UNIQUE EQUILIBRIA AND SUBSTITUTION EFFECTS IN A STOCHASTIC MODEL OF THE MARRIAGE MARKET

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Abstract. Choo-Siow (2006) proposed a model for the marriage market which allows for random identically distributed noise in the preferences of each of the participants. The randomness is McFadden-type, which permits an explicit resolution of the equilibrium preference probabilities.

In this note we exhibit a strictly convex function whose derivatives vanish precisely at the equilibria of their model. This implies uniqueness of the resulting equilibrium marriage distribution, simplifies the argument for its existence, and gives a representation of it in closed form. We go on to derive smooth dependence of this distribution on exogenous preference and population parameters, and establish sign, symmetry, and size of the various substitution effects, facilitating comparative statics. In particular, we show that an increase in the population of men of any given type in this model leads to an increase in (a) the equilibrium transfer paid by such men to their spouses, (b) the percentage of such men who choose to remain unmarried, (c) the number of unmarried men of each type, and (d) a decrease in the number of unmarried women of each type. While trends (a)-(d) are not surprising, the verification of such properties helps to substantiate the validity of the model. Moreover, we make unexpected predictions which could be tested: namely, the percentage change of type i unmarrieds with respect to fluctuations in the total number of type j men or women turns out to form a symmetric positive-definite matrix $r_{ij} = r_{ji}$ in this model, and thus to satisfy bounds such as $|r_{ij}| \leq (r_{ii}r_{jj})^{1/2}$.

JEL Classification: J12, C62, C78, C81, D03, Z13

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1. Introduction

The classic transferable utility framework which Becker used to model the marriage market was augmented by Choo and Siow [7] to allow for the possibility that agents' preferences might be only partly determined by observable characteristics, and might therefore

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The authors are grateful to Aloysius Siow for attracting their attention to this question, and for many fruitful discussions. This project developed in part from the Master's research of CD; however, the present manuscript is based a new approach which greatly extends (and largely subsumes) the results of his thesis [9], and of the 2009 preprint by three of us entitled *When do systematic gains uniquely determine the number of marriages between different types in the Choo-Siow matching model? Sufficient conditions for a unique equilibrium.* CD, RJM and BKS are pleased to acknowledge the support of Natural Sciences and Engineering Research Council of Canada (NSERC) grant 217006-03 RGPIN; CD also benefitted from an Undergraduate Student Research Award (URSA) held in the summer of 2009. EHL acknowledges a United States National Science Foundation grant PHY-0965859.

include a stochastic component depending on unobservable characteristics. The randomness was chosen to be McFadden type [19], and spreads the preferences of agents on one side of the marriage market over the entire type-distribution of agents on the other, thus yielding non-assortative matching, a feature of the marriage market that has long been observed empirically [7]. The model is non-parametric and highly tractable; given an observation of marriages between different types of agents, there is a simple closed form, 'point-identified' expression for the average utility (or 'total gains') generated by each type of marriage. Of potential use to econometricians, demographers, and economic theorists is a solution to the backwards problem: given the total gains to each type of marriage and population data, what can be said about possible distributions of marriages corresponding to those parameters? Basic questions include existence and uniqueness of such a marital distribution; computing comparative statics is even more interesting. How will the marital distribution respond to changes in composition of the population? How will it change due to policy shocks whose effects can be summarized by revising the total gains parameters? The purpose of the present note is to address these questions.

We shall do this in Theorem 1, by showing that the equilibria in this model coincide with critical points of an explicit strictly convex function depending on the logarithm of the number of unmarrieds of each type. This shows the uniqueness of such equilibria, simplifies existing arguments [6] [8] [12] for their existence, and leads to a closed form expression for the equilibrium, facilitating subsequent analysis of its properties in Theorem 2 and Corollary 3. The strategy of using variational principles to solve nonlinear systems of equations has a rich history in mathematics and physics, going back as least as far as the classical mechanical principle of least action; it is commonly used in settings much more involved than the present one, such as the study of nonlinear partial differential equations.

1.1. **The Choo-Siow Marriage Matching Model.** The Choo-Siow model [7] is designed for analyzing matching problems in a large population. Dividing the population into I observable types of men, and J observable types of women, use m_i to denote the number of men of type $i \in \{1, \dots I\}$, and f_j to denote the number of women of type $j \in \{1, \dots J\}$, and μ_{ij} to denote the number of marriages of type i men to type j women. Using μ_{i0} and μ_{0j} to denote the number of unmarried men of type i and women of type j, we have the obvious population constraints

$$\mu_{i0} + \sum_{j=1}^{J} \mu_{ij} = m_i, \tag{1}$$

$$\mu_{0j} + \sum_{i=1}^{I} \mu_{ij} = f_j, \tag{2}$$

$$\mu_{ij} \geq 0. \tag{3}$$

Choo and Siow predict a relationship between the observed marriage pattern (μ_{ij}) and the exogenous gains π_{ij} associated to a marriage between a man of observed type i and a woman of observed type j, relative to both partners remaining single. Namely, setting

$\Pi_{ij} = e^{\pi_{ij}}$, they predict

$$\frac{\mu_{ij}}{\sqrt{\mu_{i0}\mu_{0j}}} = \Pi_{ij}. \tag{4}$$

There is theoretical basis for this prediction, which we outline below, but its relevance to us is tangential. It is based on a competitive equilibrium framework with transferable-utility and unobserved heterogeneity of a specific (and highly restrictive) form. The main goal of this note will be to solve the system (1)–(4) to determine (μ_{ij}) and its properties as functions of the gains matrix (Π_{ij}) and the population vector $\nu = [m \mid f]$, whose $I + J$ components consist of the observed frequencies m_i and f_j.

1.2. **Choo and Siow's derivation.** To motivate the model, and give context for our results in the literature, let us briefly recall Choo and Siow's derivation of (4). They assume individual agents have a utility function that depends on both an endogenous deterministic component η capturing systematic utility, and an exogenous random one ϵ modeling heterogeneity within the population of each given type. Thus the utility accrued by a man of type i and specific identity g who marries a woman of type j is assumed to be:

$$V^m_{ijg} = \eta^m_{ij} + \sigma\epsilon_{ijg}; \tag{5}$$

the case $j = 0$ represents the utility of remaining single. Endogeneity of η^m_{ij} can be interpreted to reflect the possibility of interspousal transfer, as in §6 below. It is set in equilibrium, and depends explicitly on the type of the man and the type of the woman, and implicitly on market conditions, i.e. on the relative abundance or scarcity of men and women of each different type. The random term ϵ_{ijg} depends additionally on the specific identity of the man, but not on the specific identity of the woman. Hence a specific thirty-five year old man may have stronger than typical (with respect to his age group) attraction for fifty-year old women. But this attraction does not depend on whether, for example, the older woman has an especially strong attraction to younger men (assuming this latter characteristic is unobservable in the data and hence not reflected in j). Analogously, the utility accrued by a woman of type j and specific identity h who marries a woman of type i is assumed to be $V^f_{ijh} = \eta^f_{ij} + \sigma\epsilon_{ijh}$, with η^f_{ij} deterministic and endogenous.

The random variables ϵ_{ijg} (and ϵ_{ijh}) are assumed to be independent and identically distributed, with the Gumbel extreme value distribution used by McFadden [19], whose cumulative distribution function is given by $Pr(\epsilon_{ijg} < x) = \exp(-\exp(-x))$. These simplifying assumptions are severe. However, they permit one to determine the relative demand for different types of women by type i men explicitly:

$$\text{Pr(Man of type } (i,g) \text{ prefers a woman of type } j) = \frac{\exp(\frac{\eta^m_{ij}}{\sigma})}{\sum_{k=0}^{J}\exp(\frac{\eta^m_{ik}}{\sigma})}. \tag{6}$$

In the limit of an infinitely large population, this probability coincides with the percentage μ^m_{ij}/m_i of type i men who demand type j wives. Comparing this with the percentage who choose to remain unmarried yields $\mu^m_{ij}/\mu^m_{i0} = e^{(\eta^m_{ij}-\eta^m_{i0})/\sigma}$. Treating women's preferences as the supply side, the analogous computation yields

$$\mu^f_{ij}/\mu^f_{0j} = e^{(\eta^f_{ij}-\eta^f_{0j})/\sigma}. \tag{7}$$

Finally, (4) results from multiplying these expressions together and equating supply with demand by setting $(\mu^m_{ij}) = (\mu^f_{ij})$, where the exogenous gains matrix is identified as

$$\pi_{ij} := \frac{\eta^m_{ij} + \eta^f_{ij} - \eta^m_{i0} - \eta^f_{0j}}{2\sigma}. \tag{8}$$

Since all finite values of $\sigma > 0$ are equivalent to an appropriate rescaling of η^m_{ij} and η^f_{ij}, we henceforth follow Choo and Siow [7] by fixing $\sigma = 1$. We have introduced the standard deviation σ only to be able to note the conceptual limits: $\sigma \to 0$, representing an absence of randomness — in which case the problem boils down to a discrete choice model which be solved using a linear program as in [16] [4]; and $\sigma \to \infty$, which represents all partners chosen randomly — in which case marital frequencies of different types precisely mirror the abundance of these types in the population.

1.3. **Motivations for the present work.** The problem we solve is important for several reasons. First, the implicit conditions present in equation (4) are the equilibrium outcome of a competitive market. There are not so many realistic environments with finitely many agent types and many commodities which are known to generate unique competitive equilibria — except possibly generically. While there are generic uniqueness results for matching problems that can be reduced to convex programing problems such as Monge-Kantorovich matching, e.g. [16] [4] [10] [11], the stochastic heterogeneity prevents the equilibrium in our model from being formulated as such. Instead, stochasticity restores uniqueness without the need for a genericity assumption in our model — and indeed in the more general setting studied by Galichon and Salanié [15].

Second, an affirmative explicit solution to the *Choo-Siow Inverse Problem* makes the Choo-Siow model useful in econometric analysis. The matrix $\Pi = (\Pi_{ij})$ is exogenous and not directly observable, but can be estimated from an observed marriage distribution (μ_{ij}) using (4). An economic or social shock will affect the systematic utilities that agents of various types incur by marrying agents of various other types, and will therefore alter the value of Π. This effect can be approximated to form an updated matrix of aggregated systematic parameters Π'. Existence and uniqueness guarantee that there will be exactly one marriage distribution that results from the shock, making the model predictive. In the same vein, demographers are often interested in predicting how marriage distributions will change due to changing demographics, i.e. changes in the population vector ν. Our closed form solution and comparative statics make it possible to compute the sign and in some cases the magnitude of such changes explicitly.

Finally, if the model can be shown to admit a unique distribution, the estimated parameters π_{ij} are an alternative characterization of the observed marriage distribution. The recharacterization is useful because the parameters of the Choo-Siow model have a behavioural interpretation, and are not merely observed data.

1.4. **Summary of progress and related literature.** The related *local* uniqueness question was resolved by Choo and Siow in [7]. However the issue of *global* uniqueness was left open, and posed as an open problem in a subsequent working paper by Siow [23]. We resolve this question positively by introducing a variational principle and a change of variables which allows us to exploit convexity. The question of existence of (μ_{ij}) for all $\Pi = (e^{\pi_{ij}})$ was addressed in a working paper of Choo, Seitz and Siow [6] by appealing to the Tarski fixed point theorem; see also the related results of Fox [12] and Dagsvik [8]. However the proofs there are long and involved, whereas the variational proof in the present paper is simple and direct and follows from continuity and compactness by way of an elementary estimate. Moreover, it leads to an explicit representation of the solution. This allows us to rigorously confirm various desirable and intuitive features of Choo-Siow matching, whose presence or absence might in principle be used as a test to refute the validity of various alternative matching models. Among other results, we show for example that an increase in the number of men of a given type increases the equilibrium transfer paid by such men to their spouses, while also increasing the percentage of such men who choose to remain single. See Theorem 2 below for related statements and more surprising conclusions.

The assumption of independent, indentically distributed randomness upon which Choo and Siow's derivation is based is severe. Assuming women's ages are observed for example, it rules out the possibility of correlations between man g's preference for 52 year old women and his preference for 53 year old women, beyond those encoded in the deterministic preferences he shares with all men of his observed type i. This assumption is partly relaxed in work of Galichon and Salanié [14] [15], who also allow for dependence within idiosyncratic type, heteroskedasticity, and relax the assumption that the random variables have Gumbel extreme value distribution.

Independently of us, they develop a convex variational approach to their extension of the Choo-Siow model. From theoretical considerations, they derive a closed-form expression for a strictly concave social welfare function governing competitive equilibria in their models. This gives an alternate approach to existence and uniqueness of equilibria in our setting as a special case. Apart from its convex analytic nature, their approach is quite different and complementary to ours. Their convex minimization is more complicated but also much more general than ours: it takes place in an $I \times J$ dimensional space involving the variables μ_{ij}, whereas ours takes place in an $I + J$ dimensional space whose variables are $\log \mu_{i0}$ and $\log \mu_{0j}$. Due to these differences in formulation, certain endogenous variables and properties of the models are more immediately accessible from their approach, while others are more immediately accessible from ours. The comparative statics we derive, for example, are specific to the original Choo-Siow model and serve to highlight some implications of the very restrictive hypotheses upon which it is based. We do not know whether or how the variational approach we employ might extend to the more general setting of [15], a question of potential interest for future research.

1.5. **Organization.** The remainder of this note is organized as follows. Section §2 states our results formally; it is followed by a section containing derived statics. Section §4 proves the existence and uniqueness of equilibria, while §5 is devoted to the comparative statics asserted in Theorem 2. Section §6 establishes the further comparative static assertions of Corollary 3 and §3.

2. Precise Statement of Results

In the preceding section, the *Choo-Siow inverse problem* was formulated: it is the problem of finding existence and uniqueness of equilibrium μ satisfying (1)–(4) given exogenous gains Π and population data ν. As the name suggests, it is also useful to think of this problem as one of inverting a function. From this point of view, even though Π is exogenous, we may prefer instead to consider Π as the image of a marriage distribution under a transformation that we seek to invert. We assume $\mu_{i0} > 0$ and $\mu_{j0} > 0$ without loss of generality.[1]

2.1. **Preliminaries.**

Let us begin with a reformulation of the Choo-Siow inverse problem; Siow attributes this reformulation to Angelo Melino. Let $\beta_i := \sqrt{\mu_{i0}}$ and $\beta_{I+j} := \sqrt{\mu_{0j}}$ denote the number of unmarried men and women of types $i = 1, \dots, I$ and $j = 1, \dots, J$ respectively. Since the gains matrix (4) can be used to express each component $\mu_{ij} = \beta_i \beta_{I+j} \Pi_{ij}$ of the marital distribution in terms of these new variables, the population constraints (1)–(2) can be reduced to a system

$$\begin{aligned} \beta_i^2 + \sum_{j=1}^{J} \beta_i \beta_{I+j} \Pi_{ij} - \nu_i = 0, \qquad 1 \le i \le I, \\ \beta_{I+j}^2 + \sum_{i=1}^{I} \beta_i \beta_{I+j} \Pi_{ij} - \nu_{I+j} = 0, \qquad 1 \le j \le J. \end{aligned} \tag{9}$$

of $(I+J)$ quadratic polynomials in the $(I+J)$ variables $\{\beta_k\}_{k=1}$ counting the number of umarried men and women of each type.

A solution to this system of equations is a vector of amplitudes β that has $(I+J)$ components. Abstractly, its components might be real, complex, or both. The Choo-Siow Inverse Problem is equivalent to showing that the polynomial system (9) has a unique solution with real positive amplitudes for all gains matrices Π and population vectors $\nu = [\, m \mid f \,]$ with positive components. The full marital distribution satisfying (1)–(4) is then recovered by choosing $\mu_{ij} = \beta_i \beta_{I+j} \Pi_{ij}$. Our proof is variational. We construct a functional $E(\beta)$ with the property that β is a *critical point* of E — meaning a point where E has zero derivative — if and only if β satisfies equation (9). We then show that E has exactly one critical point in the positive *orthant* $(\mathbf{R}_+)^{I+J}$, and give a formula for this critical point using the Legendre transform of a related function.

The first result of this paper is the following theorem, which solves the Choo-Siow Inverse problem.

[1] From the market equilibrium point of view, the fact that the left hand-side of (4) becomes infinite when μ_{i0} or μ_{0j} is equal to zero is unproblematic. It means that no finite value of the exogenous Π is sufficient to induce all the representatives of some type to marry. However from the inverse problem point of view, it is necessary to stipulate that μ_{i0} and μ_{0j} be strictly positive.

Theorem 1 (Existence, uniqueness, and explicit representation of a real positive solution)**.** *If all the entries of* $\Pi = (\Pi_{ij})$ *are non-negative, and those of* $\nu = [\, m \mid f \,]$ *are strictly positive,*[2] *then precisely one solution* β *of* (9) *lies in the positive orthant of* $\mathbf{R}^{I+J}$*. Indeed, the solution* $b := (\log \beta_1, \ldots, \log \beta_{I+J})$ *satisfies* $b = (DH)^{-1}(\nu) = DH^*(\nu)$ *where* $H(b)$ *and* $H^*(\nu)$ *are smooth strictly convex dual functions on* $\mathbf{R}^{I+J}$ *defined by*

$$H(b) := \frac{1}{2}\sum_{k=1}^{I+J} e^{2b_k} + \sum_{i=1}^{I}\sum_{j=1}^{J} \Pi_{ij} e^{b_i + b_{I+j}} \tag{10}$$

and

$$H^*(\nu) := \sup_{b \in \mathbf{R}^{I+J}} \langle \nu, b \rangle - H(b). \tag{11}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbf{R}^{I+J}$.

Since each matrix (μ_{ij}) with non-negative entries solving (1)–(4) corresponds to a solution β of (9) having positive amplitudes $\beta_i = \sqrt{\mu_{i0}}$ and $\beta_{I+j} = \sqrt{\mu_{0j}}$, this theorem gives the sought characterization of (μ_{ij}) by Π. Moreover, this characterization facilitates computing variations in the marital arrangements in response to changes in the data (Π, ν).

The next theorem states that the percentage change of singles β_k^2 with respect to the population parameter ν_ℓ forms a symmetric and positive definite matrix $R = (r_{k\ell})$. The expected monotonicity $r_{kk} > 0$, the unexpected symmetry $r_{k\ell} = r_{\ell k}$, and more subtle constraints and inequalities relating these percentage rates of change and the corresponding substitution effects such as $|r_{k\ell}| < \sqrt{r_{kk} r_{\ell\ell}}$ are immediate consequences.

Theorem 2 (Comparative statics)**.** *(a) Let the unique solution to the Choo-Siow inverse problem with exogenous data* Π *and* ν *be given by* $\beta(\Pi, \nu)$*. Then for* $k, \ell \in \{1, \ldots, I + J\}$*, the percentage change of singles* β_k^2 *with respect to the population parameter* ν_ℓ *turns out to define a symmetric and positive definite matrix* $R = (r_{k\ell})$ *with entries*

$$r_{k\ell} := \frac{1}{\beta_k^2} \frac{\partial \beta_k^2}{\partial \nu_\ell}. \tag{12}$$

(b) Additionally we can account for the sign, and in some cases bound the magnitude, of each entry of this matrix. Assuming no column or row of Π *vanishes,*[3] *then*

$$r_{k\ell} < 0, \tag{13}$$

if $k \in \{1 \ldots I\}$ *and* $\ell \in \{I + 1 \ldots I + J\}$ *(or vice versa), whereas*

$$\frac{1}{2}(\beta_k^2 + \nu_k) r_{k\ell} > \delta_{k\ell} := \begin{cases} 0 & \text{if } k \neq \ell \\ 1 & \text{otherwise} \end{cases} \tag{14}$$

if both $k, \ell \in \{1 \ldots I\}$ *or if both* $k, \ell \in \{I + 1 \ldots I + J\}$*.*

[2] In case $m_i = 0$ or $f_j = 0$, we simply reformulate the problem in fewer than $I + J$ variables, corresponding only to the populated types. This reformulation shows the conclusions of Theorem 1 extend also to population vectors $\nu = [\, m \mid f \,]$ whose entries are merely non-negative, instead of strictly positive.

[3] This avoids trivialities by ruling out the existing of an observable type all of whose representatives are compelled to remain unmarried.

The qualitative comparative statics (b) have a simple interpretation. Increased supply of any type k coaxes more women into marriage (13) and decreases the number of men who wish to marry. The last statement of the theorem says that this decrease is not merely due to the fact that there are more men. Rather, increased competition causes men of type $\ell \neq k$ who would have chosen marriage under the old regime to choose to be single after the shock (14). The following corollary explains how the additional women are coaxed into marriage: it shows increased competition among men leads to larger equilibrium transfers to their female spouses, as is further explained in subsection §6. The corollary asserts not only monotonicity of utility transferred by men of type i, but also of the percentage who choose to remain single, as a function of their abundance in the population.

Corollary 3 (Utility transferred and non-participant fraction increase with abundance)**.** *For all $i \leq I$, $j \leq J$, and $k \leq I + J$, with the hypotheses and notation of Theorem 2,*

$$\frac{\partial}{\partial \nu_i}(\eta^f_{ij} - \eta^m_{ij}) \quad > \quad 0 \tag{15}$$

$$\text{and} \quad \frac{\partial}{\partial \nu_k}\left(\frac{\beta_k^2}{\nu_k}\right) \quad > \quad 0. \tag{16}$$

3. Derived statics

It is possible to express many quantities of interest in terms of the symmetric positive-definite matrix $R = (r_{ij})$ from Theorem 2, whose entries encode the relative change in the number of type i individuals who choose not to marry in response to a fluctuation in the total number ν_j of type j individuals in the population. For example, in the Choo-Siow model, the number of marriages μ_{ij} of type i men to type j women is given by the geometric mean (4) of the number of singles of the two given types times the corresponding entry in the gains matrix: $\mu_{ij} = \Pi_{ij}(\mu_{i0}\mu_{0j})^{1/2} = \Pi_{ij}\beta_i\beta_j$. Since Π_{ij} is exogenous, we immediately obtain a formula

$$\frac{\partial \log \mu_{ij}}{\partial \nu_k} = \frac{1}{2}(r_{ik} + r_{k,I+j}) \tag{17}$$

showing the relative change in the number of type (i, j) marriages caused by fluctuations in the total population of type k individuals is just the average of the relative changes $r_{ik} := 2\partial(\log \beta_i)/\partial\nu_k$ and $r_{I+j,k}$ in the numbers of unmarrieds of the corresponding types i and j.

We may also consider fluctuations in the number of singles of type k in response to changes in the exogenous gains parameters Π_{ij} when the population ν of each type of man and woman is held fixed. In section 6.2, the implicit function theorem is used to derive

$$\frac{\partial \beta_k}{\partial \Pi_{ij}} = -\beta_i\beta_{I+j}\left(\frac{\partial \beta_k}{\partial \nu_i} + \frac{\partial \beta_k}{\partial \nu_{I+j}}\right) \tag{18}$$

for all $i \in \{1, \ldots, I\}$, $j \in \{1, \ldots, J\}$, and $k \in \{1, \ldots, I + J\}$, or equivalently

$$\frac{\partial \log \beta_k}{\partial \Pi_{ij}} = -\frac{\mu_{ij}}{2\Pi_{ij}}(r_{ki} + r_{k,I+j}). \tag{19}$$

The equation (18) has an intuitive interpretation. An increase in the total systematic gains to an (i,j) marriage (produced, for example, by an isolated increase in the value of type j marriages to type i men, or an isolated decrease in the value of remaining single) has the same effect as decreasing the supply of the men or women of the respective types by a proportionate amount, weighted by the geometric mean of the unmarried men and women of type i and j. Theorem 2 shows the summands above to have opposite signs, so the sign of their sum $r_{ki} + r_{k,I+j}$ may fluctuate according to market conditions.

4. Variational Approach (Proof of Theorem 1)

4.1. **Variational method: existence of a solution.** Consider the function $E : \mathbf{R}^{I+J} \to \mathbf{R} \cup \{+\infty\}$, defined as follows:

$$E(\beta) := \frac{1}{2}\sum_{k=1}^{I+J} \beta_k^2 + \sum_{i=1}^{I}\sum_{j=1}^{J} \Pi_{ij}\beta_i\beta_{I+j} - \sum_{k=1}^{I+J} \nu_k \log|\beta_k|. \tag{20}$$

It diverges to $+\infty$ on the coordinate hyperplanes where the β_k vanish, but elsewhere is smooth.

We differentiate and observe that β is a critical point of E if and only if (9) holds. Notice strict positivity of the components of $\nu = [\, m \mid f\,]$ implies the corresponding component of a solution β to (9) is non-vanishing, hence no solutions occur on the coordinate hyperplanes which separate the different orthants. In words, the critical points of E are precisely those that satisfy the system of equations we wish to show has a unique real positive root. It therefore suffices to show that $E(\beta)$ has a unique real positive critical point; for then (9) admits exactly one real positive solution. Let us show at least one such solution exists, by showing $E(\beta)$ has at least one critical point: namely, its minimum in the positive orthant.

Claim 4 (Existence of a minimum)**.** *If all the entries of* $\Pi = (\Pi_{ij})$ *are non-negative, and those of* $\nu = [\, m \mid f\,]$ *are strictly positive, the function* $E(\beta)$ *on the positive orthant defined by* (20) *attains its minimum value.*

Proof. Since $E(\beta)$ is continuous, the claim will be established if we show the sublevel set $B_\lambda := \{\beta \in (\mathbf{R}_+)^{I+J} \mid E(\beta) \le \lambda\}$ is compact for each $\lambda \in \mathbf{R}$. Non-negativity of Π_{ij} combines with positivity of ν_k, β_k, and the inequality $\log \beta_k \le \beta_k - 1$ to yield

$$E(\beta) \ge \sum_{k=1}^{I+J} \frac{1}{2}\beta_k^2 - \nu_k(\beta_k - 1) \tag{21}$$

$$= \frac{1}{2}\sum_{k=1}^{I+J} (\beta_k - \nu_k)^2 - (\nu_k - 1)^2 + 1. \tag{22}$$

It follows that B_λ is bounded away from infinity. Since $E(\beta)$ diverges to $+\infty$ on the coordinate hyperplanes, it follows that B_λ is also bounded away from the coordinate hyperplanes — hence compactly contained in the positive orthant. □

4.2. **Uniqueness, convexity, and Legendre transforms.** With this critical point characterization of the solution in mind, let us observe for $\beta \in \mathbf{R}^{I+J}$ in the positive orthant, defining $b_k := \log \beta_k$ implies $E(\beta) = H(b) - \langle \nu, b \rangle$, where $H(b)$ is defined in (10). Since the change of variables $\beta_k \in \mathbf{R}_+ \longmapsto b_k = \log \beta_k \in \mathbf{R}$ is a diffeomorphism, it follows that critical points of $H(b) - \langle \nu, b \rangle$ in the whole space $\mathbf{R}^{I+J}$ are in one-to-one correspondence with critical points of $E(\beta)$ in the positive orthant.

On the other hand, $H(b)$ is manifestly convex, being a non-negative sum of convex exponential functions of the real variables b_k; in fact $\Pi_{ij} \geq 0$ shows the Hessian $D^2H(b)$ dominates what it would be in case $\Pi = 0$, namely the diagonal matrix with positive entries $diag[2e^{2b_1}, \ldots, 2e^{2b_{I+J}}]$ along its diagonal. Thus $H(b)$ is strictly convex throughout $\mathbf{R}^{I+J}$, and $E(\beta) = H(b) - \langle \nu, b \rangle$ can admit only one critical point β in the positive orthant — the minimizer whose existence we have already shown. The solution β to (9) which we seek therefore coincides with the unique point at which the minimum is attained.

This last fact means that b maximizes the right-hand side of the following equation:

$$H^*(\nu) := \sup_{b \in \mathbf{R}^{I+J}} \langle \nu, b \rangle - H(b) \tag{23}$$
$$= \sup_{\beta \in (\mathbf{R}_+)^{I+J}} -E(\beta).$$

The function H^* defined pointwise by the above equation is the Legendre transform or convex dual function of H. It follows that the solution b satisfies $\nu = DH(b)$. The following well-known lemma from convex analysis shows $b = DH^*(\nu)$ by the duality of H and H^*. This provides an explicit formula for b as the derivative of H^*.

Lemma 5 (Legendre duality)**.** *Let $H \in C^2$ be any convex function whose Hessian is strictly positive-definite throughout $\mathbf{R}^{I+J}$. Then H^* defined by* (23) *is also twice continuously differentiable on $\mathbf{R}^{I+J}$, and $\nu = DH(b)$ if and only if $b = DH^*(\nu)$.*

5. Comparative Statics (Proof of Theorem 2)

5.1. **Positive definiteness (a).** Our representation of the solution in terms of the Legendre transform of the convex function H can be used to obtain information about the derivatives of the solutions with respect to the population parameters ν.

Suppose we wish to know how the number of marriages $\mu_{ij} = \Pi_{ij}\beta_i\beta_{I+j}$ of each type (i,j) varies in response to slight changes in the population vector ν, assuming the gains matrix Π remains fixed. This is easily computed from the percentage rate of change $r_{k\ell}$ in the number β_k^2 of unmarrieds of each type, which is given in terms of the Hessian of either (10) or (11) by

$$r_{k\ell} := \frac{1}{\beta_k^2}\frac{\partial \beta_k^2}{\partial \nu_\ell} = 2D^2_{k\ell}H^*(\nu) = 2(D^2H|^{-1}_{(\log\beta_1,\ldots,\log\beta_{I+J})})_{k\ell}, \quad 1 \leq k, \ell \leq I+J. \tag{24}$$

To see that these equalities hold, observe that the solution β is the point where the maximum (11) is attained. The Legendre transform $H^*(\nu)$ of H defined by this maximum is manifestly convex, and its smoothness is well-known to follow from the positive-definiteness of $D^2H(b) > 0$ as in Lemma 5. Moreover $b = DH^*(DH(b))$, whence the maximum (11) is attained at $b = DH^*(\nu)$ and $D^2H(b)^{-1} = D^2H^*(DH(b)) = D^2H^*(\nu) > 0$. This positive definiteness implies the first half of the Theorem 2.

5.2. **Qualitative characterization of comparative statics (b).** To complete our qualitative description of the substitution effects in this section, we apply the following theorem from functional analysis to matrices $T : \mathbf{R}^n \longrightarrow \mathbf{R}^n$. We define the operator norm of such a matrix by $\|T\|_{op} := \max_{0 \neq v \in \mathbf{R}^n} |T(v)|/|v|$, where $|v| = \langle v, v\rangle^{1/2}$ denotes the Euclidean norm.

Theorem 6 (Neumann series for the resolvent of a linear contraction)**.** *If $\|T\|_{op} < 1$ for $T : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, the operator $(1-T)^{-1}$ exists and is equal to $\sum_{k=0}^{\infty} T^k$.*

Next, we consider the matrix $D^2H(b)|_{(\log\beta_1,\ldots,\log\beta_{I+J})}$, and derive properties of its inverse, whose entries give the various values of $r_{k\ell}/2$. Differentiating the known function $H(b)$ twice yields a positive-definite $(I+J)\times(I+J)$ matrix which can be factored into the form

$$2R^{-1} = D^2H|_{b=(\log\beta_1,\ldots,\log\beta_{I+J})} = \Delta \begin{pmatrix} \Delta_I & \Pi \\ \Pi^T & \Delta_J \end{pmatrix} \Delta \tag{25}$$

where $\Delta = \mathrm{diag}[e^{b_1}, \ldots, e^{b_{I+J}}] = \mathrm{diag}[\beta]$, while Δ_I and Δ_J are $I \times I$ and $J \times J$ diagonal submatrices whose diagonal entries are all larger than two:

$$\begin{aligned} (\Delta_I)_{ii} &= 2 + \frac{1}{\beta_i^2}\sum_{j=1}^{J} \Pi_{ij}\beta_i\beta_{I+j} = 1 + \frac{\nu_i}{\beta_i^2}, \\ (\Delta_J)_{jj} &= 2 + \frac{1}{\beta_{I+j}^2}\sum_{i=1}^{I} \Pi_{ij}\beta_i\beta_{I+j} = 1 + \frac{\nu_{I+j}}{\beta_{I+j}^2}. \end{aligned}$$

Here we have used the fact that the values β are critical points and therefore satisfy the first order conditions (9) to simplify these diagonal terms.

There are determinant and inverse formulae for block matrices which assert [17] that

$$\det \begin{pmatrix} \Delta_I & \Pi \\ \Pi^T & \Delta_J \end{pmatrix} = \det(\Delta_I)\det(\Delta_J)\det(1 - {\Delta_I}^{-1}\Pi{\Delta_J}^{-1}\Pi^T), \tag{26}$$

and

$$\begin{pmatrix} \Delta_I & \Pi \\ \Pi^T & \Delta_J \end{pmatrix}^{-1} = \begin{pmatrix} (\Delta_I - \Pi{\Delta_J}^{-1}\Pi^T)^{-1} & -(\Delta_I - \Pi{\Delta_J}^{-1}\Pi^T)^{-1}\Pi{\Delta_J}^{-1} \\ -\Pi^T{\Delta_I}^{-1}(\Delta_J - \Pi^T\Delta_I^{-1}\Pi)^{-1} & (\Delta_J - \Pi^T\Delta_I^{-1}\Pi)^{-1} \end{pmatrix}. \tag{27}$$

The determinant (26) is positive by (25) and Theorem 1. We will now show that the eigenvalues of the matrix $A(s) = {\Delta_I}^{-1}s\Pi{\Delta_J}^{-1}s\Pi^T$, appearing in (26)–(27) are bounded above by 1 and below by -1 for all values of $s \in [0,1]$. This will have implications respecting the signs of the entries of (27), whose $(k,\ell)^{th}$ entry is in fact equal to $\beta_k\beta_\ell r_{k\ell}/2$ hence shares the sign of the change (13)–(14) which we desire to estimate. Namely, it will allow us to apply Theorem 6 to block entries such as $(\Delta_I - \Pi{\Delta_J}^{-1}\Pi^T)^{-1} = (1-A(1))^{-1}{\Delta_I}^{-1}$ in (27).

Let $\lambda^{max}(s)$ be the largest eigenvalue of $A(s)$. Then, the smallest eigenvalue of $(1-A(s))$ is equal to $(1-\lambda^{max}(s))$. We proceed by continuously deforming from $s=0$ to $s=1$: The eigenvalues of $(1-A(0))$ are equal to 1, as $A(0)$ is in fact equal to the zero matrix. Since

$\det(1-A(s)) > 0$ for all $s \in [0,1]$, continuity of $\lambda^{max}(s)$ and the intermediate value theorem imply that $1-\lambda^{max}(s) > 0$ for all s, so that $\lambda^{max}(1) < 1$. Since no row of Π vanishes, $A(s)$ has positive entries whenever $s > 0$. The Perron-Frobenius theorem therefore implies that any negative eigenvalue λ of $A(1)$ is bounded by $|\lambda| < \lambda^{max}(1)$.

Since A has positive entries and $\|A\|_{op} < 1$, Theorem 6 indicates that the entries of $(1-A)^{-1}$ are all positive — exceeding one on the diagonal. But $\beta_k\beta_\ell r_{k\ell}/2$ coincides with the $(k,\ell)^{th}$ entry of $(1-A)^{-1}\mathrm{diag}[\beta_1^2/(\beta_1^2+\nu_1),\ldots,\beta_I^2/(\beta_I^2+\nu_I)]$, giving the desired inequalities (14) whenever $k,\ell \in \{1,\ldots,I\}$. The signs of the remaining derivatives (13)–(14) may be verified by applying the same technique to the three other submatrices present in (27), thus completing the proof of Theorem 2(a)–(b).

6. Transfer utilities (Proof of Corollary 3 and subsequent remarks)

6.1. **Varying the population vectors ν.** Given a specification of Π and $\nu = [\, m_i \mid f_j \,]$, the Choo-Siow model predicts a unique vector $\beta = [\, \mu_{i0} \mid \mu_{0j} \,]$ of unmarrieds. Given a fixed Π and a fixed β, the full marriage distribution can then be uniquely recovered. It is therefore possible to view μ as a single valued (smooth) function of Π and ν. By Theorem 2, the signs of $r_{k\ell}$ are independent of Π and ν and depend only on whether $k \in \{1,\ldots,I\}$, or $k \in \{I+1,\ldots,I+J\}$, and likewise for ℓ. It is perhaps useful to visualize these comparative statics as the entries of the matrix $D\beta$ with $D_\ell\beta_k := \frac{\partial\beta_k}{\partial\nu_\ell}$. Then, $D\beta$ is a block matrix that is positive in its upper-left and lower-right blocks, and negative in its upper-right and lower-left blocks. Schematically, (13)–(14) yield

$$D\beta = \begin{pmatrix} + & - \\ - & + \end{pmatrix}. \tag{28}$$

Reverting back to the Choo-Siow notation for unmarrieds and population vectors, we have for all k and ℓ:

$$\frac{\partial\mu_{k0}}{\partial m_\ell} > 0, \quad \frac{\partial\mu_{k0}}{\partial f_\ell} < 0, \quad \frac{\partial\mu_{0k}}{\partial f_\ell} > 0, \quad \frac{\partial\mu_{0k}}{\partial m_\ell} < 0. \tag{29}$$

These basic comparative statics yield qualitative information about other more complex quantities of interest. As indicated prior to equation (8), the quantity $\eta^m_{ij} + \eta^f_{ij} - \eta^m_{i0} - \eta^f_{0j}$ is exogenous, whereas the first two individual summands are separately endogenous and determined within the model. In the original formulation of this model, present in [7], our endogenous payoff $\eta^m_{ij} = \tilde{\eta}^m_{ij} - \tau_{ij}$ is separated into a systematic return $\tilde{\eta}^m_{ij}$ presumed to be exogenous, and a utility transfer τ_{ij} from husband to wife, which is endogenous and set in equilibrium. Similarly, $\eta^f_{ij} = \tilde{\eta}^f_{ij} + \tau_{ij}$.

In equilibrium (7), both of the following equations hold:

$$\log(\mu_{ij}) - \log(\mu_{0j}) = \eta^f_{ij} - \eta^f_{0j} = \tilde{\eta}^f_{ij} + \tau_{ij} - \tilde{\eta}^f_{0j}, \tag{30}$$

$$\log(\mu_{ij}) - \log(\mu_{i0}) = \eta^m_{ij} - \eta^m_{i0} = \tilde{\eta}^m_{ij} - \tau_{ij} - \tilde{\eta}^m_{i0}; \tag{31}$$

there is no utility transferred by remaining single. Subtracting one from the other, we see that:

$$\log(\frac{\mu_{i0}}{\mu_{0j}}) = 2\tau_{ij} + c_{ij}, \tag{32}$$

where $c_{ij} = (\tilde{\eta}^f_{ij} - \tilde{\eta}^f_{0j} - \tilde{\eta}^m_{ij} + \tilde{\eta}^m_{i0})$ is exogenous.

We denote the differentiation operator $\frac{\partial}{\partial \nu_k} f$ by $\dot{f}$ (suppressing the dependence on k). Differentiating $c_{ij} = (\eta^f_{ij} - 2\tau_{ij} - \tilde{\eta}^f_{0j} - \eta^m_{ij} + \tilde{\eta}^m_{i0})$ and (32) yields:

$$\frac{\partial}{\partial \nu_k}(\eta^f_{ij} - \eta^m_{ij}) = 2\dot{\tau_{ij}} = \frac{\dot{\mu_{i0}}}{\mu_{i0}} - \frac{\dot{\mu_{0j}}}{\mu_{0j}}.$$

The inequalities (29) now determine the sign of $\dot{\tau_{ij}}$, which depends on the differentiation variable ν_k. Since $\dot{\mu_{i0}}$ and $\dot{\mu_{0j}}$ have opposite signs, according to Theorem 2, we find

$$\frac{\partial \tau_{ij}}{\partial m_i} > 0, \tag{33}$$

which means the transfer of type i men to each type of spouse must increase in response to an isolated increase in the population of men of type i. This is expected because an increase in the number of type i men introduces additional competition for each type of women, due to the smearing present in the model. To decrease the number of type i men demanding marriage to a particular type of woman to a level that permits one-to-one matching requires an increase in the transfer to crowd out some men.

While in principle the men might re-distribute so that the proportion of married men remains the same, our next computation shows this is not the case. We consider the marital participation rate of type k individuals, or rather the non-participation rate $s_k(\nu) := \beta_k^2/\nu_k$, defined as the proportion of individuals who choose not to marry. Differentiation yields

$$\begin{aligned} \frac{\partial s_k}{\partial \nu_k} &= \frac{\beta_k^2}{\nu_k^2}\Big(\nu_k r_{kk} - 1\Big) \\ &> \frac{\beta_k^2}{\nu_k^2}\Big(\frac{\nu_k - \beta_k^2}{\nu_k + \beta_k^2}\Big), \end{aligned}$$

according to (14). But this is manifestly positive since the number β_k^2 of singles of type k cannot exceed the total number ν_k of type k individuals. This means, for example, that an increase in the total population of type k men increases the percentage of type k men who choose to remain unmarried, given a fixed population of women and men of other types (and assuming, as always, that the exogenous gains matrix Π remains fixed). It concludes the proof of Corollary 3.

6.2. **Varying the gains data Π (Proof of** (18)-(19)**).** The population vector ν is one variable of interest. However the function β also depends on the gains parameters Π_{ij}. The complete derivative $D_{(\nu,\Pi)}\beta = [D_\nu \beta \mid D_\Pi \beta]$ is an $(I+J) \times (IJ + I + J)$ matrix. As such there are linear dependencies among its rows and columns. Since the matrix $D_\nu \beta$ is invertible, its columns are linearly independent and form a basis of the column space.

Hence, the remaining columns of the complete derivative can be expressed using linear combinations of them. The implicit function theorem applied to this problem turns out to yield the simple linear relationship (18):

$$\frac{\partial \beta_k}{\partial \Pi_{ij}} = -\beta_i \beta_{I+j} \left(\frac{\partial \beta_k}{\partial \nu_i} + \frac{\partial \beta_k}{\partial \nu_{I+j}}\right) \tag{34}$$

for all $i \in \{1, \ldots, I\}$, $j \in \{1, \ldots, J\}$, and $k \in \{1, \ldots, I+J\}$.

Equilibrium (9) coincides with vanishing of the function $F(\beta, \nu, \Pi) : \mathbf{R}^{(I+J)+(I+J)+(IJ)} \to \mathbf{R}^{I+J}$ defined by

$$F_i(\nu, \Pi) = \beta_i^2 + \sum_{j=1}^{J} \beta_i \beta_{I+j} \Pi_{ij} - \nu_i, \qquad 1 \leq i \leq I$$

$$F_j(\nu, \Pi) = \beta_{I+j}^2 + \sum_{i=1}^{I} \beta_i \beta_{I+j} \Pi_{ij} - \nu_{I+j}, \qquad 1 \leq j \leq J. \tag{35}$$

The implicit function theorem stipulates that if the derivative $D_\beta F|_{\beta_0,\nu_0,\Pi_0}$ is invertible, there is a small neighbourhood around (β_0, ν_0, Π_0) inside which for each (ν, Π) there is a unique β satisfying equation (9), and further that β depends smoothly on (ν, Π). The implicit function theorem also provides a formula for the derivative of the implicit function $\beta(\nu, \Pi)$. It is obtained by applying the chain-rule to $F(\beta(\nu, \Pi), \nu, \Pi)$:

$$[D_\nu \beta \mid D_\Pi \beta]_{\nu_0,\Pi_0} = -[D_\beta F]^{-1} [D_\nu F \mid D_\Pi F]_{\beta_0,\nu_0,\Pi_0}. \tag{36}$$

Since $\frac{\partial F_k}{\partial \nu_\ell} = -\delta_{k\ell}$, and $\frac{\partial F_\ell}{\partial \Pi_{ij}} = \beta_i \beta_{I+j}(\delta_{i\ell} + \delta_{I+j,\ell})$, the first part of the preceding formula yields $[D_\beta F]^{-1} = D_\nu \beta$, and the second part then implies (34). Theorem 2 shows $D_\nu \beta$ is invertible, so the hypotheses of the implicit function theorem are globally satisfied and our calculations are valid. This concludes the proof of (18)-(19).

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