Long-time asymptotic expansions for nonlinear diffusions in Euclidean space*

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The purpose of this announcement is to describe a few recent advances [12] [23] in our understanding of the long-time behavior of the nonlinear diffusion equation

$$\frac{\partial \rho}{\partial \tau} = \nabla \cdot (\rho^{m-1} \nabla \rho), \tag{1}$$

which governs the evolution of a density $\rho(\tau, \cdot) \geq 0$ on $\mathbf{R}^n$. For $m \neq 1$ this dynamics generalizes the linear heat equation to the case in which the thermal conductivity (or diffusion coefficient) is given by a power ρ^{m-1} of the diffusing density. It can also be viewed as a scalar conservation law

$$\frac{\partial \rho}{\partial \tau} = \nabla \cdot (\rho \nabla (\frac{\rho^{m-1}}{m-1})) \tag{2}$$

in which the density ρ is advected by the gradient of the pressure $\frac{1}{m-1}\rho^{m-1}$. The ranges $m > 1$ and $m < 1$ are known as the *porous medium* and *fast diffusion* regimes respectively, depending on whether the rate of diffusion

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(or pressure) varies directly or inversely with density; their phenomenology, history and motivating applications are described in the book of Vázquez [25].

The advances described hereafter involve understanding the long-time behavior of solutions starting from integrable initial data of sufficiently rapid decay; to fix ideas we shall call the initial profile

$$\rho_0(\cdot) = \lim_{\tau \to 0} \rho(\tau, \cdot) \tag{3}$$

nice if it is integrable, non-negative, and compactly supported; the sense in which this limit holds needs to be made precise by specifying an appropriate topology. We are especially interested in the rates at which the dynamics causes different aspects of the initial profile to be dissipated / suppressed / forgotten. To understand what is possible in this direction, let us begin by recalling the familiar situation for the linear heat equation on $\mathbf{R}^n$. There a well-known conjugacy to the quantum harmonic oscillator yields an expansion (6) to all orders which describes the decay of the various modes, as we now recall; c.f. Bartier et al [3] and the references there.

1 Long-time asymptotics for the heat equation on $\mathbf{R}^n$

Fourier transforming the heat equation $\frac{\partial \rho}{\partial \tau} = \Delta \rho$ on $\mathbf{R}^n$ yields an exact formula $\hat{\rho}(\tau, k) = \hat{\rho}(0, k) e^{-|k|^2 \tau}$ for the rate of decay of the k-th Fourier mode

$$\hat{\rho}(\tau, k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{ik \cdot x} \rho(\tau, x) dx.$$

Only the zeroth Fourier mode fails to decay — since net mass is invariant under the heat flow. This description reflects the fact that nice initial data decay to zero under the heat flow, in any $L^p(\mathbf{R}^n)$ norm with $p > 1$.

However, this description misses many of the salient aspects of the evolution which are apparent either from its description in terms of Brownian motion, or from its explicit solution, expressed as a convolution of the initial data with the heat kernel:

$$\rho(\tau, y) = \frac{1}{(4\pi\tau)^{n/2}} \int_{\mathbf{R}^n} \rho_0(z) e^{-|y-z|^2/4\tau} dz.$$

Either perspective shows that mass spreads in all directions from its initial location a distance proportional to $\tau^{1/2}$ in time τ, and moreover that the shape of this spreading mass will necessarily become more and more Gaussian as time evolves, and details of the initial data are averaged away. It is the rate of this *averaging away* that we are interested in quantifying.

To do so, let us renormalize the flow by setting

$$\rho(\tau, y) = \frac{1}{\tau^{n/2}} u(\log \tau, \frac{y}{\tau^{1/2}}). \tag{4}$$

Changing dependent variables from ρ to u corresponds to viewing the evolving mass distribution from a receding perspective: at each instant in time, the density $u(\log \tau, \cdot)$ has the same $L^1(\mathbf{R}^n)$ mass as ρ_0, and corresponds to the density $\rho(\tau, \cdot)$ viewed from distance $\tau^{1/2}$.

A standard computation

$$\frac{\partial \rho}{\partial \tau} - \Delta \rho = \tau^{-\frac{n+2}{2}} \left[-\frac{n}{2} u + \frac{\partial u}{\partial t} - \frac{1}{2} x \cdot \nabla u - \Delta u \right]_{(t,x)=(\log \tau, y/\tau^{1/2})}.$$

shows ρ to be a solution of the heat equation if and only if

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{2} \nabla \cdot (xu) =: -Lu.$$

This evolution fixes the Gaussian $u(t,x) = e^{-x^2/4} =: u_\infty(x)$, corresponding to a self-similar solution of the original dynamics, which is proportional to the heat kernel: $\rho(\tau, y) = \tau^{-\frac{n}{2}} e^{-\frac{y^2}{4\tau}}$. The variables $(t,x) = (\log \tau, y/\tau^{1/2})$ are sometimes called self-similar coordinates.

Unlike the generator $-\Delta$ of the original dynamics, the operator L is not self-adjoint on $L^2(\mathbf{R}^n)$, though it is self-adjoint on the weighted space $L^2(\mathbf{R}^n, u_\infty^{-1} d^n x)$. Notice the related quantity $v_\theta(t,x) = u_\infty^{-\theta}(x) u(t,x)$ evolves according to a dynamics generated by $L_\theta := u_\infty^{-\theta} L u_\infty^{\theta}$, namely

$$\begin{aligned} -\frac{\partial v_\theta}{\partial t} &= L_\theta v_\theta \\ &= -\Delta v_\theta + (\theta - \tfrac{1}{2}) x \cdot \nabla v_\theta - (1-\theta) \tfrac{n}{2} v_\theta + \theta(1-\theta) \tfrac{|x|^2}{4} v_\theta. \end{aligned} \tag{5}$$

Choosing $\theta = 1$, we see the evolution of the relative density $v_1 = u/u_\infty$ is generated by a self-adjoint operator $L_1 = u_\infty^{-1} H u_\infty$ on the weighted space

$L^2(\mathbf{R}^n, e^{-|x|^2/4}dx)$ as in [3]. More remarkably, choosing $\theta = \frac{1}{2}$ we see the dynamics of $v_{1/2}$ is generated by

$$L_{1/2}v = -\Delta v - \tfrac{n}{4}v + \tfrac{1}{16}|x|^2 v$$

which acts self-adjointly on the *unweighted* $L^2(\mathbf{R}^n)$. Notice that $L_{1/2}$ is essentially the Hamiltonian of the quantum harmonic oscillator, whose spectrum $\sigma(L_{1/2})$ is well-known to consist of the non-negative integers and half-integers: $\sigma(L_{1/2}) = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. For $\vec{k} = (k_1, \ldots, k_n)$ with non-negative integer components, the normalized eigenfunction corresponding to eigenvalue $\lambda_{\vec{k}} := \frac{1}{2}\sum^n k_i$ is

$$\psi_{\vec{k}}(x) = \frac{1}{(4\pi)^{n/4}} e^{-|x|^2/8} \prod_{i=1}^{n} H_{k_i}(x_i/2)$$

where $H_k(x) = \frac{(-1)^k}{\sqrt{2^k k!}} e^{x^2} \frac{d^k}{dx^k}(e^{-x^2})$ is the k-th Hermite polynomial. Thus we can expand $v_{1/2}(t,x) = \sum c_{\vec{k}}(t)\psi_{\vec{k}}(x)$ in $L^2(\mathbf{R}^n)$, where $c_{\vec{k}}(t) = e^{-\lambda_{\vec{k}}t}c(0)$ and

$$\begin{aligned} c_{\vec{k}}(0) &= \int_{\mathbf{R}^n} \psi_{\vec{k}}(x) v_{1/2}(0,x)dx \\ &= \int_{\mathbf{R}^n} \psi_{\vec{k}}(x) u(0,x) e^{|x|^2/8} dx. \end{aligned}$$

Equivalently,

$$\left\| e^{|x|^2/8}u(t,x) - \sum_{\{0 \le k_i \in \mathbf{N} \mid \frac{1}{2}\sum^n k_i < \Lambda\}} c_{\vec{k}}(0) e^{-t\sum^n k_i/2} \psi_{\vec{k}}(x) \right\|_{L^2(\mathbf{R}^n)} \le Ce^{-\Lambda t} \qquad (6)$$

as $t \to \infty$, where $C \le (\sum_{\Lambda \le \lambda_{\vec{k}}} c_{\vec{k}}(0)^2)^{1/2} \le \|u(0,x)e^{|x|^2/8}\|_{L^2(\mathbf{R}^n)}$. The factor $u_\infty^{-1/2}$ multiplying the solution $u(t,x)$ is reciprocal to the Gaussian factor in the eigenfunctions and suggests the convenience of expressing the convergence in appropriately weighted spaces; also, additional eigenfunctions with known coefficients lead to faster and faster rates of decay.

2 Nonlinear diffusion

If one is interested in the effects produced by a density dependent rate ρ^{m-1} of diffusion (1), it is natural to wonder whether there is a description of the long-time behavior of this nonlinear evolution analogous to the linear case, in spite of the fact that the available tools for investigating the nonlinear problem must necessarily be quite different.

Since the behavior depends crucially on the exponent m, let us set $m_p = 1 - \frac{2}{n+p}$, where p is moment index introduced in [13]. Three distinct ranges of interest are: the porous medium regime $m > m_\infty = 1$, the extinction regime $m < m_0 = 1 - \frac{2}{n}$ and the (conservative) fast diffusion regime $m \in]m_0, 1[$.

In each of these three ranges, there is an explicit family of solutions discovered by Barenblatt [2], Zeldovich, Kompaneetz [27], and Pattle [22]: the self-similar BPKZ family

$$\rho_B(\tau, y) = \frac{1}{\tau^{n\beta}} u_B(\frac{y}{\tau^\beta})$$

with $\beta = \frac{1}{2+n(m-1)} = \frac{1}{2}(1 + \frac{n}{p})$ and

$$u_B(y) := \left[B + \frac{1-m}{2}|y|^2\right]_+^{\frac{1}{m-1}} = \left[B + \frac{|y|^2}{n+p}\right]_+^{-\frac{n+p}{2}}$$

where $[\lambda]_+ = \max\{\lambda, 0\}$. Here $B > 0$ is a positive constant used to adjust the mass of the solution, which is finite for $m > m_0$. The behavior manifested by these solutions varies across the three regimes mentioned above. In the porous medium regime it is a classical solution where positive, but because the rate of diffusion slows down where the density is small, the property of having compact support is preserved by the flow, and one has to understand the equation at the free boundary where ρ vanishes as prescribing that the free boundary move with a velocity given by the gradient of the pressure $\rho^{m-1}/(m-1)$, which is consistent with the conservation law (2) and typically incorporated into a suitable definition of weak solution. In the fast diffusion regime on the other hand, the rate of diffusion diverges at low densities, so that compactly supported initial data instantaneously develop thick tails whose moments are finite only up to order p; the BPKZ solution is a classical solution for $t > 0$, which has finite mass if $m > m_0$, and infinite mass otherwise.

Clearly the BPKZ solutions are poor models for the behavior of nice initial data under the flow in the range $m < m_0$, where the phenomenology of the equation is quite different. In this range of nonlinearities, finite mass initial profiles instantaneously develop tails so fat that mass leaks out at infinity, draining completely in finite time. This extinction phenomenon has to be modeled using different approach, as in [10] [5]. We shall have nothing to say about this case.

For the range $m > m_0$ on the other hand, it has been known since the work of Friedman and Kamin [16] that ρ_B acts as an global attractor in L^1 for the flow starting from nice initial data: $\|\rho(\tau, \cdot) - \rho_B(\tau, \cdot)\|_1 = o(1)$ as $\tau \to \infty$, where B is chosen so the initial mass of the solutions being compared coincides. For the one-dimensional porous medium equation $n = 1 < m$, Angenent [1] was able to provide a complete description of the long time asymptotic behavior based on a linearization which Bareblatt and Zeldovich [26] had diagonalized; he noted the possibility of resonances for rational m. Little was known in the complementary range of (n, m) for some years after that.

The turn of millenium marked two directions of progress on this question: Koch's habilitation established a potential framework for extending the results of Angenent to higher dimensions, if the linear problem could be diagonalized [18]. At the same time, three groups of authors [8] [11] [21] were able to quantify the L^1-convergence rate sharply in the range $m \geq m_n = \frac{n-1}{n}$, showing $\|\rho(\tau, \cdot) - \rho_B(\tau, \cdot)\|_1 = O(\tau^{-\beta})$. Otto's method for doing this has proved particularly influential. Rescaling the solution

$$\rho(\tau, y) = \frac{1}{\tau^{n\beta}} u(\log \tau, \frac{y}{\tau^{\beta}}). \tag{7}$$

in analogy with the linear case (4), he was able to show the rescaled dynamics

$$\frac{\partial u}{\partial t} = \frac{1}{m}\Delta(u^m) + \frac{1}{2}\nabla \cdot (xu) \tag{8}$$

to be the gradient flow of an entropy

$$E(u) = \frac{2}{m(m-1)} \int_{\mathbf{R}^n} u^m(x) dx + \frac{1}{2} \int_{\mathbf{R}^n} u(x)|x|^2 dx$$

with respect to the 2-Wasserstein distance

$$d_2(u, \tilde{u})^2 = \inf_{\gamma \in \Gamma} \int_{\mathbf{R}^n \times \mathbf{R}^n} |x - y|^2 d\gamma(x, y).$$

This infimum is taken over all joint measures $\gamma \geq 0$ on $\mathbf{R}^n \times \mathbf{R}^n$ with marginals u and $\tilde{u}$ respectively. Obviously $d_2 = +\infty$ unless u has the same mass of $\tilde{u}$. Note the Barenblatt profile u_B minimizes $E(u)$ among densities u with fixed mass. For $m \geq m_n$, the entropy was known to be convex along 2-Wasserstein geodesics since McCann [19]; its modulus of convexity translates into a sharp rate of d_2 contraction produced by the flow, which through suitable analysis can be converted into an L^1 rate of convergence [21].

These analyses inspired various developments. On the one hand, Otto's gradient flow formulation suggested that the linearization of the rescaled dynamics around the fixed profile u_B would be governed by the Hessian of $E(u)$ at u_B. This Hessian acts self-adjointly on the tangent space to the set of probability measures, metrized by the weighted Hilbert space norm $W^{1,2}(\mathbf{R}^n, u_B)$. According to Benamou and Brenier, this norm plays the role of a metric tensor generating the 2-Wasserstein distance [4]. In the fast diffusion regime $m < 1$, the spectrum of this Hessian was computed by Denzler and McCann [13] [14]. It consists of a finite number of eigenvalues

$$\begin{aligned}\lambda_{\ell k} &= \frac{\ell + 2k + (m-1)(2\ell + 2k + n - 2)k}{2 + n(m-1)} \\ &= \frac{1}{2p}[(\ell + 2k)p + n\ell + 4k(1 - \ell - k)] \end{aligned} \tag{9}$$

plus a semi-infinite interval of continuous spectrum beginning at

$$\lambda_0^{cts} = \frac{1}{2 - n(1-m)} \frac{[(1-m)(1-\frac{n}{2}) + 1]^2}{2(1-m)} = \frac{1}{2p}(\frac{p}{2} + 1)^2. \tag{10}$$

Here $\ell, k \in \mathbf{N}$ are non-negative integers the corresponding eigenfunctions are polynomials of degree $\ell + 2k < \frac{p}{2} + 1$ — just small enough to lie in the weighted space $W^{1,2}(\mathbf{R}^n, u_B)$. The multiplicity of $\lambda_{\ell k}$ coincides with the multiplicity of the ℓ-th spherical harmonic on $\mathbf{S}^{n-1}$ except at eigenvalue crossings (where $\lambda_{\ell k} = \lambda_{\ell' k'}$ with $(\ell, k) \neq (\ell', k')$). The lowest lying eigenvalues λ_{01} and λ_{10} correspond to translations in time and space, which commute with the flow (1); the next higher eigenvalue λ_{20} corresponds to affine shears which do not.

Concerning the nonlinear problem, it was shown that the L^1 rate of convergence can be improved to $O(\tau^{-1})$ for initial data which is radially symmetric [9] (by Carrillo and Vázquez) or at least has its center of mass at the origin [20] (by McCann and Slepcev); the faster rate turns out to extend all the way to the threshold of the extinction regime $m > m_0$ in these cases;

see Carrillo and Vázquez for the radial case [9], Kim and McCann for the case $m \in]m_0, m_2]$ [17] and Bonforte, Dolbeault, Grillo and Vázquez [6] or Denzler, Koch and McCann [12] for the general case. Sharp rates of convergence in entropy and L^1 senses were eventually found in the full range of m by Blanchet, Bonforte, Dolbeault, Grillo and Vazquez [5]. In the fast-diffusion regime $m \in]m_0, 1[$, Vazquez also observed that convergence occurs in a stronger topology: the ratio of any two solutions tends to a constant in $L^\infty(\mathbf{R}^n)$, at a rate which has subsequently quantified by various groups of the authors above [9] [17] [12] [6].

Although a further improvement becomes possible by centering the data in time as well as in space [15], what has remained elusive is a statement analogous to (6). Very recently, Christian Seis diagonalized the Hessian $D^2E(u)$ in the porous medium regime $m > 1$. In contrast to the fast diffusion setting [14], which is plagued by the presence of continuous spectrum (9)–(10), he obtains a complete basis of eigenfunctions. However, it remains to be seen whether his diagonalization can be married to Koch's framework [18] to produce a description of porous medium asymptotics in higher dimensions analogous to Angenents results on the line [1]. In the present manuscript we describe how such a marriage has been accomplished in the the fast diffusion regime $m \in]m_0, 1[$ by Denzler, Koch and McCann [12].

3 A dynamical systems approach

Departing for a moment from the (infinite-dimensional) PDE setting, let us review what we are trying to achieve in the context of a (finite-dimensional) ODE setting. If we are interested in the long-time behavior of the initial value problem

$$x'(t) = -V(x(t)) \in \mathbf{R}^n \quad \text{with} \quad x(0) = x_0,$$

we can linearize the flow near near each fixed point $V(x_\infty) = 0$:

$$(x(t) - x_\infty)' = -DV(x_\infty)(x(t) - x_\infty)' + O(x(t) - x_\infty)^2; \tag{11}$$

the eigenvalues of $DV(x_\infty)$ then determine the flow behavior nearby. If, in addition, the vector field $V(x) = DE(x)$ has a gradient structure, then $DV(x) = D^2E(x)$ is a symmetric matrix and its eigenvalues are real; denote them by $\sigma(DE^2(x_\infty)) = \{\lambda_1 \le \lambda_2 \le \dots \lambda_n\}$. Then it is natural to expect

$$x(t) - x_\infty = \sum_{i=1}^{n} c_i e^{-\lambda_i t} + O(e^{2\lambda_1 t}),$$

which is in fact what happens unless the resonance $2\lambda_1 \in \sigma(D^2E(x_\infty))$ occurs between the linear and quadratic terms in (11), in which case the error term might be larger by a polynomial factor in t. Notice however, that this heuristic requires differentiable dependence of the vector field $V(x)$ or equivalently of the flow $X(t; x_0)$ on its initial condition x_0, at least near x_∞. In the PDE context, this will mean we will need a well-posedness result which guarantees differentiable (as opposed to continuous) dependence on initial conditions.

4 The result

The strategy of [12] is to adapt the finite-dimensional procedure caricatured above to the infinite-dimensional evolution of interest. The first challenge is to identify functional spaces in which the nonlinearity of the problem can be controlled, to yield a well-posedness result which includes differentiable dependence of the flow on initial conditions. This requires confronting — among other things — the degenerate parabolicity of the equation (8). Moreover, it turns out that the spaces in which this can be achieved are quite different from the spaces in which the linearized problem diagonalizes, a mismatch which must be reconciled. Finally, the possibilities of eigenvalue resonances and continuous spectrum must be addressed.

As a sample of the results obtained: let us restrict our attention to initial conditions with center of mass at the origin so the low lying mode λ_{10} is not excited; the lowest remaining mode is then λ_{01}. Fix a desired rate Λ of expontential decay as in (6). To avoid resonances and continuous spectra, assume Λ lies in the interval $2\lambda_{01} > \Lambda \in [\lambda_{01}, \lambda_0^{cts}]$.

Theorem 4.1 (Fast diffusion asymptotics in weighted spaces) *Fix $p = 2(1-m)^{-1} - n > 2$ and $2\lambda_{01} > \Lambda \in [\lambda_{01}, \lambda_0^{cts}]$. There is a sequence of polynomials $\{\phi_{\ell k}(x)\}$ — with $\phi_{\ell k}(x)$ having degree $\ell + 2k \in]1, \frac{p}{2}+1[$ — such that: For each solution $u(t,x)$ with integrable, compactly supported initial data u_0*

and center of mass at the origin, there are coefficients $c_{\ell k}$ such that

$$\left\| \frac{[\frac{u(t,x)}{u_B(x)} - 1] - \frac{1}{B+\frac{|x|^2}{n+p}} \sum_{0<\lambda_{\ell k}<\Lambda} c_{\ell k}\phi_{\ell k}(x) e^{-\lambda_{\ell k} t}}{(B + \frac{|x|^2}{n+p})^{\left(p-2-\sqrt{(p+2)^2-4\Lambda}\right)/4}} \right\|_\infty = O(e^{-\Lambda t}) \qquad (12)$$

as $t \to \infty$, where the sum is over non-negative integers $k, \ell \in \mathbf{N}$ for which $\lambda_{\ell k} = (1 + \frac{n}{p})\ell/2 + k + 2(1 - \ell - k)k/p$ lies in the interval $]0, \Lambda[$ (and for which $\ell \leq 1$ if $n = 1$).

Let us remark on several aspects of this result beyond its resemblance to (6). Here the space $L^\infty(\mathbf{R}^n)$ satisfies the algebra property $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ which is relevant for controlling nonlinear corrections. The degree $\ell + 2k$ polynomials $\phi_{\ell k}(x)$ are the eigenfunctions of $D^2 E(u_B)$; even after division by $B + \frac{|x|^2}{n+p}$ they cannot lie in unweighted L^∞ unless $\ell + 2k < 2$. Thus the more terms which appear in the sum approximating u/u_B, the more severely the weighted norm must discount growth at infinity to ensure the sum remains in the space. The $c_{\ell k}$ represent the amplitudes of each excited mode in the range $]0, \Lambda[$. One may naturally wonder how many distinct modes fall into this range? For appropriate choices of m and λ the answer can be as many as eight; it is possible to access even more modes by translating u in time to ensure the mode λ_{01} is not excited [12]. In contrast to the linear case, it is not possible to read the amplitudes $c_{\ell k}$ off the initial data in any obvious way except when $k = 0$; in this case the eigenfunction $\phi_{\ell 0}(x)$ is a harmonic polynomial, whose integral against the solution is therefore a conserved quantity of the original flow $m\rho_\tau = \Delta \rho^m$.

5 A few ideas from the proof

While we do not attempt even to sketch a proof here, we can never the less mention a few of its key ingredients.

Since the solution $u(t, x)$ decays to zero at spatial infinity, it does not stay a uniform distance from the singularity at zero of the nonlinearity $u \mapsto u^m$. To overcome this lack of smoothness, we reexpress the dynamics in terms of the *relative density* $v(t, x) := u(t, x)/u_B(x)$; unlike the density, the relative density stays bounded above and below according to maximum principle type arguments of Vázquez; it tends uniformly to the constant 1 for an appropriate choice of B [24].

The relative density satisfies an evolution equation whose second-order term

$$v_t = \frac{1}{m}\nabla\cdot[(B+\frac{|x|^2}{n+p})\nabla v^m] + l.o.t.(Dv, v, |x|, B). \tag{13}$$

appears degenerate parabolic as $|x| \to \infty$. To cure this degenerate parabolicity of the dynamics linearized at $v(t,x)=1$, we view $\mathbf{R}^n$ as a (conformally flat) Riemannian manifold (M,g) with the so-called *cigar* metric

$$ds^2 = \frac{1}{B+\frac{|x|^2}{n+p}}\sum_{i=1}^{n}(dx_i)^2,$$

introduced to this context independently by [7] and [12]. The second-order term in the dynamics (13) is then given by the Laplace-Beltrami operator $\Delta_{(M,g)}v^m/m$. This allows us to combine DeGiorgi-Nash-Moser regularity with the implicit function theorem to get differentiability of the flow $v_0 \in C^{k,\alpha}(M)\cap B^{L^\infty}_\epsilon(1) \longmapsto v \in C^{k,\alpha}([0,\infty[\times M)$ with respect to appropriate Hölder norms on the cigar — at least in a small uniform neighborhood of the fixed point $v_\infty = 1$.

The linearized dynamics $(v-1)_t = -L(v-1)+o(v-1)$ are generated by an operator $L : C^{k,\alpha}(M) \longrightarrow C^{k,\alpha}(M)$ given in the coordinates $ds = \frac{dr}{\sqrt{B+\frac{r^2}{n+p}}}$ where $r=|x|$ by an expression like

$$\begin{aligned} L_\theta &= (\cosh s)^{-\theta}\circ L\circ(\cosh s)^{\theta} \\ &= -\Delta_{(M,ds^2)} + 2(\tfrac{p}{2}-1-\theta)\tanh s\frac{\partial}{\partial s} + (\tfrac{p}{2}+1)^2 \\ &\quad -(\tfrac{p}{2}-1-\theta)^2 - ((\tfrac{n}{2}+\tfrac{p}{2}+1)^2-(\tfrac{n}{2}+\tfrac{p}{2}-1-\theta)^2)\tfrac{1}{\cosh^2 s}. \end{aligned}$$

Here θ is selecting the strength of the weight, as in (5). Choosing $\theta=\theta_{cr} := \frac{p}{2}-1$ suppresses the drift term, reducing L_θ to a Schrödinger operator on the cigar manifold with a universal potential. This operator is related to $H = D^2E|_{u_B} : W^{1,2}(\mathbf{R}^n, u_B) \longrightarrow W^{1,2}(\mathbf{R}^n,u_B)$ through conjugation $L_{\theta_{cr}}\circ\Lambda = \Lambda\circ H$ by the differential operator $\Lambda\phi = \frac{1}{u_B}\nabla\cdot(u_B\nabla\phi) = \frac{1}{B+\frac{|x|^2}{n+p}}\circ H$ and also by the multiplication operator $L_{\theta_{cr}}\circ\frac{1}{B+\frac{|x|^2}{n+p}} = \frac{1}{B+\frac{|x|^2}{n+p}}\circ H$. Here

$$H\phi = -(B+\frac{|x|^2}{n+p})\Delta_{\mathbf{R}^n} + (p+n)x\cdot\nabla\phi$$

is the operator diagonalized by Denzler and McCann [13] [14] and s is geodesic distance along the cigar.

The decay rate Λ of the error term in (12) determines the relevant choice of $\theta \neq \theta_{cr}$. Thus we actually work in weighted Hölder spaces on the cigar, but the weighted Hölder norms also control weighted L^∞.

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