SMOOTH OPTIMAL TRANSPORTATION ON HYPERBOLIC SPACE

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ABSTRACT. In this paper, we will show that the cost $-\cosh \circ d_{\mathbb{H}^n}$ is a regular cost, meaning that minimizing this cost on hyperbolic space yields a smooth optimal map between two given distributions of mass which satisfies suitable hypotheses. We show this by proving this cost satisfies Ma-Trudinger-Wang's conditions and by investigating notions of convexity under this cost.

1. Introduction

Let M^+ and M^- be Borel subsets of compact separable metric spaces that are equipped with Borel probability measures ρ^+ and ρ^-. Let c : $\text{cl}(M^+ \times M^-) \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous transport cost. The Kantorovich problem [1] is to find measure $\gamma \geq 0$ on $M^+ \times M^-$ whose total cost is minimal among $\Gamma(\rho^+, \rho^-)$. Here total cost is

$$\int_{M^+ \times M^-} c(x, y) d\gamma(x, y)$$

and $\Gamma(\rho^+, \rho^-)$ denotes the set of joint probability measures having the same left and right marginals as $\rho^+ \otimes \rho^-$. It can be shown that such a minimizer exists; such γ is called *optimal*.

The Monge problem of optimal transport is to find a Borel map F : $M^+ \to M^-$ and optimal measure γ vanishing outside $\text{Graph}(F) = \{(x, y) \in M^+ \times M^- : y = F(x)\}$. If such F exists, it is called an *optimal map*. When M^+ and M^- are subsets of a smooth manifold and ρ^+ vanishes on Lipshitz submanifolds of lower dimension and the cost function $c(x, y)$ satisfies the twist condition (see (A1) below), an optimal map F exists and is unique; see Gangbo [3] or Levin [4]. Then one can study the regularity of the optimal map F and how it is influenced by the choice of the cost function. Caffarelli [5] [6] [7] studied the smoothness of optimal map under the cost of Euclidean distance squared $c(x, y) = |x - y|^2/2$. This cost is also studied by Delanöe [8] in the case of $\mathbb{R}^2$ and Urbas [9] in higher dimensions. Ma, Trudinger and Wang [10] used analytic methods in partial differential equations to give a

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sufficent condition (known as Ma-Trudinger-Wang's conditions) on the cost function and the domain for its optimal map to be smooth, that is,

Theorem 1.1. *If ρ^+ and ρ^- are smooth densities that have positive upper and lower bounds on bounded subsets M^+ and M^- in $\mathbb{R}^n$ respectively, and $c \in C^4(M^+ \times M^-)$ with (A1)-(A3) satisfied, then the optimal map F is smooth.*

(**A1**) *(Twist condition):*

$$x \in M^+ \hookrightarrow -D_y c(x, y_0) \in T^*_{y_0} M^-$$
$$y \in M^- \hookrightarrow -D_x c(x_0, y) \in T^*_{x_0} M^+$$

are smooth embeddings for all $x_0 \in M^+$ and $y_0 \in M^-$.

(**A2**) *(Convexity condition):*
$-D_y c(M^+, y_0) \subset T^*_{y_0} M^-$ and $-D_x c(x_0, M^-) \subset T^*_{x_0} M^+$ are convex for all $x_0 \in M^+$ and $y_0 \in M^-$.

(**A3**)*: There exists $C_0 \geq 0$ such that for $p, q \in \mathbb{R}^n$, $p^i c_{ij} q^j = 0$,*

$$(c^{k,l} c_{ij,k} c_{l,st} - c_{ij,st}) p^i p^j q^s q^t \geq C_0 |p|^2 |q|^2,$$

where $c_{ij,st}$ denotes $D_{x^i x^j y^s y^t} c$, and $c^{k,l}$ denotes the k, l entry of the inverse matrix of $D_{xy} c$.

The above condition was first linked to curvature by Loeper [13], who observed that when it is satisfied by the Riemannian distance square $c := d_g^2/2$ on $M^+ = M^-$, then the underlying manifold must have non-negative sectional curvature. Subsequently Kim and McCann [11] gave a geometric interpretation by putting a pseudo-Riemannian metric induced by the cost on $M^+ \times M^-$, and then (A2) turns into geodesic convexity condition of the product manifold, and (A3) into positivity condition of certain sectional curvature on $M^+ \times M^-$.

Definition 1.2. We call a smooth cost function which satisfies (A1) and (A3) *regular.*

As for example of weakly regular cost functions, Loeper [13] has shown the Riemannian distance squared on sphere is regular. Riemannian distance squared on other positively curved manifolds were studied by Delonöne and Ge [14], Figalli and Rifford [15], Loeper and Villani [16], and Kim [17]. Lee and McCann [18] have found examples that arise from mechanics. However, in [13], Loeper showed that Riemannian distance squared is not regular for manifold that is negatively curved somewhere. Consequently, for a while the negatively curved spaces were considered incompatible with regularity. However, it could be the case that Riemannian distance squared is not always the most appropriate cost to consider; for more general spaces, we need to study more general cost functions.

The main purpose of this paper is to show

Theorem 1.3. $-\cosh \circ d_{\mathbb{H}^n}$ *on hyperbolic space is regular.*

The proof is provided in section 2. In section 3 we discuss (A2) condition with respect to this cost.

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2. Hyperbolic space viewed as a graph

In this section we prove Theorem 1.3. By Lemma 4.4 of [12] and the fact that the hyperbolic space has no cut locus, we conclude $\cosh \circ d_{\mathbb{H}^n}$ satisfies (A1) twist condition. Then it follows that $-\cosh \circ d_{\mathbb{H}^n}$ satisfies (A1). Therefore it suffices to show $-\cosh \circ d_{\mathbb{H}^n}$ satisfies condition (A3).

One particular example discussed by Ma, Trudinger and Wang [10] corresponds to the cost function determined by the distance squared between points on graphs of functions over $\mathbb{R}^n$. In this case, a simple sufficient condition for (A3) is found. Here we view hyperbolic space as the graph of hyperbola in Minkowski space, and derive a similar result in this setting.

Minkowski space $\mathbb{M}^{n+1}$ is the space of $\mathbb{R}^{n+1}$ equipped with inner product

$$\langle (x, x_{n+1}), (y, y_{n+1}) \rangle_m = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

In the hyperboloid model, hyperbolic space can be viewed as the graph of hyperbola: $\mathbb{H}^n = \{u \in \mathbb{M}^{n+1} : \|u\|_m^2 = -1, u_{n+1} > 0\} = \{(x, \sqrt{1+|x|^2}) : x \in \mathbb{R}^n\}$, with its metric induced from Minkowski space. Even though the Minkowski metric is not positive definite, its restriction to $\mathbb{H}^n$ is, which makes hyperbolic space a Riemannian manifold. Also there is a nice relation between hyperbolic distance and (ambient) Minkowski inner product, that is: $-\cosh(d_{\mathbb{H}^n}(u, v)) = \langle u, v \rangle_m$. (For this identity, see [19].)

Proof of Theorem 1.3. We start with a general case. Let $M^+ = \{(x, f(x)) : x \in \Omega^+ \subset \mathbb{R}^n\}$, $M^- = \{(y, g(y)) : y \in \Omega^- \subset \mathbb{R}^n\}$ be two graphs in $\mathbb{M}^{n+1}$. Consider the cost function determined by the Minkowski inner product

$$c(x, y) := \langle (x, f(x)), (y, g(y)) \rangle_m = x \cdot y - f(x)g(y).$$

We can carry out a direct computation similar to that of page 22 of [10]:

$$
\begin{aligned}
c_{i,j} &= \delta_{i,j} - f_i g_j, \\
c^{i,j} &= \delta_{i,j} + \frac{f_i g_j}{1 - \nabla f \cdot \nabla g} \quad \text{if } \nabla f \cdot \nabla g \neq 1, \\
c_{ij,k} &= -f_{ij} g_k, \\
c_{l,st} &= -f_l gst, \\
c_{ij,st} &= -f_{ij} g_{st}, \\
\sum_{k,l} c^{k,l} c_{ij,k} c_{l,st} - c_{ij,st} &= \frac{f_{ij} g_{st}}{1 - \nabla f \cdot \nabla g}.
\end{aligned}
$$

Hence if f and g are convex, with gradients satisfying $\nabla f \cdot \nabla g < 1$, then c will satify (A3).

In the problem of optimal transportation on hyperbolic space, $f(x) = g(x) = \sqrt{1+|x|^2}$ and $c(x,y) = x \cdot y - \sqrt{1+|x|^2}\sqrt{1+|y|^2}$. Since the hyperbola is a strongly convex function, f_{ij} is positive definite. Also it can be easily checked by using Cauchy-Schwartz inequality that $1 - \nabla f \cdot \nabla g = 1 - \frac{x \cdot y}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} > 0$. Therefore $-\cosh \circ d_{\mathbb{H}^n}$ is regular. □

3. Notion of convexity

In this section we investigate the convexity requirement (A2) for the compact domains M^+ and M^-.

Definition 3.1 (cost exponential)**.** For cost functions satisfying (A1), the map $x \in M^+ \hookrightarrow -D_y c(x, y_0) \in T^*_{y_0} M^-$ is a smooth embedding. We call its inverse *the c-exponential map based at* y_0 and denote it by c-$\exp_{y_0}$. If $U \subset M^+$ is the image of a convex set in $T^*_{y_0} M^-$ under c-$\exp_{y_0}$, then U is said to be *c-convex with respect to* y_0. Notions of c-$\exp_{x_0}$ and c-convex with respect to x_0 can be defined similarly.

Remark 3.2. The convexity condition (A2) requires M^+ and M^- to be c-convex with respect to every y_0 in M^- and every x_0 in M^+ respectively. To understand (A2) condition geometrically, it is enough to understand the image of c-exponential map of a half space in the cotangent space of x_0, i.e, $\{p \in T^*_{x_0}\mathbb{H}^n : a \cdot p \geq b\}$, because every convex set in $T^*_{x_0}\mathbb{H}^n$ is the intersection of half spaces.

We identify $(x, \sqrt{1+|x|^2}) \in \mathbb{H}^n$ with its coordinate x. In local coordinates, the cost function $c = -\cosh \circ d_{\mathbb{H}^n}$ is $c(x,y) = x \cdot y - \sqrt{1+|x|^2}\sqrt{1+|y|^2}$. The inverse of c-$\exp_{x_0}$ is

$$-\partial_{x_i} c(x_0, y) = -y_i + \frac{\sqrt{1+|y|^2}}{\sqrt{1+|x_0|^2}} (x_0)_i.$$

Fix a hyperplane $P = \{p \subset T^*_{x_0}\mathbb{H}^n : a \cdot p = b\}$ of $T^*_{x_0}\mathbb{H}^n$. Then its image

$$\begin{aligned}
\text{c-exp}_{x_0}(P) &= \{(y, \sqrt{1+|y|^2}) : p = -y + \frac{\sqrt{1+|y|^2}}{\sqrt{1+|x_0|^2}} x_0,\ a\cdot p = b\} \\
&= \{(y, \sqrt{1+|y|^2}) : a\cdot y - \frac{a\cdot x_0}{\sqrt{1+|x_0|^2}}\sqrt{1+|y|^2} = -b\} \\
&= \{(y, \sqrt{1+|y|^2}) : a\cdot y - \frac{a\cdot x_0}{\sqrt{1+|x_0|^2}} y_{n+1} = -b\} \\
&= \mathbb{H}^n \cap \bar{P},
\end{aligned}$$

where $\bar{P}$ is the hyperplane $\{(y, y_{n+1}) : a\cdot y - \frac{a\cdot x_0}{\sqrt{1+|x_0|^2}} y_{n+1} = -b\}$ in the Minkowski space.

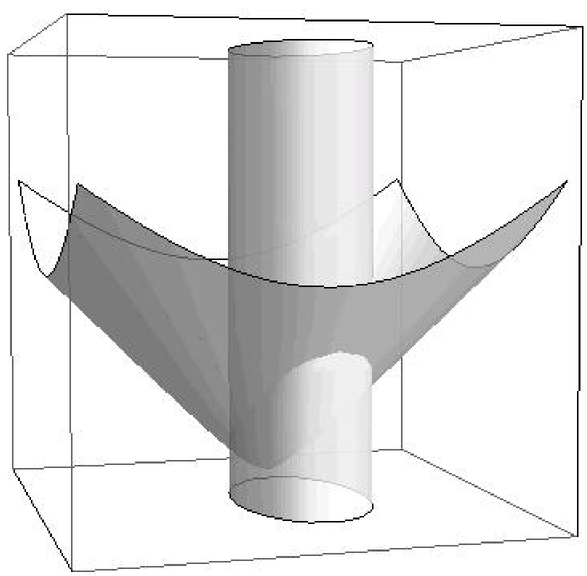
It is not hard too see how $P \subset T^*_{x_0}\mathbb{H}^n$ and x_0 geometrically determines $\bar{P} \subset \mathbb{M}^{n+1}$. First of all, $\bar{P} \cap \{x_{n+1} = 0\} = \{(x, 0) : a\cdot x = -b\}$, which is the reflection of P. Secondly, the normal of $\bar{P}$ is $(a, -\frac{a\cdot x_0}{\sqrt{1+|x_0|^2}})$ and it is perpendicular to $(x_0, \sqrt{1+|x_0|^2})$, so $(x_0, \sqrt{1+|x_0|^2})$ is parallel to $\bar{P}$. These two features determine the hyperplane $\bar{P}$.

Therefore, if U is a convex set in $T^*_{x_0}\mathbb{H}^n$, then

$$\text{c-exp}_{x_0}(U) = \{(-U, 0) + t(x_0, \sqrt{1+|x_0|^2}) : t \in \mathbb{R}\} \cap \mathbb{H}^n,$$

which is a convex solid cylinder intersecting the hyperbolic space.

FIGURE 1. A c-convex set in $\mathbb{H}^2$ with respect to $(0, 0, 1)$



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