\frac{r}{R}\right)\cdot D_r w + w\cdot\sinh^{n-1}\left(\frac{r}{R}\right)\right] drd\Theta$$
$$+ \int_{B\times[0,\pi]^{n-2}\times[0,2\pi]} \left[\int_{\bar r}^{r} \sinh^{n-1}\left(\frac{t}{R}\right) dt\cdot D_r w + w\cdot\sinh^{n-1}\left(\frac{r}{R}\right)\right] drd\Theta.$$

Use the integration by parts formula, then drop the term with non-negative integrand

$$\int_{[0,\pi]^{n-2}\times[0,2\pi]} \int_0^{\tilde r} w\left[2\sinh^{n-1}\left(\frac{r}{R}\right) + \frac{n-1}{R} r\sinh^{n-2}\left(\frac{r}{R}\right)\cosh\left(\frac{r}{R}\right)\right] drd\Theta.$$

Thus,

$$\begin{aligned}
(8.1.12) \geq & \int_{[0,\pi]^{n-2}\times[0,2\pi]} \left\{ \int_0^{\tilde r} w\left[2\sinh^{n-1}\left(\frac{r}{R}\right) + \frac{n-1}{R} r\sinh^{n-2}\left(\frac{r}{R}\right)\cosh\left(\frac{r}{R}\right)\right] dr \right. \\
& \left. - \left[r\sinh^{n-1}\left(\frac{r}{R}\right)\cdot w\right]\Big|_{r=0}^{\tilde r} \right\} d\Theta \\
& + \int_{[0,\pi]^{n-2}\times[0,2\pi]} \left[\int_{\bar r}^{r} \sinh^{n-1}\left(\frac{t}{R}\right) dt\cdot w\right]\Big|_{r=\tilde r}^{\bar r} d\Theta \\
\geq & \int_{[0,\pi]^{n-2}\times[0,2\pi]} \left[-\tilde r\sinh^{n-1}\left(\frac{\tilde r}{R}\right) - \int_{\bar r}^{\tilde r}\sinh^{n-1}\left(\frac{t}{R}\right) dt\right]\cdot w(\tilde r, \theta_1, ..., \theta_{n-1}) d\Theta \\
= & \ 0.
\end{aligned}$$

The last integral equals to 0 by the definition of $\tilde r$. Therefore, $\mathcal{L}(\bar u + w) \geq \mathcal{L}(\bar u)$ for any $w \in U_3$. For any G-convex $u \geq u_\emptyset$, $u - \bar u \in U_3$. So

$$\bar u \in \underset{\substack{u\geq u_\emptyset \\ u \text{ is } G\text{-convex}}}{\operatorname{argmin}} \mathcal{L}(u).$$

If, in addition, $\mathcal{L}(\bar u+w)-\mathcal{L}(\bar u) = 0$, then the above inequalities must be equalities. Thus $D_r w(x) = 0$, for almost every $x \in \mathcal{D}$. Since both $\bar u + w$ and $\bar u$ are G-convex, $w \in C^{1,1}$, thus $D_r w(x) \equiv 0$, for all $x \in \mathcal{D}$. So, one can write $w(x) = w(\theta_1, ..., \theta_{n-1})$. Since for $x \in A \times [0,\pi]^{n-1} \times [0, 2\pi]$, $w(x) \geq 0$, we have $w \geq 0$ on $\mathcal{D}$. Then from the above inequalities, we get

$$0 = \mathcal{L}(\bar u + w) - \mathcal{L}(\bar u) \geq \int_{\mathcal{D}} w(x) R^{n-1}\sinh^{n-1}(\frac{r}{R})\sin^{n-2}\theta_1\cdots\sin\theta_{n-2} drd\theta_1\cdots d\theta_{n-1} \geq 0$$

Thus $w(x)R^{n-1}\sinh^{n-1}(\frac{r}{R})\sin^{n-2}\theta_1\cdots\sin\theta_{n-2} = 0$, for almost every $x \in \mathcal{D}$. This implies $w \equiv 0$ on $\mathcal{D}$. So $\bar u$ is the unique minimizer, i.e.

$$\bar u = \underset{\substack{u\geq u_\emptyset \\ u \text{ is } G\text{-convex}}}{\operatorname{argmin}} \mathcal{L}(u).$$

□

Remark 8.1.4. We also have uniqueness results with different explicit solutions on $\mathbf{S}^n$ and $\mathbf{R}^n$, where the uniqueness is also ensured by Figalli-Kim-McCann[11]. Moreover, the solutions on $\mathbf{S}^n$, $\mathbf{H}^n$ converge to those on $\mathbf{R}^n$, as curvatures go to 0.

The unique minimizer of the principal-agent problem on $\mathbf{R}^n$ is given by

$$\bar{u}_{\mathbf{R}^n}(r) = \begin{cases} -\dfrac{1}{2}r^2 & , 0 \le r \le \tilde{r}_{\mathbf{R}^n}; \\ \dfrac{(\bar{r})^n}{n(n-2)} r^{2-n} + \dfrac{r^2}{2n} - \dfrac{(\bar{r})^2}{2(n-2)(n+1)^{\frac{2-n}{n}}} & , \tilde{r}_{\mathbf{R}^n} < r \le \bar{r}. \end{cases}$$

Here $\tilde{r}_{\mathbf{R}^n} = \frac{\bar{r}}{(n+1)^{\frac{1}{n}}}$.

Moreover, the unique minimizer on $\mathbf{S}^n$ is given by

$$\bar{u}_{\mathbf{S}^n}(r) = \begin{cases} -\dfrac{1}{2}r^2 & , 0 \le r \le \tilde{r}_{\mathbf{S}^n}; \\ \displaystyle\int_{\tilde{r}_{\mathbf{S}^n}}^{r} \sin^{1-n}\left(\frac{t}{R}\right) \int_0^t \sin^{n-1}\left(\frac{\sigma}{R}\right) d\sigma dt & \\ \quad \displaystyle - \int_0^{\bar{r}} \sin^{n-1}\left(\frac{\sigma}{R}\right) d\sigma \int_{\tilde{r}_{\mathbf{S}^n}}^{r} \sin^{1-n}\left(\frac{t}{R}\right) dt - \frac{(\tilde{r}_{\mathbf{S}^n})^2}{2} & , \tilde{r}_{\mathbf{S}^n} < r \le \bar{r}. \end{cases}$$

Here $\tilde{r}_{\mathbf{S}^n}$ satisfies

$$\int_{\bar{r}}^{\tilde{r}_{\mathbf{S}^n}} \sin^{n-1}\left(\frac{\sigma}{R}\right) d\sigma + \tilde{r}_{\mathbf{S}^n} \sin^{n-1}\left(\frac{\tilde{r}_{\mathbf{S}^n}}{R}\right) = 0.$$

The proofs are similar to that of Theorem 8.1.3.

8.2 Convexity results on several examples for the non-quasilinear case

We close with several examples, which are established by computing two derivatives of $\pi(x, y_t, z_t)$ along an arbitrary G-segment $t \in [0,1] \longmapsto (x, y_t, z_t)$. These computations are tedious but straightforward.

For specific non-quasilinear agent preferences, we use the explicit expression in Lemma 5.2.8 for the desired second derivative to establish the following examples, which assume the principal is indifferent to whom she transacts business with and that her preferences depend linearly on payments. These examples give conditions under which the principal's program inherits concavity or convexity from the agents' price sensitivity. Although the resulting conditions appear complicated, they illustrate the subtle interplay between the preferences of the agent and the principal for products in the first example, and between the preferences of the agents for products as opposed to prices in the second.

Example 8.2.1 (Nonlinear yet homogeneous sensitivity of agents to prices)**.** Take $\pi(x, y, z) = z - a(y)$, $G(x, y, z) = b(x, y) - f(z)$, satisfying (G0)-(G6), $G \in C^3(cl(X \times Y \times Z))$, $\pi \in C^2(cl(X \times Y \times Z))$, and assume $\bar{z} < +\infty$.

1. If $f(z)$ is convex [respectively concave] in $cl(Z)$, then $\mathbf{\Pi}(u)$ is concave [respectively convex] for all $\mu \ll \mathcal{L}^m$ if and only if there exists $\varepsilon \geq 0$ such that each $(x, y, z) \in X \times Y \times Z$ and $\xi \in \mathbf{R}^n$ satisfy

$$\pm \left\{ a_{kj}(y) - \frac{b_{,kj}(x,y)}{f'(z)} + \Big(\frac{b_{,l}(x,y)}{f'(z)} - a_l(y) \Big) b^{i,l}(x,y) b_{i,kj}(x,y) \right\} \xi^k \xi^j \ \geq \varepsilon \mid \xi \mid^2 . \tag{8.2.1}$$

2. In addition, $\mathbf{\Pi}(u)$ is uniformly concave [respectively uniformly convex] on $W^{1,2}(X, d\mu)$ uniformly for all $\mu \ll \mathcal{L}^m$ if and only if $\pm f'' > 0$ and (8.2.1) holds with $\varepsilon > 0$.

Proof. From Lemma 5.2.8, $\mathbf{\Pi}(u)$ is concave for all $\mu \ll \mathcal{L}^m$ if and only if $(\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}} \bar{G}^{\bar{i},\bar{l}} \bar{G}_{\bar{i},\bar{k}\bar{j}})\big|_{x_0=-1}$ is non-positive definite, and uniformly concave uniformly for all $\mu \ll \mathcal{L}^m$ if and only if this matrix is uniform negative definite.

In this example, we have $\pi(x, y, z) = z - a(y)$, $\bar{G}(x, x_0, y, z) = x_0 G(x, y, z) = x_0(b(x, y) - f(z))$. Thus,

$$\pi_{,\bar{k}\bar{j}} = \begin{pmatrix} -a_{kj} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}, \quad \pi_{,\bar{l}} = (-a_l, 1), \quad \bar{G}_{\bar{i},\bar{l}}\big|_{x_0=-1} = \begin{pmatrix} -b_{i,l} & \mathbf{0} \\ b_{,l} & -f'(z) \end{pmatrix}.$$

By (G4), $f'(z) > 0$ for all $z \in cl(Z)$. By (G6), since $\bar{G}_{\bar{i},\bar{l}}\big|_{x_0=-1}$ has the full rank, the matrix $(b_{i,l})$ also has its full rank. Taking $b^{i,l}$ as its left inverse, we have

$$\bar{G}^{\bar{i},\bar{l}}\Big|_{x_0=-1} = \begin{pmatrix} -b^{i,l} & \mathbf{0} \\ -\frac{b_{,l} b^{i,l}}{f'(z)} & \frac{1}{-f'(z)} \end{pmatrix}, \quad \bar{G}_{\bar{i},\bar{k}\bar{j}}\big|_{x_0=-1} = \begin{pmatrix} -b_{i,k\bar{j}} & \mathbf{0} \\ b_{,k\bar{j}} & (-f'(z))_{\bar{j}} \end{pmatrix}.$$

Therefore,

$$\Big(\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}} \bar{G}^{\bar{i},\bar{l}} \bar{G}_{\bar{i},\bar{k}\bar{j}}\Big)\Big|_{x_0=-1}$$

$$= \begin{pmatrix} -a_{kj} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} - \Big((-a_l b^{i,l} + \tfrac{b_{,l}}{f'(z)} b^{i,l}) b_{i,k\bar{j}} - \tfrac{b_{,k\bar{j}}}{f'(z)}, \tfrac{(f'(z))_{\bar{j}}}{f'(z)} \Big)$$
$$= - \begin{pmatrix} a_{kj} + (-a_l + \frac{b_{,l}}{f'(z)}) b^{i,l} b_{i,kj} - \frac{b_{,kj}}{f'(z)} & \mathbf{0} \\ \mathbf{0} & \frac{f''(z)}{f'(z)} \end{pmatrix}.$$

Since (G4) and f is convex, we have $f'(z) > 0$ and $f''(z) \geq 0$, for all $z \in cl(Z)$. Thus, $\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}} \bar{G}^{\bar{i},\bar{l}} \bar{G}_{\bar{i},\bar{k}\bar{j}}$ is non-positive definite if and only if $a_{kj} + (-a_l + \frac{b_{,l}}{f'(z)}) b^{i,l} b_{i,kj} - \frac{b_{,kj}}{f'(z)}$ is non-negative definite, i.e., there exist $\varepsilon \geq 0$ such that each $(x, y, z) \in X \times Y \times Z$ and $\xi \in \mathbf{R}^n$ satisfy

$$\Big\{ a_{kj}(y) - \frac{b_{,kj}(x,y)}{f'(z)} + \Big(\frac{b_{,l}(x,y)}{f'(z)} - a_l(y) \Big) b^{i,l}(x,y) b_{i,kj}(x,y) \Big\} \xi^k \xi^j \ \geq \varepsilon \mid \xi \mid^2 .$$

In addition, $\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}} \bar{G}^{\bar{i},\bar{l}} \bar{G}_{\bar{i},\bar{k}\bar{j}}$ is uniform negative definite if and only if $f'' > 0$ and $\varepsilon > 0$, which is equivalent to that $\mathbf{\Pi}(u)$ is uniformly concave uniformly for all $\mu \ll \mathcal{L}^m$. Similarly, one can show equivalent conditions for $\mathbf{\Pi}(u)$ being convex or uniformly convex. □

Although the next two examples are not completely general, they have the following economic interpretation. The same selling price impacts utility differently for different types of agents. In other words, it models the situation where agents have different sensitivities to the same price. In Example 8.2.2, the principal's utility is linear and depends exclusively on her revenue, which is a simple special case of Example 8.2.3.

Example 8.2.2 (Inhomogeneous sensitivity of agents to prices, zero cost)**.** Take $\pi(x, y, z) = z$, $G(x, y, z) = b(x, y) - f(x, z)$, satisfying (G0)-(G6), $G \in C^3(cl(X \times Y \times Z))$, $\pi \in C^2(cl(X \times Y \times Z))$, and assume $\bar{z} < +\infty$. Suppose $D_{x,y} b(x, y)$ has full rank for each $(x, y) \in X \times Y$, and denote its left inverse $b^{i,l}(x, y)$.

1. If $(x, y, z) \longmapsto h(x, y, z) := f(x, z) - b_{,l}(x, y) b^{i,l}(x, y) f_{i,}(x, z)$ is strictly increasing and convex [respectively concave] with respect to z, then $\mathbf{\Pi}(u)$ is concave [respectively convex] for all $\mu \ll \mathcal{L}^m$ if and only if there exists $\varepsilon \geq 0$ such that each $(x, y) \in X \times Y$ and $\xi \in \mathbf{R}^n$ satisfy

$$\pm \Big\{ - b_{,kj}(x,y) + b_{,l}(x,y) b^{i,l}(x,y) b_{i,kj}(x,y) \Big\} \xi^k \xi^j \geq \varepsilon \mid \xi \mid^2 . \tag{8.2.2}$$

2. In addition, $\mathbf{\Pi}(u)$ is uniformly concave [respectively uniformly convex] on $W^{1,2}(X, d\mu)$ uniformly for all $\mu \ll \mathcal{L}^m$ if and only if $\pm h_{zz} > 0$ and (8.2.2) holds with $\varepsilon > 0$.

Example 8.2.3 (Inhomogeneous sensitivity of agents to prices)**.** Take $\pi(x, y, z) = z - a(y)$, $G(x, y, z) = b(x, y) - f(x, z)$, satisfying (G0)-(G6), $G \in C^3(cl(X \times Y \times Z))$, $\pi \in C^2(cl(X \times Y \times Z))$, and assume $\bar{z} < +\infty$. Suppose $D_{x,y} b(x, y)$ has full rank for each $(x, y) \in X \times Y$, and $1 - (f_z)^{-1} b_{,\beta} b^{\alpha,\beta} f_{\alpha,z} \neq 0$, for all $(x, y, z) \in X \times Y \times Z$.

1. If $(x, y, z) \longmapsto h(x, y, z) := a_l b^{i,l} f_{i,zz} + \frac{(a_\beta b^{\alpha,\beta} f_{\alpha,z} - 1)(b_{,l} b^{i,l} f_{i,zz} - f_{zz})}{f_z - b_{,\beta} b^{\alpha,\beta} f_{\alpha,z}} \geq 0$ $[\leq 0]$, then $\mathbf{\Pi}(u)$ is concave [respectively convex] for all $\mu \ll \mathcal{L}^m$ if and only if there exists $\varepsilon \geq 0$ such that each $(x, y, z) \in X \times Y \times Z$ and $\xi \in \mathbf{R}^n$ satisfy

$$\pm \Big\{ \ a_{kj} - a_l b^{i,l} b_{i,kj} + \frac{1 - a_\beta b^{\alpha,\beta} f_{\alpha,z}}{1 - (f_z)^{-1} b_{,\beta} b^{\alpha,\beta} f_{\alpha,z}} \Big(- \frac{b_{,kj}}{f_z} + \frac{b_{,l}}{f_z} b^{i,l} b_{i,kj} \Big) \Big\} \xi^k \xi^j \ \geq \varepsilon \mid \xi \mid^2 . \tag{8.2.3}$$

2. If in addition, $\mathbf{\Pi}(u)$ is uniformly concave [respectively uniformly convex] on $W^{1,2}(X,d\mu)$ uniformly for all $\mu \ll \mathcal{L}^m$ if and only if $\pm h > 0$ and (8.2.3) holds with $\varepsilon > 0$.

Proof. Similar to the proof of Example 8.2.1, $\mathbf{\Pi}(u)$ is concave for all $\mu \ll \mathcal{L}^m$ if and only if $(\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}}\bar{G}^{\bar{i},\bar{l}}\bar{G}_{\bar{i},\bar{k}\bar{j}})$ is non-positive definite, and uniformly concave uniformly for all $\mu \ll \mathcal{L}^m$ if and only if this tensor is uniform negative definite.

Since $D_{x,y}b(x,y)$ has full rank for each $(x,y) \in X \times Y$, and for all $(x,y,z) \in X \times Y \times Z$, $1 - (f_z)^{-1}b_{,\beta}b^{\alpha,\beta}f_{\alpha,z} \neq 0$, for $\pi(x,y,z) = z - a(y)$, $\bar{G}(x,x_0,y,z) = x_0(b(x,y) - f(x,z))$, we have

$$
\begin{aligned}
&-(\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}}\bar{G}^{\bar{i},\bar{l}}\bar{G}_{\bar{i},\bar{k}\bar{j}}) \\
&= \begin{pmatrix} a_{kj} - a_l b^{i,l} b_{i,kj} + \dfrac{(a_\beta b^{\alpha,\beta} f_{\alpha,z} - 1)(b_{,kj} - b_{,l}b^{i,l}b_{i,kj})}{f_z - b_{,\beta}b^{\alpha,\beta}f_{\alpha,z}} & \mathbf{0} \\ \mathbf{0} & h(x,y,z) \end{pmatrix},
\end{aligned}
$$

where $h(x,y,z) = a_l b^{i,l} f_{i,zz} + \frac{(a_\beta b^{\alpha,\beta} f_{\alpha,z}-1)(b_{,l}b^{i,l}f_{i,zz} - f_{zz})}{f_z - b_{,\beta}b^{\alpha,\beta}f_{\alpha,z}}$. Since $h(x,y,z) \geq 0$, then $(\pi_{,\bar{k}\bar{j}} - \pi_{,\bar{l}}\bar{G}^{\bar{i},\bar{l}}\bar{G}_{\bar{i},\bar{k}\bar{j}})$ is non-positive definite if and only if there exist $\varepsilon \geq 0$ such that each $(x,y,z) \in X \times Y \times Z$ and $\xi \in \mathbf{R}^n$ satisfy

$$
\left\{ a_{kj} - a_l b^{i,l} b_{i,kj} + \frac{(a_\beta b^{\alpha,\beta} f_{\alpha,z} - 1)(b_{,kj} - b_{,l}b^{i,l}b_{i,kj})}{f_z - b_{,\beta}b^{\alpha,\beta}f_{\alpha,z}} \right\} \xi^k \xi^j \ \geq \varepsilon \mid \xi \mid^2 .
$$

In addition, $\mathbf{\Pi}(u)$ is uniformly concave uniformly for all $\mu \ll \mathcal{L}^m$ if and only if $h > 0$ and $\varepsilon > 0$. $\square$

Example 8.2.4 asserts the concavity of monopolist's maximization in the zero-sum setting, where the agent's utilities are relatively general but the principal's profit is extremely special. Also, more non-quasilinear examples could be discovered by applying Lemma 5.2.8.

Example 8.2.4 (Zero sum transactions)**.** Take $\pi(x,y,z) = -G(x,y,z)$, satisfying (G0)-(G5) and $\mu \ll \mathcal{L}^m$, which means the monopolist's profit in each transaction coincides exactly with the agent's loss. From (4.2.1), since G is linear on G-segments, we know $\mathbf{\Pi}(u)$ is linear.

Bibliography

[1] M. Armstrong. Multiproduct nonlinear pricing. *Econometrica*, 64:51–75, 1996.

[2] E. J. Balder. An extension of duality-stability relations to non-convex optimization problems. *SIAM J. Control Optim.*, 15:329–343, 1977.

[3] D. P. Baron and R. B. Myerson. Regulating a monopolist with unknown costs. *Econometrica*, 50:911–930, 1982.

[4] S. Basov. *Multidimensional screening.* Springer-Verlag, Berlin, 2005.

[5] G. Carlier. A general existence result for the principal-agent problem with adverse selection. *J. Math. Econom.*, 35:129–150, 2001.

[6] G. Carlier and T. Lachand-Robert. Regularity of solutions for some variational problems subject to convexity constraint. *Comm. Pure Appl. Math.*, 54:583–594, 2001.

[7] S. Dolecki and S. Kurcyusz. On ϕ-convexity in extremal problems. *SIAM J. Control Optim.*, 16:277–300, 1978.

[8] I. Ekeland and R. Temam. *Analyse convexe et problémes variationnels.* Dunod (Libraire), Paris, 1976.

[9] K.-H. Elster and R. Nehse. Zur theorie der polarfunktionale. *Math. Operationsforsch. Statist.*, 5:3–21, 1974.

[10] L. C. Evans. *Partial Differential Equations.* American Mathematical Society, Providence, Rhode Island, 1998.

[11] A. Figalli, Y.-H. Kim, and R. J. McCann. When is multidimensional screening a convex program? *J. Econom. Theory*, 146:454–478, 2011.

[12] W. Gangbo and R. J. McCann. The geometry of optimal transportation. *Acta Math.*, 177:113–161, 1996.

[13] N. Gigli. On the inverse implication of brenier-mccann theorems and the structure of $(p_2(m), w_2)$. *Methods Appl. Anal.*, 18:127–158, 2011.

[14] R. Guesnerie and J.-J. Laffont. Taxing price makers. *J. Econom. Theory*, 19:423–455, 1978.

[15] N. Guillen and J. Kitagawa. On the local geometry of maps with c-convex potentials. *Calc. Var. Partial Differential Equations*, 52(1-2):345–387, 2015.

[16] N. Guillen and J. Kitagawa. Pointwise estimates and regularity in geometric optics and other generated jacobian equations. *Comm. Pure Appl. Math.*, 70:1146–1220, 2017.

[17] O. Kadan, P. J. Reny, and J. M. Swinkels. Existence of optimal mechanisms in principal-agent problems. *Econometrica*, 85(3):769–823, 2017.

[18] Y.-H. Kim and R. J. McCann. Continuity, curvature, and the general covariance of optimal transportation. *J. Eur. Math. Soc.*, 12:1009–1040, 2010.

[19] S. S. Kutateladze and A. M. Rubinov. Minkowski duality and its applications. *Russian Math. Surveys*, 27:137–192, 1972.

[20] G. Loeper. On the regularity of solutions of optimal transportation problems. *Acta Math.*, 202:241–283, 2009.

[21] X.-N. Ma, N. S. Trudinger, and X.-J. Wang. Regularity of potential functions of the optimal transportation problem. *Arch. Ration. Mech. Anal.*, 177:151–183, 2005.

[22] J. E. Martínez-Legaz. Generalized convex duality and its economic applications. In *Handbook of generalized convexity and generalized monotonicity*, pages 237–292. Springer, New York, 2005.

[23] E. Maskin and J. Riley. Monopoly with incomplete information. *Rand J. Econom.*, 15:171–196, 1984.

[24] R. P. McAfee and J. McMillan. Multidimensional incentive compatibility and mechanism design. *J. Econom. Theory*, 46:335–354, 1988.

[25] R.J. McCann and K.S. Zhang. On concavity of the monopolist's problem facing consumers with nonlinear price preferences. *To appear in Comm. Pure and Applied Math.*

[26] J. A. Mirrlees. An exploration in the theory of optimum income taxation. *Rev. Econom. Stud.*, 38:175–208, 1971.

[27] P. K. Monteiro and F. H. Page Jr. Optimal selling mechanisms for multiproduct monopolists: incentive compatibility in the presence of budget constraints. *J. Math. Econom.*, 30:473–502, 1998.

[28] J.-J. Moreau. Inf-convolution, sous-additivité, convexité des fonctions numériques. *J. Math. Pures Appl.*, 49:109–154, 1970.

[29] M. Mussa and S. Rosen. Monopoly product and quality. *J. Econom. Theory*, 18:301–317, 1978.

[30] R. B. Myerson. Incentive compatibility and the bargaining problem. *Econometrica*, 47:61–73, 1979.

[31] R. B. Myerson. Optimal auction design. *Math. Oper. Res.*, 6:58–73, 1981.

[32] G. Nöldeke and L. Samuelson. The implementation duality. *Econometrica*, 86(4):1283–1324, 2018.

[33] M. Quinzii and J.-C. Rochet. Multidimensional screening. *J. Math. Econom.*, 14:261–284, 1985.

[34] K. W. S. Roberts. Welfare considerations of nonlinear pricing. *The Economic Journal*, 89:66–83, 1979.

[35] J.-C. Rochet. The taxation principle and multitime Hamilton-Jacobi equations. *J. Math. Econom.*, 14:113–128, 1985.

[36] J.-C. Rochet. A necessary and sufficient condition for rationalizability in a quasi-linear context. *J. Math. Econom.*, 16:191–200, 1987.

[37] J.-C. Rochet and P. Choné. Ironing sweeping and multidimensional screening. *Econometrica*, 66:783–826, 1998.

[38] J.-C. Rochet and L. A. Stole. The economics of multidimensional screening. In M. Dewatripont, L. P. Hansen, and S. J. Turnovsky, editors, *Advances in economics and econometrics*, pages 150–197. Cambridge University Press, Cambridge, 2003.

[39] A. M. Rubinov. *Abstract convexity and global optimization*, volume 44 of *Nonconvex optimization and its applications.* Kluwer Academic Publ., Boston-Dordrecht-London, 2000.

[40] A. M. Rubinov. Abstract convexity: examples and applications. *Optimization*, 47:1–33, 2000.

[41] I. Singer. *Abstract convex analysis.* Wiley-Interscience, New York, 1997.

[42] M. Spence. Competitive and optimal responses to signals: an analysis of efficiency and distribution. *J. Econom. Theory*, 7:296–332, 1974.

[43] M. Spence. Multi-product quantity-dependent prices and profitability constraints. *Rev. Econom. Stud.*, 47:821–841, 1980.

[44] N. S. Trudinger. On the local theory of prescribed jacobian equations. *Discrete Contin. Dyn. Syst.*, 34:1663–1681, 2014.

[45] R. V. Vohra. *Mechanism design: a linear programming approach.* Cambridge University Press, Cambridge, 2011.

[46] R. Wilson. *Nonlinear pricing.* Oxford University Press, Oxford, 1993.