SUPERVISED MSc RESEARCH
Dept. of Mathematics, University of Toronto

Gonzalo García Alarcón Estrada
1004229676
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- REPORT -
On generalizations of the Lorentzian distance function

Contents

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1 Three approaches to Lorentzian distance

One of the most relevant objects of my study was the Lorentzian distance and some related concepts. This is, however confusingly, not a distance. It's usual definition in the setting of a smooth Lorentzian manifold is the analogue to it's Riemannian counterpart, with the principal distinction that it does not satisfy the triangle inequality, but the reverse triangle inequality. This is a direct consequence of the signature of the Lorentzian metric. In this section I will present the 3 different approaches that will be discussed.

Note 1.1. The objects defined in this section are closely related and have confusingly similar names. I will do my best effort to call them by their proper names -*Lorentzian distance function*, *Lorentz-distance*, and *time separation function*- along this report, since their definitions are different. **Keep in mind that none of them is actually a distance.**

1.1 Lorentzian distance function on a smooth spacetime

The following is taken from [BEE96].

Consider a smooth, paracompact[1], and Hausdorff n-dimensional manifold M . A **semi-Riemannian metric** g for M is a smooth symmetric tensor field of type (0,2) on M which assigns to each point $p \in M$ a nondegenerate[2] inner product $g|_p : T_pM \times T_pM \to \mathbb{R}$ of signature $(-, \dots, -, +, \dots, +)$[3]. A Lorentzian manifold is semi-Riemannian manifold (M,G) of signature $(-, +, \cdots +)$. Tangent vectors $v \in T_pM$ are classified as:

$$\begin{aligned} &\textbf{timelike} \text{ if} && g(v,v) < 0 && \text{(1a)} \\ &\textbf{nonspacelike or causal} \text{ if} && g(v,v) \leq 0 && \text{(1b)} \\ &\textbf{null or lightlike} \text{ if} && g(v,v) = 0 && \text{(1c)} \\ &\textbf{spacelike} \text{ if} && g(v,v) > 0 && \text{(1d)} \end{aligned}$$

A vector field X on M is timelike if $g(X,X) < 0$ at ll points of M. A Lorentzian manifold with a given timelike vector field X is said to be **time oriented** by X. The timelike vector field X divides all nonspacelike tangent vectors into **future** and **past directed**. A nonspacelike tangent vector $v \in T_pM$ is **future (past) oriented** if $g_p(X(p), v) < 0$ ($g_p(X(p), v) > 0$) A **spacetime** is a time oriented Lorentzian manifold.

Some *elementary causality theory* is required to define the Lorentzian distance function. Consider the following standard relations in a spacetime (M, g): let $p, q \in M$,

- $p \ll q$ if there is a future directed piecewise smooth *timelike* curve in M from p to q
- $p \leq q$ if $p = q$ or if there is a future directed piecewise smooth *nonspacelike* curve in M from p to q

($p < q$ will mean that $p \leq q$ and $p \neq q$) and define the following sets:

$$\begin{aligned} &\textbf{chronological past of } p && I^-(p) := \{q \in M : q \ll p\} && \text{(2a)} \\ &\textbf{chronological future of } p && I^+(p) := \{q \in M : p \ll q\} && \text{(2b)} \\ &\textbf{causal past of } p && J^-(p) := \{q \in M : q \leq p\} && \text{(2c)} \\ &\textbf{causal future of } p && J^+(p) := \{q \in M : p \leq q\} && \text{(2d)} \end{aligned}$$

Definition 1.1 (Lorentzian arc length)**.** Given $p, q \in M$ with $p \leq$ and let $\Omega_{p,q}$ be the path space of all future directed nonspacelike curves $\gamma : [0,1] \to M, \gamma(0) = p, \gamma(1) = q$. The **Lorentzian arc length functional** $L = L_g : \Omega_{p,q} \to \mathbb{R}$ is defined as: given a piecewise smooth curve $gamma \in \omega_{p,q}$, choose a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\gamma|_{t_i, t_{i+1}}$ is smooth for all $i = 0, \dots, n-1$, and define

$$L(\gamma) = L_g(\gamma) = \inf \sum_{i=0}^{n-1} \int_{t=t_i}^{t_{i+1}} \sqrt{-g(\gamma'(t), \gamma'(t))} dt \tag{3}$$

Remark 1.1. If $p \ll q$ there are timelike curves from p to q of arbitrary small Lorentzian arc length. However, if p and q are in a geodesically convex neighborhood[4] U, the future directed timelike

[1] From [Lee10]: Let X be a topological space. $\mathcal{A} \subseteq \mathcal{P}(X)$ is **locally finite** if each $x \in X$ has a neighborhood that intersects at most finitely many of the sets in $\mathcal{A}$. Given a cover $\mathcal{A}$ of X, another cover $\mathcal{B}$ is a *refinement of* $\mathcal{A}$ if $\forall B \in \mathcal{B}\ \exists A \in \mathcal{A}$ such that $B \subseteq A$. It is an **open refinement** if every $B \in \mathcal{B}$ is open. X is **paracompact** if every open cover of X admits a locally finite open refinement.

[2] Here **nondegenerate** means that for each nontrivial vector $v \in T_pM\ \exists w \in T_pM$ such that $g_p(v,w) \neq 0$.

[3] If g has s negative eigenvalues and $r = n - s$ positive eigenvalues, then the **signature of g** will be denoted by (s,r). For each fixed $p \in M\ \exists$ local coordinates $(U, (x^1, \dots, x^n))$ such that g_p can be represented as the diagonal matrix $diag\{-, \dots, -, +, \dots, +v\}$.

[4] *i.e.* all points are joint by geodesic segments

geodesic segment in U from p to q has the largest Lorentzian arc length among all nonspacelike curves in U from p to q. Hence the following definition:

Definition 1.2 (Lorentzian distance function)**.** Given $p \in M$,

$$d(p,q) = \begin{cases} \sup\{L_g(\gamma) : \gamma \in \Omega_{p,q}\} & \text{if } q \in J^+(p) \\ 0 & \text{if } q \notin J^+(p) \end{cases} \tag{4}$$

Remark 1.2. Therefore $d(p,q) > 0 \iff q \in I^+(p)$ and $d(q,p) > 0 \iff q \in I^-(p)$ which in turn implies that the Lorentzian distance function determines the chronological past and future of any point. However, $d(p,q) = 0 \not\Rightarrow q \in J^+(p) \setminus I^+(p)$, but $d(p,q) = 0$ if $q \in M \setminus I^+(p)$. Therefore the Lorentzian distance function **DOES NOT** (in general) determine the causal past and future of p.

Remark 1.3. The following properties can be proved:

1. both relations are transitive, $\leq$ is reflexive, and $x \ll y \Rightarrow x \leq y$
2. if $p \leq q \leq r \Rightarrow d(p,r) \geq d(p,q) + d(q,r)$
3. if (M,g) is chronological, then $\forall p \in M,\ d(p,p) = 0$
4. if $0 < d(p,q) < \infty \Rightarrow d(q,p) = 0$

1.2 Noldus' Lorentz-distance

Noldus establishes the following definitions (remarking that this is not the standard definition in the literature):

Definition 1.3 (Lorentz-distance)**.** Let X be a set, $x, y, z \in X$, a **Lorentz-distance** is a function $d : X \times X \to \mathbb{R}^+ \cup \{\infty\}$ which satisfies:

(i) $\forall x \in X, d(x,x) = 0$

(ii) *(antisymmetry)* if $d(x,y) > 0 \Rightarrow d(y,x) = 0$

(iii) *(reverse triangle inequality)* if $d(x,y)d(y,z) > 0 \Rightarrow d(x,z) \geq d(x,y) + d(y,z)$

Every chronological spacetime determines a canonical Lorentz-distance, d_g, as in Def. 1.2. If (M,g) is a globally hyperbolic spacetime, the metric g induces a continuous Lorentzian distance function d_g[5]. However, the intention of the authors in [Nol04a], [Nol04b], [BN04], is not to study the usual Lorentzian distance function, but to study the pairs (M,d) where M is a compact interpolating spacetime (see § 2) and d is a Lorentz-distance in order to establish a Lorentzian analogue to the Gromov-Hausdorff distance between Riemannian manifolds.

In this setting, a partial order, $\ll$ is established on M by defining $x \ll y \iff d(x,y) > 0$. This relation is interpreted as a *chronological relation* on (M,d).

Remark 1.4. Note that an important consequence of choosing (M,d) as the object of study is that a chronological relation is directly recovered, however recovering a causal relation will be much more problematic. This is coherent with Rmk. 1.2.

Remark 1.5. Observe that for any chronological and spacetime (M,g), if its Lorentzian distance function is finite, it is a Lorentz-distance (in the Noldus sense) (see Rmk. 1.3). Noldus and Bombelli work with globally hyperbolic spacetimes, which satisfy the *finite distance condition.*

[5] Which is a Lorentz-distance, see Rmk. 1.5.

1.3 Kunzinger and Sämann's time separation function

The approach in [KS17] is different than the previous two. Kunzinger and Sm̈annaim to work not with spacetimes (*i.e.* not necessarily with manifolds), but with general sets in order to introduce an analogue of the theory of length spaces into the setting of Lorentzian geometry and causality theory. So their main object of study are sets with two order relations which intend to recover causal and chronological relations.

Definition 1.4 (Causal space)**.** Let X be a set with two order relations:

1. $\leq$ a reflexive and transitive relation on X
2. $\ll$ a transitive relation on X contained in $\leq$ (*i.e.* $\ll \subseteq \leq$ as subsets of $X \times X$, *i.e.* if $x \ll y \Rightarrow x \leq y$)

Write $x < y$ if $x \leq y$ and $x \neq y$. The triplet $(X, \ll, \leq)$ is called a **causal space**.

The future and past chronological and causal sets $(I^+(x), I^-(x), J^+(x), I^-(x))$ are defined, using these order relations, as in § 1.1.

Assume that X is a metric space.

Definition 1.5 (Time separation function and Lorentzian pre-length space)**.** Let $(X, \ll, \leq)$ be a causal space and d a metric on X. Let $\tau : X \times X \to [0, \infty]$ be a lower semicontinuous map (wrt. the metric topology induced by d) that satisfies:

(i) $\tau(x, y) = 0$ if $x \nleq y$

(ii) $\forall x, y, z \in X$ such that $x \leq y \leq z$, $\quad \tau(x, z) \geq \tau(x, y) + \tau(y, z)$ *(reverse triangle inequality for* τ*)*

(iii) $\tau(x, y) > 0 \iff x \ll y$

Then $(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* and τ is called *time separation function.*

Remark 1.6. Observe that for any spacetime (M, g), its Lorentzian distance function together with the usual causal and chronological relation, is a time separation function (see Rmk. 1.2, Rmk. 1.3).

Notice from Rmk. 1.5 and Rmk. 1.6 that indeed both, a Lorentz-distance and a time separation function, are generalizations of the usual Lorentzian distance function. However, a relation between a Lorentz-distance and a time separation function is not clear beginning with the fact that a Lorentz-distance does not (always) induce a causal relation, $\ll$.

2 About the Noldus' articles

In this section I will present a summary of the the articles [Nol04a], [Nol04b], and [BN04].[6] I will refer to these colectively as *the Noldus' articles.*

The Noldus' articles intend to begin the research programme on the structure of the *moduli space of isometry classes of globally hyperbolic spacetimes* (which they also refer to as the *space of Lorentzian geometries*). They propose studying this space via a Lorentzian analogue of the Gromov-Hausdorff theory for Riemannian manifolds. The central idea is to define a notion of closeness between spacetimes to define limit spaces and to study the topology of the moduli space itself. From the physical point of view, the intended field of application is quantum gravity, in particular its path integral formulation, by addressing questions such as when a sequence of spacetimes converges to another space, when two geometries are close or how to calculate an integral over them.

[6] This series of three articles work as a combined longer article. They are mostly sections taken from [Nol04c].

2.1 Assumptions and basic definitions

Recall from the comments after the definition of a Lorentz-distance (Def. 1.3), a globally hyperbolic spacetime (M, g) induces a continuous Lorentz-distance d_g. They henceforth stick to globally hyperbolic spacetimes to guarantee the continuity of the induced Lorentz-distance. They also assume that the spacetimes are compact. In order to have globally hyperbolic compact spacetimes, the existence of spacelike boundaries is required. These spacetimes will be referred to as *interpolating spacetimes.*

Definition 2.1 (Interpolating spacetime and cobordism)**.** Let S_1, S_2 be $n-1$-manifolds (possibly with boundary).

1. A connected n-manifold with boundary M is called **interpolating between S_1 and S_2** iff $\exists$ a further $n-1$-manifold T (possibly empty) such that $\partial M = S_1 \cup S_2 \cup T$. If $T \neq \emptyset$ it is required that $T \cap S_i = \partial S_i (i = 1, 2), \partial S_1$ and ∂S_2 are diffeomorphic, and that $T = \partial S_1 \times [0, 1]$.

2. M is an **interpolating spacetime** iff $\exists$ a smooth Lorentz metric on M such that S_1 and S_2 are spacelike and T is timelike or empty.

3. If S_1 and S_2 are closed (*i.e.* compact with boundary), then a compact interpolating manifold M is a **cobordism** and S_1 and S_2 are **cobordant**. (Note that in this case $T = \emptyset$.)

From now on, consider in this section only compact interpolating spacetimes with spacelike boundaries, together with a Lorentz-distance, (M, d), where $d = d_g$ is the Lorenz-distance induced by g. The subindex will be dropped when not necessary.

The authors define two notions of closeness between spacetimes:

Definition 2.2. (Lorentzian Gromov-Hausdorff distance) Consider two spacetimes (M_1, d_1) and (M_2, d_2).

1. given $\epsilon > 0$, **(M_1, d_1) and (M_2, d_2) are ϵ-close** iff $\exists\ \psi : M_1 \to M_2$ and $\zeta : M_2 \to M_1$ such that $\forall p_1, q_1 \in M_1, p_2, q_2 \in M_2$:

$$|d_2(\psi(p_1), \psi(q_1)) - d_1(p_1, q_1)| < \epsilon \tag{5a}$$

$$|d_1(\zeta(p_2), \zeta(q_2)) - d_2(p_2, q_2)| < \epsilon \tag{5b}$$

2. define the Lorentzian Gromov-Hausdorff distance as:

$$d_{GH}((M_1, d_1), (M_2, d_2)) := \inf\{\epsilon > 0 : (M_1, d_1) \text{ and } (M_2, d_2) \text{ are } \epsilon\text{-close}\} \tag{6}$$

Compare this definition with (20). Note that (5) suggest that ψ and ζ are "almost isometries".

Theorem 2.1. $d_{GH}((M_1, d_1), (M_2, d_2)) = 0 \iff (M_1, d_1)$ *and* (M_2, d_2) *are isometric.*

The second definition of closeness is given at the beginning of the 2nd article ([Nol04b]), which is a stronger version. It includes a restriction on the maps ψ and ζ given by the following strong metric:

Definition 2.3 (Strong metric)**.** Let (M, d) be a compact interpolating spacetime, the **strong metric D_M** is defined as

$$D_M(p, q) := \max_{r \in M} |d(p, r) + d(r, p) - d(q, r) - d(r, q)| \tag{7}$$

(Note that this strong metric could be defined on any set with a Lorentz-distance replacing max by sup.)

Recall that only one of each pair in (7) is nonzero, *i.e.* if $d(p,r) > 0 \Rightarrow d(r,p) = 0$. The authors claim that this strong metric is a positive-definite distance on on M, although it's not proved. It would be a good idea to check this.

Proposition 2.1. *Given $p, q \in (M,d)$, then $\forall r \in M$:*

$$|d(p,r) + d(r,p) - d(q,r) - d(r,q)| < \delta \iff |d(p,r) - d(q,r)| < \delta \text{ and } |d(r,p) - d(r,q)| < \delta$$

Definition 2.4 ((ϵ,δ)-close)**.** Consider two spacetimes (M_1, d_1) and (M_2, d_2). Given $\epsilon, \delta > 0$, **(M_1, d_1) and (M_2, d_2) are (ϵ,δ)-close** iff $\exists$ maps ψ and ζ as in Def. 2.2, satisfying additionally that:

$$D_M(p_1, \zeta \circ \psi(p_1)) < \delta \tag{8a}$$
$$D_M(p_2, \psi \circ \zeta(p_2)) < \delta \tag{8b}$$

$\forall p_1 \in M_1$ and $p_2 \in M_2$.[7]

Loosely speaking, note that (8) suggests that $\zeta \circ \psi$ and $\psi \circ \zeta$ are "almost surjections", making ψ and ζ "approximate inverses".

They proceed to call this closeness measure the **generalized Gromov-Hausdorff uniformity (GGH)**. In [Nol04b], it is showed that "(ϵ,δ)-closeness is a uniformity" by showing that (ϵ,δ)-closeness is symmetric and satisfies the triangle inequality. I do not understand what they mean by this statement. In the appendix D of that same paper, one can find an introduction to uniformities, which is a topological structure (a collection of covers) that induces a topology. They mention that a uniformity can be generated by a family of pseudometrics. I am guessing that the correct way to interpret their statement is: "(ϵ,δ)-closeness is a family of pseudometrics which define a uniformity on the space of pairs (M,d)".

Another statement that seems suspicious to me is that they claim that "(M_1, d_1) and (M_2, d_2) are isometric $\iff$ they cannot be distinguished by the GGH uniformity" which can be proved as a consequence of Theo. 2.1. I need to try to make this clearer. The theorem is an analogous version for ϵ-closeness, so what is the role of the extra D_M condition and of the extra parameter δ?

Theorem 2.2. *The strong metric D_M is never a path metric for any spacetime (M,g).*

Idea of the proof. By contradiction to the *lemma*[8]: if d is a strictly intrinsic metric, then for every two points x, y there exists a midpoint z. □

2.2 Construction of the limit spaces

Noldus provides in [Nol04b] two constructions of candidate limit spaces for Cauchy sequences of spaces (M_i, d_i) using the (ϵ,δ)-closeness (or GGH uniformity, as they call it). The constructions are somehow unclear and leave several sketchy details. The first construction is done using a (not so clearly defined) Alexandrov-type topology. This construction will not be satisfactory, hence a second one using a topology induced by the strong metric M is presented. A big part of this article is devoted to construct toy examples (based on the cylinder spacetime) to show how some of the

[7] (8) can be replaced by:

$$|d_1(\zeta \circ \psi(p_1), q_1) + d_1(q_1, \zeta \circ \psi(p_1)) - d_1(p_1, q_1) - d_1(q_1, p_1)| < \delta, \ \forall p_1, q_1 \in M_1$$

and its analogue for d_2. Given the max in the definition of D_M, these equations are equivalent.

[8] Lemma 2.4.8 in [BBI01].

constructions fail and to exhibit some of the counter intuitive and pathological behavior of these limit spaces. However, for most of the examples provided, it is not evident to me how they are using the definitions of closeness and their general construction (in specific the ser $\mathcal{S}$, see definition below).

The authors state that the addition of (8) was a necessary condition to add to their proposed Lorentzian Gromov-Hausdorff distance to be able to construct the limit spaces they suggest. They do not know if this condition is actually necessary, however they mention that they were unable to show the results without them. Note that the (ϵ, δ)-closeness is a two parameter condition. However, most of the examples they provide use only one parameter. (Note that (ϵ, δ)-close $\Rightarrow$ ϵ-close.) I need to go over them in detail again. Even though I am skeptical about the constructions (and still confused due to the not-so-easy-to-follow presentation of the articles), I will present here an a brief outline of the construction and the main related results.

Note 2.1. It is not entirely clear to me what the authors mean by *limit space*. (ϵ, δ)-closeness does not provide a distance on the set of spaces (M, d). However, I am guessing they refer to a limit as $\epsilon \to 0$ and $\delta \to 0$.

2.2.1 First construction: Alexandrov (type) topology

Remark 2.1. This construction is an analogue to the construction of the *completion of a metric space*. However, the space they are trying to complete is not metric and is not even necessarily T_0, hence the extra complication of the process and the extra amount of quotients that needed to be taken.

Let (M_i, d_i) be a sequence of compact interpolating spacetimes ($d_i = d_{g_i}$ for the corresponding metric g_i) such that there are mappings

$$\psi^i_{i+1} : M_i \to M_{i+1} \text{ and } \zeta^{i+1}_i : M_{i+1} \to M_i$$

such that make (M_i, d_i) and (M_{i+1}, d_{i+1}) $(1/2^i, 1/2^i)-$close. Introduce the maps

$$\psi^i_{i+k} = \psi^{i+k-1}_{i+k} \circ \psi^{i+k-2}_{i+k-1} \circ \cdots \circ \psi^{i+2}_{i+1} \circ \psi^{i+1}_i : M_i \to M_{i+k},$$

$$\zeta^{i+k}_i = \zeta^{i+1}_i \circ \zeta^{i+2}_{i+1} \circ \cdots \circ \zeta^{i+k-1}_{i+k-2} \circ \zeta^{i+k}_{i+k-1} : M_{i+k} \to M_i$$

This maps make (M_i, d_i) and M_{i+k}, d_{i+k} $(1/2^{i-1}, 3/2^{i-1})-$close.

Now define the set $\mathcal{S}$ of sequences $(x_i)_{i\in\mathbb{N}}, x_i \in M_i$ such that there exists an $m \in \mathbb{N}$ such that $\forall i > m,\ x_i = \psi^m_i(x_m)$. Hence $x_i = \psi^j_i(x_j) \forall i > j \geq m$.[9]

Define the following Lorentz-distance on $\mathcal{S}$:

$$d((x_i), (y_i)) := \lim_{i\to\infty} d_i(x_i, y_i) \tag{9}$$

Recall that this Lorentz-distance will induce a partial order ('chronological relation'):

$$(x_i) \ll (y_i) \iff d((x_i), (y_i)) > 0$$

Define the Alexandrov (type) topology on $\mathcal{S}$ as the topology *generated* by the sets:

- $\mathcal{S}, \emptyset$
- $I^+((x_i))$ and $I^+((x_i))$

[9]Note that this is some sort of Cauchy condition.

- $I^+((x_i)) \cap I^+((x_i)), \ \forall (x_i),(y_i) \in \mathcal{S}$ such that $(x_i) \ll (y_i)$

Note 2.2. The authors indicate the following after this definition: "I stress the word 'generated' since in general the above sets do not constitute a basis of the Alexandrov topology as will become clear in examples 2 and 3, where specific intersections of generating sets do not contain any generating set." However, if this is the case, the family of sets defined above, does not constitute a subbase for the Alexandrov topology on $\mathcal{S}$. Call this family $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$. $\mathcal{F}$ will indeed define a topology, simply the topology generated by $\mathcal{F}$ on $\mathcal{S}$, but this will not be the Alexandrov topology. Therefore I will call this an **Alexandrov-type topology**. As a matter of fact, if that statement holds, are those spaces topological spaces?

Note 2.3. I will quote the first example from [Nol04b] to point out how it is not very clear how the (ϵ, δ)-closeness and the set $\mathcal{S}$ play a role:[10]

> **Example 1.** Take the 'cylinder universe' $S^1 \times \mathbb{R}$ with metric $-dt^2 + d\theta^2$ and let $p = (0, -T/2)$ and $q = (0.T/2)$ with $T > 0$.
>
> In the notation of [BEE96], let $K^+(q,\epsilon) = \{r | d(q,r) = \epsilon\}$ be the 'future ball' of radius ϵ centered at q and $K^-(p,\epsilon) = \{r | d(r,p) = \epsilon\}$ be the 'past ball' centered at p. Consider the spacetimes $(J^+(K^-(p,\epsilon)) \cap J^-(K^+(q,\epsilon)), -dt^2 + d\theta^2)$ then a candidate Gromov-Hausdorff limit space for $\epsilon \to 0$ is
> $$(J^+(E^-(p)) \cap J^-(E^+(q)), -dt^2 + d\theta^2),$$
> which is T_0, but not T_1, in the Alexandrov topology.

Inspired by this example, this limit spaces are at most T_0 but not T_1. The set $\mathcal{S}$ with the Alexandrov-type topology could possibly not even be T_0, so they take the T_0 quotient of $\mathcal{S}$ to ensure that it is T_0.

Definition 2.5 (topologically indistinguishability)**.** Consider (X, τ) a topological space:

1. Let $N_x := \{U \in \tau : U \text{ is a neighborhood of } x\}$. $x, y \in \tau$ are **topologically indistinguishable** iff they have the same neighborhoods, *i.e.* if $N_x = N_y$.

2. X is $\boldsymbol{T_0}$ if all points are topologically distinguishable, *i.e.* $\forall x, y \in X \ \exists \ U_x$ neighborhood of x such that $y \notin U_x$, or the other way around.

3. Notice that topologically indistinguishability is an equivalence relation. I will denote $KQ(X)$ the quotient of X wrt. this equivalence relation, called the $\boldsymbol{T_0}$ **quotient** (or Kolmogorov quotient).

Observe that if X is T_0, then $X = KQ(X)$.

Now they take the T_0 quotient of $\mathcal{S}$. Let:

$$T_0\mathcal{S} := KQ(\mathcal{S}) \text{ wrt. the Alexandrov-type topology on } \mathcal{S} \tag{10}$$

The next step is to construct the *timelike closure* of $T_0\mathcal{S}$. For this, consider the following definitions:

Definition 2.6 (future (past) timelike Cauchy sequences)**.** A sequence $(x^i) \subset T_0\mathcal{S}$ is called **future timelike Cauchy** iff $x^i \ll x^j \ \forall i < j$ and $\forall \epsilon > 0 \ \exists \ m \in \mathbb{N}$ such that $\forall k > j \geq m : 0 < d(x^j, x^k) < \epsilon$.

A past **timelike Cauchy sequence** is defined dually.

Definition 2.7 (equivalence of Cauchy sequences)**.** 1. Two *future* timelike Cauchy sequences $(x^i), (y^i) \subset T_0\mathcal{S}$ are equivalent iff $\forall k \in \mathbb{N} \exists m \in \mathbb{N}$ such that if $i \geq m \Rightarrow x^k \ll y^i$ and $y^k \ll x^i$

[10] $E^+(p) := J^+(p) \setminus I^+(p)$ is the **future horismos of** $\boldsymbol{p}$.

2. Two *past* timelike Cauchy sequences are equivalent iff ... (dual definition).

3. A *future* timelike Cauchy sequence $(x^i) \subset T_0\mathcal{S}$ and a *past* timelike Cauchy sequence $(y^i) \subset T_0\mathcal{S}$ are equivalent iff $x^k \ll y^l \forall k, l \in \mathbb{N}$ and $\nexists\, z^1, z^2 \in T_0\mathcal{S}$ such that :

$$x^k \ll z^1 \ll z^2 \ \ \forall k \quad \text{and } z^2 \notin \bigcup_{j\in\mathbb{N}} I^+(y^j)$$

or

$$z^2 \ll z^1 \ll y^k \ \ \forall k \quad \text{and } z^2 \notin \bigcup_{j\in\mathbb{N}} I^-(x^j)$$

Write $(x^i) \sim_{tc} (y^i)$ iff the sequences are **timelike Cauchy equivalent** according to the above definitions.

$\sim_{tc}$ is an equivalence relation. Similarly to the completion of a metric space, we want to take the quotient wrt. to equivalence of Cauchy sequences. However, due to how the timelike Cauchy sequences were defined, doing this would leave out the corresponding points that are not the "limit" of a sequence, for example the past boundary. The authors write the following: "define $\widetilde{T_0\mathcal{S}}$ as the union of $T_0\mathcal{S}$ with all timelike Cauchy sequences in $T_0\mathcal{S}$". This is unclear. I believe that what they mean is to define $\widetilde{T_0\mathcal{S}}$ as $T_0\mathcal{S}/\sim_{tc}$ and to somehow add all the points that are not the "limit" of a sequence. Following this idea, and abusing notation, let:

$$\widetilde{T_0\mathcal{S}} := T_0\mathcal{S}/\sim_{tc} \bigcup T_0\mathcal{S}/(\neg \sim_{tc}) \tag{11}$$

Now define again an Alexandrov (type) topology on $\widetilde{T_0\mathcal{S}}$ as the topology *generated* by the sets $E \subset \widetilde{T_0\mathcal{S}}$ such that:[11]

- $E \cap T_0\mathcal{S}$ is a generating set for the Alexandrov (type) topology of $T_0\mathcal{S}$
- a future (past) timelike Cauchy sequence $(p^i) \subset T_0\mathcal{S}$ belongs to E iff
  - $\exists q \in T_0\mathcal{S}$ such that $E \cap T_0\mathcal{S} = I^+(q)\ (I^-(q))$ and $\exists m \in \mathbb{N}$ such that if $i \geq m \Rightarrow p^i \in E \cap T_0\mathcal{S}$
    or
  - $\exists r^1, r^2 \in E \cap T_0\mathcal{S}, m \in \mathbb{N}$ such that $p^i \ll r^2 \ll r^1$ $(r^1 \ll r^2 \ll p^i)$ and $p^i \in E \cap T_0\mathcal{S}\ \forall i \geq m$.

Note 2.4. Similar to Note 2.2, they indicate that this family of subsets does not really satisfy the properties for being a subbase of the Alexandrov topology on $\widetilde{T_0\mathcal{S}}$.

Now take again the T_0 quotient. Define the **timelike closure of $\mathcal{S}$** as:

$$\overline{T_0\mathcal{S}} := KQ(\widetilde{T_0\mathcal{S}}) \text{ wrt. the Alexandrov-type topology on } \widetilde{T_0\mathcal{S}} \tag{12}$$

[11] I get the feeling that this definition is written a bit sloppily. For example, in the first condition, in $E \cap T_0\mathcal{S}$, aren't they mixing different types of objects? Since $E \subset \widetilde{T_0\mathcal{S}}$ (*i.e.* a subset of equivalence classes of $T_0\mathcal{S}$) and $I^+(q) \subset T_0\mathcal{S}$. Additionally, why is there a condition on the sequences as part of the definition of the 'generating' sets? What does it mean that (p^i) belongs to E? Perhaps that the equivalence class of (p^i) is an element of E, namely $[(p^i)]_{\sim_{tc}} \in E$. A great deal of this confusion comes from the unclear definition of $\widetilde{T_0\mathcal{S}}$ and the abuse in ignoring the quotients, regarding how they define themselves quotient topologies and how the distance d and the order $\ll$ behave after taking repeated quotients all the way from $\mathcal{S}$ to $\overline{T_0\mathcal{S}}$.

The article provides very little intuition or motivation for the definition of $\overline{T_0\mathcal{S}}$ and of the topologies mentioned above, not to mention the incongruences that have been commented, the lack of clarity in some of the definitions, and the lack of a rigorous notation to keep track of the several quotients taken. All this definitions seem to me to be tweaked to work as they intend in the examples they provide. I will not cover in detail the examples (in part because I do not understand them fully), but I will quote a pragraph in [Nol04b] that summarizes the conclusions from the examples and that (sort of) motivates the second construction:

> Let us summarize our preliminary results: examples 2 and 3 show that we have to allow degenerate metrics[12] and that, moreover, the Alexandrov topology has bad separation properties on the 'degenerate area'. The aforementioned results show that the candidate limit space has the required behavior on the timelike continuum.[13] However, by a judicious choice of mappings ψ and ζ, one can give examples where $\overline{T_0\mathcal{S}}$ is *not* compact in the Alexandrov topology while $\overline{T_0\mathcal{S}}$ is compact in the Alexandrov topology for another set of mappings, see [Nol04c]! All this shows, in my opinion, that the Alexandrov topology is not appropriate and I shall concentrate on the strong topology from now on.

2.2.2 Second construction: strong topology

Recall that the strong metric D_M associated to a given (M,d) is a distance on M.

Theorem 2.3. *Let* (M_1,d_1) *and* (M_2,d_2) *be* (ϵ,δ)*-close* $\Rightarrow d_{GH}((M_1,d_1),(M_2,d_2)) \le \epsilon + \frac{3\delta}{2}$.

Sketch of proof. Let $\psi : M_1 \to M_2$ and $\zeta : M_2 \to M-1$ the maps that make (M_1,d_1) and (M_2,d_2) be (ϵ,δ)-close, then, from the definition of (ϵ,δ)-close:

$$D_{M_1}(\zeta\circ\psi(p),p) < \delta, \quad D_{M_2}(\psi\circ\zeta(q),q) < \delta$$

from which, one can get that:

$$|D_{M_2}(\psi(p_1),\psi(q_1)) - D_{M_1}(p_1,q_1)| < 2(\epsilon+\delta) \quad \forall p_1,q_1 \in M_1 \tag{13}$$

$$|D_{M_1}(\zeta(p_2),\zeta(q_2)) - D_{M_2}(p_2,q_2)| < 2(\epsilon+\delta) \quad \forall p_2,q_2 \in M_2 \tag{14}$$

Define an admissible metric D on $M_1 \sqcup M_2$ by:

$$D(p,q) = \min_{r\in M_1, s\in M_2} \frac{1}{2}\left(D_{M_1}(p,r) + D_{M_2}(\psi(r),q) + D_{M_2}(q,s) + D_{M_1}(\zeta(s),p)\right) + (\epsilon+\delta) \tag{15}$$

By using (13) and (14), one can show that D is effectively an admissible metric that satisfies the required bound. □

After this theorem, the authors write:

[12] *i.e.* such that $\exists x \neq y$ such that $d(x,y)=0$

[13] Two different definitions are given of the **timelike continuum (*TCON*)**:

1. (from [Nol04b]) The timelike continuum of $\overline{T_0\mathcal{S}}$ as the subset of all points r such that $\exists$ timelike past *and* future Cauchy sequences in $T_0\mathcal{S}$ wich are T_0 equivalent with r in the Alexandrov topology.
2. (reformulated in [BN04]) Let (M,d) be a Lorentz space (which they define as a pair (M,d) with M a set and d a Lorentz-distance, such that $M, D_M)$ is a compact metric space), the $TCON$ is defined as the set of points $p \in M$ which are limit points of a future and past time-like Cauchy sequence.

Theo. 2.3 reveals that any compact **limit space (in the strong topology), (M^{str}, d)** of the modified Gromov-Hausdorff sequence (M_i, g_i) must be isometric, wrt. $D_{M^{str}}$, to the limit space of the Gromov-Hausdorff sequence (M_i, D_{M_i}) due to the well-known result ot Gromov[14] [Gromov M., 1997, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser], [Petersen P., 1988, *Riemannian Geometry*, Springer].

I do not understand what they are referring to with "*limit space (in the strong topology), (M^{str}, d) of the modified Gromov-Hausdorff sequence (M_i, g_i)*" since (ϵ, δ)-closeness is NOT a topological concept. Do they mean constructing M^{str} as they did $\overline{T_0\mathcal{S}}$ but replacing the Alexandrov-type topology by the topology induced by the strong metric on $\mathcal{S}$?

Instead of doing that, they will construct a set via the 'classical Gromov construction' whose T_0 quotient will be the sought after limit space M^{str}:

Theorem 2.4. *The Gromov-Hausdorff limit space of the sequence (M_i, D_{M_i}) equipped with a suitably defined Lorentz-distance d (see (19)), is a limit space[15] of the sequence (M_i, d_i).*

Sketch of proof. Let $\psi^i_{i+1} : M_i \to M_{i+1}$ and $\zeta^{i+1}_i : M_{i+1} \to M_i$ as in the beginning of § 2.2.1, and let $D_{i,i+1}$ be the admissible metric on $M_i \sqcup M_{i+1}$ as constructed on (15).

Define a metric on $\bigsqcup_{i\in\mathbb{N}} M_i$, $\forall i, k > 0$, by:

$$G(p_i, \psi^i_{i+k}(p_i)) := \min_{p_{i+j}\in M_{i+j}, j=1,\ldots,k-1} \left(\sum_{j=0}^{k-1} D_{(i+j),(i+j)+1}(p_{i+j}, p_{i+j+1}) \right) \tag{16}$$

Now construct the limit space as the completion of $(\bigsqcup_{i\in\mathbb{N}} M_i, G)$. Define:

$$\widehat{M} := \{(p_i) : p_i \in M_i \text{ and } G(p_i, p_j) \xrightarrow{i,j\to\infty} 0\} \tag{17}$$

$\widehat{M}$ has the following *pseudometric*:

$$D((p_i), (q_i)) := \lim_{i\to\infty} G(p_i, q_i) \tag{18}$$

and the following *Lorentz-metric*:

$$d((p_i), (q_i)) := \lim_{i\to\infty} d_i(p_i, q_i) \tag{19}$$

It results that (the strong metric induced by d on $\widehat{M}$) $D_{\widehat{M}}$ equals D Hence, $(\widehat{M}, d)$ is a compact limit space in the strong topology,[16] since $\widehat{M}$ is compact wrt. D. The T_0 quotient of $(\widehat{M}, d)$ is the desired limit space (M^{str}, d). □

The authors mention on [BN04], that in order to keep the presentation accessible, they will omit some proofs and some technical details. I was unable to find any other continuation to these articles where those details are to be found. On the other hand, I did find on L. Bomelli's website other articles that suggest that they kept trying to define the limit of sequences of spacetimes in other ways, and that they abandoned this idea. One of the newer proposals from these authors is [BNT12]. A previous, apparently unsuccessful (judging from the comments on these articles), attempt from L. Bombelli can be found in [Bom00] which develops the idea of *statistical Lorentzian geometry*.

[14] which!?

[15] in which sense!?

[16] same question again, limit space in which sense? What does the strong topology have to do? Aren't we precisely constructing the limit space (in the GGH sense)?

FUTURE READINGS:

- Necessary background material: [BEE96], [BBI01], [Wal84]
- Other related articles: [SV16], [BNT12] [MP06]

A APPENDIX

A.1 Some definitions from causality theory

The following is taken from [BEE96]. Consider (M, g) a spacetime as defined in § 1.1.

Definition A.1 (Some causality conditions)**.** (M, g) is:

1. **chronological** iff $\forall p \in M, p \notin I^+(p)$ (*i.e.* (M, g) contains no closed timelike curves)
2. **causal** iff $\nexists p, q \in M (p \neq q)$ such that $p \leq q \leq p$ (this is equivalent to (M, g) containing no closed nonspacelike curves)

Remark A.1. (M, g) is causal $\Rightarrow$ (M, g) is chronological.

Proposition A.1. *Any* compact *spacetime (M, g) contains a closed timelike curve, and therefore fails to be chronological (and hence also fails to be causal) .*

In the setting of general relativity, every point in an spacetime represents an event. Hence, the existence of a closed timelike curve would imply the possibility of traversing spacetime into the future returning to the same point. Therefore, the chronological condition is the weakest causality condition which is usually required from spacetimes.

An open set $U \subseteq M$ is **causally convex** iff no nonspacelike curve intersect U in a disconnected set.

Given $p \in M$, (M, g) is **strongly causal at *p*** iff p has arbitrarily small causally convex neighborhoods (thus p has arbitrarily small neighborhoods such that no nonspacelike curve that leaves one of these neighborhoods ever returns).

Definition A.2 (Some more causality conditions)**.** (M, g) is:

3. **strongly causal** iff it is strongly causal $\forall p \in M$
4. **globally hyperbolic** iff it is strongly causal *and* $\forall p, q \in M$, the set $J^+(p) \cap J^-(q)$ is compact

Theorem A.1 (Alexandrov topology and strongly causal spacetimes)**.** *The **Alexandrov topology** is the topology given by the basis $\{I^+(p) \cap I^-(q) : p, q \in M\}$*

The following are equivalent:

i) *(M, g) is strongly causal*

ii) *The Alexandrov topology induced on M agrees with the given manifold topology*

iii) *The Alexandrov topology is Hausdorff*

A.2 Gromov-Hausdorff distance

Consider a metric space (X, d_X). Let $U, V \subseteq X$ and define:

$$B(U, \epsilon) := \{x \in X : \exists\ a \in U \text{ such that } d_X(x, a) < \epsilon\}$$

and define the **Hausdorff distance** between subsets of X as:

$$d_H(U, V) := \inf\{\epsilon > 0 : U \subset B(V, \epsilon), V \subset B(U, \epsilon)\}$$

Definition A.3 (Gromov-Hausdorff distance)**.** Consider two compact metric spaces $(X, d_X), (Y, d_Y)$ and define a metric d on the disjoint union $X \sqcup Y$ to be **admissible** iff $d|_X = d_X$ and $d|_Y = d_Y$.

The **Gromov-Hausdorff distance** between two compact metric spaces (X, d_X) and (Y, d_Y) is:

$$d_{GH}\left((X, d_X), (Y, d_Y)\right) := \inf\{d_H(X, Y) : d_H \text{ wtr. all admissible metrics on} X \sqcup Y\}$$

Suppose d is an admissible metric on $X \sqcup Y \Rightarrow \exists\, f : X \to Y$ and $g : Y \to X$ such that $d(x, f(x)) \leq d_H(X, Y)$ and $d(y, g(y)) \leq d_H(X, Y)$, and it can be shown that:

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \leq 2d_H(X, Y) \tag{20a}$$
$$|d_X(g(y_1), g(y_2)) - d_Y(y_1, y_2)| \leq 2d_H(X, Y) \tag{20b}$$

and

$$d_X(x, g \circ f(x)) \leq 2d_H(X, Y) \tag{21a}$$
$$d_Y(y, f \circ g(x)) \leq 2d_H(X, Y) \tag{21b}$$

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