The spectrum of a family of fourth-order nonlinear diffusions near the global attractor*

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Abstract

The thin film and quantum drift diffusion equations belong to a fourth-order family of evolution equations proposed in [21] to be analogous to the (second-order) porous medium family. They are 2-Wasserstein ($= d_2$) gradient flows of the generalized Fisher information $I(v)$ just as the porous medium family was shown to be the d_2 gradient flow of the generalized entropy $E(v)$ by Otto [40]. The identity $aI(v) = bE(v) + |\nabla_{d_2} E(v)|^2/2$ implies $a \operatorname{Hess}_{d_2} I(v_*) = \operatorname{Hess}_{d_2} E(v_*)(b + \operatorname{Hess}_{d_2} E(v_*))$ formally, when the equation is rescaled and linearized around the resulting self-similar critical profile v_*. We couple this relation with the diagonalization of $\operatorname{Hess}_{d_2} E(v_*)$ for the porous medium flow computed in [45]. This yields information about the leading- and higher-order asymptotics of the equation on $\mathbf{R}^N$ which — outside of special cases — was inaccessible previously.

1 Introduction

In this manuscript, we investigate the long time asymptotic behaviour of certain nonnegative solutions to a family of nonlinear fourth-order equations in $\mathbf{R}^N$, namely

$$\partial_t u + \nabla \cdot \left(u \nabla \left(u^{m-3/2} \Delta u^{m-1/2} \right) \right) = 0 \tag{1}$$

with exponent $m \geq 1$. Our main contribution is a powerful heuristic which leads to a complete asymptotic expansion (35) for these solutions as time tends to infinity. In the special case $(N, m) = (1, 3/2)$, such a prediction was made by Bernoff and Witelski using a completely different method; our calculation not only supports their result, but vastly extends it to cover at least the full range of parameters $(N, m) \in \mathbf{N} \times [1, 3/2]$ for which weak solutions to (1) have been constructed [35].

This family has two prominent members which are of physical relevance: If $m = 3/2$ the above equation is the thin film equation (with linear mobility), which is a model for the capillarity-driven evolution of a viscous thin film over a solid substrate [38, 37] and can also be seen as a lubrication approximation of a Hele–Shaw flow [27]. In this case, the unknown $u(t, x)$ represents the height of the thin film. The other physically relevant case is $m = 1$, which models statistical mechanical fluctuations around a one-dimensonsal interface [22, 23], and has also been proposed as a simplified quantum drift-diffusion model for electrons in semiconductors; see [33] for references.

Solutions to (1) are expected to feature different phenomena, depending on the nonlinearity exponent. If $m = 1$, the equation can be viewed as a fourth-order analog for the heat equation and thus u is expected to become strictly positive instantaneously. On the other hand, if $m > 1$, there are compactly supported solutions ("droplets") which exhibit a slowly propagating contact line $\partial \operatorname{spt}(u)$. In this case, (1) involves a free boundary problem for the support of u and the equation shows some similarities to the porous medium

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flow. It is this similarity to the heat and porous medium equations — which is not purely phenomenological — that we will exploit in this paper to study the long time behavior of solutions.

The Cauchy problem for the nonlinear fourth-order equation (1) is still not completely solved. As the equation is only degenerate parabolic, i.e., it is parabolic only in regions where $u > 0$, solutions to (1) will in general not be classical — at least when $m > 1$. The theory of existence of weak solutions in the thin film case $m = 3/2$ was initiated by Bernis & Friedman [8] and is by now quite complete, see [11, 4, 30] and references therein. On the other hand, regularity and uniqueness of solutions are only partially understood, even in the one-dimensional setting; see Giacomelli, Knüpfer & Otto [25] and John [31] for example. If $m = 1$, existence of nonnegative solutions was established in [33, 28]. The first existence results in the complete range $1 \leq m \leq 3/2$ (this is the range in which the generalized Fisher information, see Eq. (7) below, is convex in the ordinary sense) were found by Mattes, McCann and Savare [35], and they are completely unknown for $m > 3/2$. The major difficulty in the analysis of (1) is the lack of a comparison principle for fourth-order equations. Still, the weak solutions which have been constructed are known to preserve non-negativity.

To properly discuss the existence theory for (1) in the case $m > 1$, one should be more precise. Since a fourth-order stationary problem in a fixed domain typically requires two boundary conditions, we expect the fourth-order free-boundary problem (1) to require three spatial boundary conditions for well-posedness. These are the defining condition $u = 0$ for the free-boundary $\partial\,\mathrm{spt}(u)$, the no-flux condition

$$\nu \cdot u\nabla(u^{m-3/2}\Delta u^{m-1/2}) = 0 \quad \text{on } \partial\,\mathrm{spt}(u), \tag{2}$$

where ν is the outer unit normal vector to the free boundary $\partial\,\mathrm{spt}(u)$, and the Neumann condition

$$\nu \cdot \nabla \left(u^{m-1/2}\right) = 0 \quad \text{on } \partial\,\mathrm{spt}(u). \tag{3}$$

Condition (3) can also be viewed as a regularity or order-of-vanishing condition; it seems not to have been previously articulated outside of the thin film case $m = 3/2$, where it was discussed e.g. by Giacomelli and Otto [27] (with $N = 1$) among others. There it corresponds to the complete wetting regime. In that context, different choices of contact angles also have physical significance, but the zero contact angle implied by (3) is apparently the simplest one [39].

Suitable weak solutions of (1)–(2) preserve mass

$$\int u(t,x)\,dx \;=\; \int u(0,x)\,dx \;=:\; M \quad \text{for any } t > 0; \tag{4}$$

those that satisfy (3) also preserve center of mass

$$\int xu(t,x)\,dx \;=\; \int xu(0,x)\,dx \qquad \text{for any } t > 0. \tag{5}$$

This is true in particular for the Generalized Minimizing Movements (GMM) constructed by Matthes, McCann and Savare for $m \in [1, 3/2]$. These are shown in [35] to satisfy (4) and a weak form (16) of the flow (1); as we presently explain, it is possible to deduce (5) and weak forms of (2)–(3) from these facts.

As already remarked, the evolution equation (1) represents a fourth-order analog for the porous medium equation

$$\partial_t u - \Delta u^m \;=\; 0 \tag{6}$$

on $\mathbf{R}^N$. This equation is best known for modeling the flow of an isentropic gas through a porous medium if $m > 1$; it becomes the ordinary heat equation if $m = 1$. We refer to Vázquez's monograph [50] for a recent survey on the analytical treatment of this equation. In fact, both the fourth-order equation (1) and the porous medium equation (6) can be interpreted as gradient flows for the 2-Wasserstein distance. (Indeed, Denzler and the first author initially *introduced* (1) as a 2-Wasserstein gradient flow [21, Section 4.3].) In the case of the fourth-order equation (1), the dissipating functional is the *generalized Fisher information*

$$\frac{1}{2m-1}\int |\nabla \left(u^{m-1/2}\right)|^2\,dx, \tag{7}$$

while in the case of the porous medium equation, the dissipating functional is the *entropy*

$$\int e(u)\,dx, \quad \text{where } e(z) = \begin{cases} z\ln z & \text{if } m = 1, \\ \frac{1}{m-1} z^m & \text{if } m > 1, \end{cases} \tag{8}$$

a fact that was established in OTTO's seminal work [40]. The derivation of (1) in [21] was also foreshadowed by OTTO's discovery of this 2-Wasserstein gradient flow structure in the thin film case $m = 3/2$ [39, 26]. However, besides this structural similarity, there are further close links between (1) and (6), one of which is via the following identity:

$$\frac{d}{dt}\int e(u(t,\,\cdot\,))\,dx \;=\; -\left(\frac{2m}{2m-1}\right)^2 \int |\nabla\left(u(t,\,\cdot\,)^{m-1/2}\right)|^2\,dx$$

for any sufficiently smooth solution u of (6). That is, the generalized Fisher information encodes the limiting dissipation mechanism for the porous medium flow. A certain rescaled version of this identity, the entropy-information relation (23), is at the heart of our analysis, cf. Section 4 and 5. The gradient flow structure of (1) will be reviewed in Sections 3 and 4 below.

Both equations (1) and (6) allow for a family of self-similar solutions. The former, first discovered by SMYTH & HILL [47] in the particular case $m = 3/2$ and $N = 1$ and then generalized in [24, 21], is given by

$$u_*(t,x) = \frac{1}{t^{N\alpha}}\hat{u}_*\left(\frac{|x|}{t^\alpha}\right), \tag{9}$$

where

$$e'(\hat{u}_*(r)) = \begin{cases} 1 + \log \hat{u}_*(r) = \left(\sigma_M - \gamma r^2/2\right) & \text{if } m = 1, \\ \frac{m}{m-1}\hat{u}(r)^{m-1} = \left(\sigma_M - \gamma r^2/2\right)_+ & \text{if } m > 1. \end{cases}$$

Here, $(\,\cdot\,)_+ = \max\{\,\cdot\,,0\}$, and

$$\alpha = \frac{1}{N(2m-2)+4} \quad \text{and} \quad \gamma^2 = \frac{2\alpha m^2}{(2m-1)(N(m-1)+1)}.$$

Moreover, σ_M is a real number (positive if $m > 1$) that is determined by the mass constraint

$$\int u_*\,dx \;=\; M.$$

These self-similar solutions have a delta measure located at the origin as initial data, hence are often called *source-type solutions*. For the thin film case it is shown that u_* is the only radially symmetric self-similar solution of (1) that satisfies the auxiliary condition (3), cf. [9, 24]; we expect the same to be true also for the general case $m \geq 1$. We also remark that u_* resembles the Barenblatt solution (except for the specific values of α and γ), which is well-known to be the self-similar solution of the porous medium equation [52, 5, 41]. Moreover, just as the Barenblatt solution describes the long time behavior of any solution to the porous medium equation [49], we conjecture that u_* is the asymptotic limit of any solution to the fourth-order equation. More precisely, for any solution of (1) with total mass M (and satisfying (3)) we expect that

$$u(t,\,\cdot\,) \approx u_*(t,\,\cdot\,) \quad \text{as } t \gg 0. \tag{10}$$

This asymptotic behavior, however, is only partially understood. In the case $m = 3/2$, CARRILLO & TOSCANI prove this asymptotics for strong solutions in one space dimension and obtain the sharp rate of convergence. In the range $1 \leq m \leq 3/2$, the long time behavior and rate of convergence is established by MATTHES, SAVARÉ and the first author [35] for GMM solutions of arbitrary dimension.

In the present work, we build upon (10) and go further: we show the 2-Wasserstein gradient structure and relation to the porous medium flow leads to a powerful heuristic allowing us to predict the the complete spectrum and the corresponding eigenfunctions of the displacement Hessian operator obtained by linearizing a certain rescaled version of (1) around its global attractor u_*. This new approach confirms and vastly extends the heuristic derivation of the spectrum given by Bernoff & Witelski [10] for the case of the one-dimensional thin film equation ($m = 3/2$, $N = 1$). The knowledge of the spectrum and the corresponding eigenfunctions provides information not only on the slowest rates of convergence that saturate the optimal bounds; it also allows us to extract further information on large time behaviour of the flow to all orders. In particular, we will make explicit conjectures about the role of solutions generated by translations, shears, and dilations of u_*. We obtain the precise information on the spectrum and the corresponding eigenfunctions directly from the spectrum of the displacement Hessian of the linearized (confined) porous medium equation via an entropy-information relation to be described in Section 4 below. This entropy-information relation provides us with an formula that associates eigenvalues and eigenfunctions of the porous medium equation with those of the fourth-order equations considered in this paper. The former were computed by the second author in [45] using techniques from mathematical physics developed for the spectral analysis of Schrödinger operators. An analogous earlier study for the fast diffusion equation (that is (6) with $m < 1$) is due to Denzler and the first author [20].

We finally caution the reader that even though the spectral analysis presented in the Section 6 of this manuscript is rigorous, the linearization that gives rise to the displacement Hessian operator is not. That is why the results presented in Corollary 3 below position us to *conjecture* (rather than prove) the higher-order asymptotics of solutions to the nonlinear equation (1). In the fast-diffusion case ($m < 1$ of (6)), an argument for closing the analogous gap is provided by work of Denzler, Koch, and the first author [19], who find a rigorously controlled linearization of the fast diffusion equation and a similarity transform that relates the operator which appears in this controlled linearization to the displacement Hessian operator from [20]. As a result, the authors derive higher-order asymptotics for the fast diffusion equation in weighted Hölder spaces.

The remainder of the paper is organized as follows: In Section 2, we provide a convenient rescaling of the equation following a standard procedure. Section 3 recalls the definition and construction of weak solutions using the GMM scheme and explains why they preserve center of mass and obey (2)–(3). Section 4 describes the formal gradient flow structure of the resulting dynamics and the aforementioned entropy-information relation. In Section 5 we finally linearize the rescaled equation around its global attractor. Section 6 describes the diagonalization of the resulting linear operator, while Section 7 explores the dynamical and analytic implications of its spectrum. The paper concludes with an appendix on the spectral analysis of the Ornstein–Uhlenbeck operator in a Gaussian weighted Sobolev space.

2 Rescaling

In order to study the long time behavior of solutions to (1), it is convenient to rescale equation (1) in such a way that the global attractor becomes a fixed point of the dynamics. That is, we choose to view the dynamics from perspective receding at a rate $|x| \sim t^\alpha$ inspired by (9). Indeed, setting

$$\hat{x} = \frac{x}{At^\alpha}, \quad \hat{t} = \alpha \log t, \quad \text{and} \quad v = \frac{t^{N\alpha}}{B} u,$$

and

$$u(t, x) = \frac{B}{t^{N\alpha}} v\left(\alpha \log t, \frac{x}{At^\alpha}\right),$$

yields

$$\partial_{\hat{t}} v - \hat{\nabla} \cdot (\hat{x} v) + \frac{\gamma^2}{\alpha} \hat{\nabla} \cdot \left(v \nabla \left(v^{m-3/2} \hat{\Delta} v^{m-1/2}\right)\right) \;=\; 0, \tag{11}$$

where we have chosen

$$A = \begin{cases} 2^{1/4} & \text{if } m = 1, \\ \sqrt{2\sigma_M/\gamma} & \text{if } m > 1, \end{cases} \quad \text{and} \quad B = \begin{cases} \exp\left(\sigma_M - \frac{1}{2}\right) & \text{if } m = 1, \\ (2\sigma_M)^{\frac{1}{m-1}} & \text{if } m > 1. \end{cases}$$

We call equation (11) the *confined* equation; the parameter γ^2/α can be interpreted as the relative strength of diffusion compared to confinement.

Under this change of variables, the self-similar solution u_* from (9) transforms to a fixed point $v_*(x)$ with a particular simple form (due to our choices of A and B):

$$e'(v_*(\hat{r})) = \begin{cases} \log v_*(\hat{r}) + 1 & = \frac{1}{2}\left(1 - \hat{r}^2\right) & \text{if } m = 1, \\ \frac{m}{m-1} v_*(\hat{r})^{m-1} & = \frac{1}{2}\left(1 - \hat{r}^2\right)_+ & \text{if } m > 1. \end{cases} \tag{12}$$

By a slight abuse of language, we will call v_* the *Barenblatt profile*. We remark the above choice of the prefactor in the Barenblatt profile is motivated by the particular formula of the global attractor of the confined porous medium equation considered in [45], the benefit of which we will become apparent in the subsequent analysis.

Working with the confined equation (11) instead of (1) has the advantage that the global attractor is a stationary solution of the equation. This change of perspective proved very useful in the study of long time asymptotics of both the porous medium equation and the thin film equation, e.g., [14, 40, 18, 15]. Also, as described in the following sections, the above rescaling makes the Barenblatt profile a ground state of the dissipating functional for which the rescaled evolution remains a 2-Wasserstein gradient flow.

We finally remark that under the above rescaling, the initial data of (1) and (11) must be evaluated at different times. More precisely, the initial datum $v(0, \cdot)$ corresponds to $u(1, \cdot)$. However, as we are interested in the long time behavior of solutions, this drawback is merely of aesthetic nature.

The remainder of this paper is exclusively devoted to the analysis of the confined equation. Hence, to simplify the notation we drop from here on the hats from the time and space variables.

3 Weak solutions and their properties

In this section, we recall the weak solutions to (1) for $m \in [1, 3/2]$ constructed by Matthes, McCann and Savare using Generalized Minimizing Movements [35]. The GMM framework was proposed in a very general setting by DeGiorgi [17], and specialized to the 2-Wasserstein setting by Jordan, Kinderlehrer and Otto [32] in the context of heat flow dynamics. Otto exposed its gradient flow interpretation in the context of the porous medium equation [40], and its pertinence to the thin film equation first alone [39] and then with Giacomelli [26, 27].

To define our minimizing movement scheme we require an energy functional defined on a metric space. We take our energy functional to be proportional to generalized Fisher information (20), plus an extra term to account for the rescaling (11):

$$I_{\gamma^2/\alpha}(v) = \frac{\gamma^2/\alpha}{2m-1} \int |\nabla \left(v^{m-1/2}\right)|^2 \, dx + \frac{1}{2} \int |x|^2 v \, dx.$$

It is an extended real-valued functional on the set of all nonnegative functions on $\mathbf{R}^N$ with finite second moments and fixed total mass M:

$$\mathcal{M} := \left\{ v : \mathbf{R}^N \to [0, \infty) : \int v \, dx = M, \qquad \int |x|^2 v dx < \infty \right\}.$$

The distance between two such functions w_0 and w_1 in $\mathcal{M}$ is the so called 2-Wasserstein distance:

$$d_2(w_1, w_0)^2 := \inf_{\pi \in \Gamma(w_1, w_0)} \iint |x - y|^2 \, d\pi(x, y), \tag{13}$$

where the set $\Gamma(w_1, w_0)$ consists of all Borel joint measures $\pi \geq 0$ on the product space $\mathbf{R}^N \times \mathbf{R}^N$ with marginals w_1 and w_0.

Given a sequence $\tau = \{t_\tau^n\}_n$ increasing monotonically from $t_\tau^0 = 0$ to $\lim_{n\to\infty} t_\tau^n = +\infty$, we define a *discrete solution* $v_\tau : [0, \infty) \longrightarrow \mathcal{M}$ from initial condition $v^0 \in \mathcal{M}$ by choosing

$$v^n \in \operatorname*{arg\,min}_{v \in \mathcal{M}} 2I_{\gamma^2/\alpha}(v) + \frac{d_2^2(v^{n-1}, v)}{t_\tau^n - t_\tau^{n-1}} \tag{14}$$

recursively, and setting $v_\tau(t) = v^n$ for $t \in (t_\tau^{n-1}, t_\tau^n]$. A *Generalized Minimizing Movement* (GMM) refers to any continuous curve $v : [0,\infty) \longrightarrow \mathcal{M}$ which arises as a pointwise limit $v(t) = \lim_{k\to\infty} v_{\tau_k}(t)$ of a sequence of discrete solutions v_{τ_k} whose time steps shrink to zero locally uniformly

$$0 = \lim_{k\to\infty} \sup_{\{1\le i\le n | t^n_{\tau_k} \le T\}} t^i_{\tau_k} - t^{i-1}_{\tau_k}$$

for each $T < \infty$, and whose initial conditions converge to $v(0)$ not only in the metric d_2, but also in entropy and information senses:

$$E(v(0)) = \lim_{k\to\infty} E(v_{\tau_k}(0)) \qquad \text{and} \qquad I_{\gamma^2/\alpha}(v(0)) = \lim_{k\to\infty} I_{\gamma^2/\alpha}(v_{\tau_k}(0)).$$

Here

$$E(v) = \int e(v)\, dx + \frac{1}{2}\int |x|^2 v\, dx, \tag{15}$$

is the generalized entropy defined using (8).

The main result of [35] is the existence of GMM solutions to (11) in the range $m \in [1, 3/2]$, with arbitrary initial condition $v(0) \in \mathcal{M}$ having finite logarithmic entropy. It is also shown in [35] that these GMM solutions v arise as weak limits in $L^2_{loc}((0,T), H^2(\mathbf{R}^N))$ of discrete solutions $v_{\tau_k}^{m-1/2} \to v^{m-1/2}$, and satisfy

$$0 = \int_0^\infty \int_{\mathbf{R}^N} [(-\partial_t\varphi + x\cdot\nabla\varphi)v + \frac{\gamma^2}{\alpha}(v^{m-1/2}\Delta\varphi + \nabla\frac{v^{m-1/2}}{m-1/2}\cdot\nabla\varphi)\Delta(v^{m-1/2})]dxdt \tag{16}$$

for all compactly supported smooth test functions $\varphi \in C_0^\infty((0,\infty)\times\mathbf{R}^N)$. Here $H^2(\mathbf{R}^N) = W^{2,2}(\mathbf{R}^N)$ denotes the usual Sobolev space of L^2 functions whose first two distributional derivatives lie in $L^2(\mathbf{R}^N)$. This is a weak form of (1).

Since $v(t) \in \mathcal{M}$, it is evident that GMM solutions preserve non-negativity and mass (4). The no-flux boundary condition (2) is implicit in (16). And just as the derivative of any differentiable nonnegative function must vanish wherever the function does, (3) follows from non-negativity and $v^{m-1/2}(t) \in H^2(\mathbf{R}^N)$ (which holds for almost all $t > 0$). Finally let us show why weak solutions (16) which evolve continuously in $(\mathcal{M}, d_2)$ — as GMM solutions do — preserve zero center of mass.

Lemma 1 (Center of mass evolution)**.** *If $v^{m-1/2} \in L^2_{loc}((0,t), H^2(\mathbf{R}^N))$ satisfies* (16) *for all $\varphi \in C_0^\infty((0,\infty)\times\mathbf{R}^N)$ and $v \in C([0,\infty),\mathcal{M})$, then*

$$\int_{\mathbf{R}^N} xv(t,x)dx = e^{-t}\int xv(0,x)dx. \tag{17}$$

In particular, the center of mass of $v(t)$ vanishes for all $t > 0$ if it vanishes when $t = 0$.

Proof. Recall that convergence in d_2 is equivalent to weak convergence of measures plus convergence of all moments up to order two [51, Theorem 7.12]. This shows the center of mass (17) of $v \in C([0,\infty),\mathcal{M})$ is finite, well-defined, and depends continuously on $t \ge 0$. Formally, letting e_i denote a unit vector and setting $\varphi(x) = \psi(t)x\cdot e_i$ in (16) yields

$$\begin{aligned}
\int_0^\infty [\psi'(t) - \psi(t)]\int_{\mathbf{R}^N} x\cdot e_i v dxdt &= \frac{\gamma^2}{\alpha}\int_0^\infty \psi(t)\int_{\mathbf{R}^N}(\partial_i\frac{v^{m-1/2}}{m-1/2})\Delta(v^{m-1/2})dxdt \\
&= B_1 - \frac{\gamma^2}{\alpha(m-1/2)}e_i\cdot\int_0^\infty\psi(t)\int_{\mathbf{R}^N}\nabla\frac{|\nabla(v^{m-1/2})|^2}{2}dxdt \\
&= B_2 \\
&= 0
\end{aligned}$$

where the boundary integrals B_1 and B_2 vanish due to the no-flux condition (2) and Neumann condition (3) respectively. Conclusion (17) then follows from the fact that the center of mass varies continuously in time, and satisfies a weak form of the exponential decay equation since $\psi \in C_0^\infty((0,\infty))$ is arbitrary.

Although $\varphi(x) = \psi(t)x\cdot e_i$ is not compactly supported in space, the foregoing argument can be made rigorous by choosing $\varphi(x) = \psi(t)\eta_R(x)x\cdot e_i$ where $\eta_R(x)$ is a cut-off function supported on a ball of radius R such that $|\nabla\eta_R| \le c_1 R^{-1}$ and $|D^2\eta_R| \le c_2 R^{-2}$, and using $v^{m-1/2} \in H^2(\mathbf{R}^N)\cap\mathcal{M}$ to control the errors that this cut-off introduces to the preceding argument (in place of the boundary terms) in the usual way. ☐

4 Gradient flow interpretation

It is well-known that Generalized Minimizing Movements can be understood formally as producing a gradient flow of the energy $I_{\gamma^2/\alpha}(v)$ with respect to the distance d_2. Indeed, if $\mathcal{M}$ were a Riemannian manifold and $I_{\gamma^2/\alpha}$ a smooth function on $\mathcal{M}$ this would be rigorously true. In our more general setting, such an interpretation requires much more care to substantiate analytically [2]. On the other hand it provides powerful geometric intuition, leading to analytical statements which in many cases have proven rigorously verifiable a posteriori by other means. Two examples of such statements include the explicit rate of d_2 contraction by the porous medium flow [40, 42, 16, 2], and the higher-order asymptotics of the fast diffusion equation from [20] and [19].

Our purpose here is to use this intuition to extract the analogous predictions concerning the fourth-order evolution (1). Before doing so, let us briefly recall the salient ingredients of the gradient flow formalism developed by Otto [40].

The Benamou–Brenier formula [7]

$$d_2(v_0, v_1)^2 = \inf\left\{\int_0^1 \int_{\mathbf{R}^N} v(s,x)|\nabla\psi(s,x)^2|dxds : \partial_s v + \nabla\cdot(v\nabla\psi) = 0\right\} \tag{18}$$

indicates the 2-Wasserstein distance can be understood as a Riemannian distance induced by the weighted H^{-1} metric

$$g_v(\delta v, \delta v) = \int_{\mathbf{R}^N} v|\nabla\psi|^2 dx = \langle\psi,\psi\rangle_{H^1_v} \tag{19}$$

on the tangent space $T_v\mathcal{M}$. Here ψ is related to $\delta v = \partial_t v$ by the linear elliptic boundary value problem

$$\begin{array}{rcll} -\nabla\cdot(v\nabla\psi) & = & \delta v & \text{in } \operatorname{spt}(v), \\ \nu\cdot v\nabla\psi & = & 0 & \text{on } \partial\operatorname{spt}(v), \end{array} \tag{20}$$

whose solution we denote by $\psi = \mathcal{L}_v(\delta v)$. When $\operatorname{spt}(v) \neq \mathbf{R}^N$, the boundary conditions on ψ also have to be understood in a limiting sense, cf. (29) below. Since tangent fields with $\operatorname{spt}(\delta v) \not\subset \operatorname{spt}(v)$ are not physically relevant and do not arise in (18), we simply set $g_v(\delta v, \delta v) = \infty$ for such fields. Notice that we can define the metric tensor also variationally (and simultaneously for all m) by

$$\frac{1}{2}g_v(\delta v, \delta v) = \sup_\varphi\left\{-\frac{1}{2}\int v|\nabla\varphi|^2\,dx + \int \varphi\delta v\,dx\right\},$$

where the supremum is taken over all smooth functions φ on $\mathbf{R}^N$. Indeed, the right hand side is finite only if $\operatorname{spt}(\delta v) \subset \operatorname{spt}(v)$ and in this case, the maximizer ψ satisfies (20) with or without the boundary condition, depending on whether v is strictly positive (as for $m = 1$) or not (as for $m > 1$).

A short, purely formal, computation now allows us to identify the gradient of a functional $F : \mathcal{M} \longrightarrow \mathbf{R}\cup\{+\infty\}$ in this representation as follows. Along any (sufficiently smooth) curve $v(s) \in \mathcal{M}$,

$$\frac{d}{ds}F(v(s)) = \int_{\mathbf{R}^N} \frac{\delta F}{\delta v}\frac{\partial v}{\partial s}dx = \int_{\mathbf{R}^N} v\nabla\frac{\delta F}{\delta v}\cdot\nabla\psi dx = \langle\frac{\delta F}{\delta v},\psi\rangle_{H^1_v}$$

in view of (20). On the other hand, abstractly

$$\frac{d}{ds}F(v(s)) = g_v(\operatorname{grad} F(v), \dot{v}(s)).$$

Comparing the abstract and concrete expressions above shows the abstract tangent vector $\operatorname{grad} F$ is represented by a variational derivative $\delta F/\delta v$ under the identification of $T_v\mathcal{M}$ with the Sobolev space $H^1_v(\mathbf{R}^N)$ (just as the tangent vector $\dot{v}(0)$ is represented by $\psi(x)$ and not $\partial_s v(0,x)$ under the same identification). For the curve $v(s)$ to be a gradient flow $\dot{v}(s) = -\operatorname{grad} F(v(s))$, we therefore want $\psi = -\frac{\delta F}{\delta v}$, or equivalently

$$\begin{array}{rcll} \dfrac{\partial v}{\partial s} & = & \nabla\cdot(v\nabla\dfrac{\delta F}{\delta v}) & \text{in} \quad \operatorname{spt} v, \\ 0 & = & v\nabla\dfrac{\delta F}{\delta v}\cdot\nu & \text{on} \quad \partial\operatorname{spt} v \end{array}$$

in view of (20). Choosing $F = I_{\gamma^2/\alpha}$ we recover the rescaled flow (11) subject to the no flux condition (2).

Alternately, choosing $F = E$, the generalized entropy (15) yields the confined porous medium equation

$$\partial_t v - \nabla \cdot (xv) - \Delta (v^m) \; = \; 0, \tag{21}$$

subject to the no flux condition

$$\nu \cdot \nabla v^m = 0 \qquad \text{on } \partial \operatorname{spt} v, \tag{22}$$

cf. Otto [40].

Finally, we are in a position to state the entropy-information relation

$$(N(m-1)+1) I_{\gamma^2/\alpha}(v) - \frac{1}{2} |\operatorname{grad} E(v)|^2_{T_v\mathcal{M}} = \begin{cases} NM & \text{if } m = 1, \\ N(m-1)E(v) & \text{if } m > 1, \end{cases} \tag{23}$$

which is at the heart of the subsequent analysis. It is possible to verify this from the foregoing development, which implies

$$|\operatorname{grad} E(v)|^2_{T_v\mathcal{M}} \; = \; \left\| \frac{\delta E}{\delta v} \right\|^2_{H^1_v} \; = \; \int v |\nabla \left(e'(v) + \frac{1}{2}|x|^2 \right) |^2 \, dx. \tag{24}$$

However, the entropy-information relation is by now well-known: see for example McCann, Matthes, Savaré [35, Corollary 2.3], where it is stated with a different normalization, and builds upon an observation exploited by Carrillo & Toscani in [14].

In fact, (23) establishes a link between the generalized entropy and the generalized Fisher information, that is, between the dissipating functionals in the Wasserstein gradient flow formulation of the confined porous medium equation and the confined fourth-order equation, respectively. As a consequence, it turns out that linearizing (11) around the global attractor v_* yields an explicit formula for the fourth-order displacement Hessian in terms of the porous medium displacement Hessian computed and studied by the second author in [45]. Moreover, eigenvalues and eigenfunctions of the porous medium displacement Hessian will immediately translate into those of the fourth-order displacement Hessian.

5 Linearization

Near a critical point $v_* \in \mathcal{M}$ of a (C^2-smooth) functional $F(v)$, Taylor expansion shows the gradient flow $\dot{v} = -\operatorname{grad} F(v(t))$ to be governed by the Hessian of F:

$$(v(t) - v_*)' = -\operatorname{Hess} F(v_*)(v(t) - v_*) + o(|v(t) - v_*|). \tag{25}$$

Here $\operatorname{Hess} F(v_*)$ is a linear operator on the Hilbert tangent space $T_{v_*}\mathcal{M}$, and should formally be computed using the Levi–Civita connection corresponding to the Riemannian metric structure inducing d_2, and is therefore called the *displacement Hessian*; its spectrum determines the behaviour of the flow near v_*. Setting aside the poorly formulated question of C^2 smoothness, our present goal is to identify the displacement Hessian of $I_{\gamma^2/\alpha}$ on $(\mathcal{M}, d_2)$ at the Barenblatt profile (12), and its spectral properties.

As is well-known, v_* minimizes both $I_{\gamma^2/\alpha}(v)$ and $E(v)$ on $\mathcal{M}$ [40]; (the former follows from the latter via (23)). Therefore, it is natural to expect both $\operatorname{Hess} I_{\gamma^2/\alpha}(v_*)$ and $\operatorname{Hess} E(v_*)$ to be self-adjoint nonnegative definite operators on $T_{v_*}\mathcal{M}$. Furthermore, the relationship between these two operators can be computed abstractly, starting from the energy-entropy formula (23). Differentiating this identity twice yields

$$(1 + N(m-1)) \operatorname{Hess} I_{\gamma^2/\alpha}(v) = D^3 E(v) \operatorname{grad} E(v) + (\operatorname{Hess} E)^2(v) + N(m-1) \operatorname{Hess} E(v).$$

At the critical point v_* however, a small miracle occurs: $\operatorname{grad} E(v_*) = 0$ so we obtain

$$(1 + N(m-1)) \operatorname{Hess} I_{\gamma^2/\alpha}(v_*) = (\operatorname{Hess} E)^2(v_*) + N(m-1) \operatorname{Hess} E(v_*) \tag{26}$$

without needing to know the third derivative of E. This simplification is crucial. It shows the complete spectral information concerning Hess $I_{\gamma^2/\alpha}(v_*) = f(\text{Hess}\, E(v_*))$ is contained in that of Hess $E(v_*)$. Moreover, since $m \geq 1$, the quadratic function

$$f(\lambda) = \frac{\lambda^2 + N(m-1)\lambda}{1 + N(m-1)} \tag{27}$$

relating them is strictly increasing on the nonnegative real axis, which contains the spectrum of Hess $E(v_*)$. Thus the ordering and multiplicity of all eigenvalues of Hess $I_{\gamma^2/\alpha}$ are the same as those of Hess $E(v_*)$. This would not necessarily be the case for $m < 1$.

Fortunately, a rigorous spectral analysis for the operator Hess $E(v_*)$ in the range $m > 1$ was recently completed by one of us [45]. The range $m < 1$ had been previously analyzed by the other author, together with Denzler [20]. We devote an appendix to the case $m = 1$; although this operator is familiar in this case, the setting which our analysis requires is somewhat different from any of the standard ones.

To make predictions concering the evolutions (1) and (11), we shall need to recall the conclusions of Seis's analysis. This we do in the next section. A point which distinguishes the case of present interest is that the operator Hess $E(v_*)$ possesses a complete basis of eigenfunctions in the range $m \geq 1$. This simplifies the remaining analysis considerably; it would not be the case for $m < 1$. As a result, we can already anticipate that there will be a complete basis for $T_{v_*}\mathcal{M}$ consisting of simultaneous eigenfunctions of Hess $I_{\gamma^2/\alpha}(v_*)$ and Hess $E(v_*)$. Moreover, the eigenvalues $\mu_{\ell k} = f(\lambda_{\ell k})$ of the former are related to those of the latter $\lambda_{\ell k}$ by the quadratic increasing function (27) on $[0, \infty)$.

6 Rigorous spectral results

To make further progress, we shall need concrete representations $\mathcal{H}_I$ and $\mathcal{H}_E$ of the Hessian operators Hess $I_{\gamma^2/\alpha}(v_*)$ and Hess $E(v_*)$ of interest; these should act on the concrete representation $H^1_{v_*}$ of the tangent space $T_{v_*}\mathcal{M}$ to $\mathcal{M}$ at the Barenblatt profile v_*. Here the Sobolev space $H^1_{v_*}$ denotes the class of all locally integrable functions on $\text{spt}(v_*)$ such that

$$\|\psi\|^2_{H^1_{v_*}} := \int v_* |\nabla \psi|^2 \, dx \; < \; \infty,$$

with the identification of functions that only differ by an additive constant. We recall that in the case $m > 1$ the Barenblatt profile v_* is compactly supported on a ball of radius one, while for $m = 1$ the Barenblatt profile is a Gaussian and thus positive everywhere, see (12). In any of these cases, the weighted Sobolev space $H^1_{v_*}$ is a separable Hilbert space with respect to the topology induced by $\|\cdot\|_{H^1_{v_*}}$.

For functions $\psi \in C_b^\infty := C^\infty \cap L^\infty$ which are smooth and bounded on the support of the Barenblatt profile v_*, the desired concrete representation of Hess $E(v_*)$ derived in [45] (c.f. [20]) is

$$\mathcal{H}_E \psi(x) \;=\; -m v_*^{m-2} \nabla \cdot (v_* \nabla \psi) \;=\; \begin{cases} -\Delta\psi(x) + x \cdot \nabla\psi(x) & \text{if } m = 1, \\ -\frac{m-1}{2}(1 - |x|^2)\Delta\psi(x) + x \cdot \nabla\psi(x) & \text{if } m > 1; \end{cases} \tag{28}$$

(the case $m = 1$ is discussed in the appendix below, where the operator $\mathcal{H}_E$ appears identical in form to the well-known Ornstein–Uhlenbeck operator). Notice that $C_b^\infty(\text{spt}(v_*))$ equals $C_b^\infty(\mathbf{R}^N)$ if $m = 1$ and $C^\infty(\bar{B}_1)$ if $m > 1$. In either case, $C_b^\infty(\text{spt}(v_*))$ is a dense subspace of $H^1_{v_*}$, see, e.g., Lemma 14 in the appendix or [45, Lemma 2], and the operator $\mathcal{H}_E$ is nonnegative and symmetric, and thus closable in $H^1_{v_*}$.

Its nonnegative symmetric closure, still denoted by $\mathcal{H}_E$, is self-adjoint with domain

$$\mathcal{D}(\mathcal{H}_E) \;=\; \left\{ \psi \in H^3_{\text{loc}}(\text{spt}(v_*)) : \; \psi, \, \mathcal{H}_E\psi \in H^1_{v_*} \right\},$$

as established in [45] (for $m > 1$) and Proposition 12 below (for $m = 1$). (Self-adjointness is well-known for the Ornstein-Uhlenbeck operator on the Gaussian-Lebesgue space $L^2(e^{-|x|^2/2}dx) = L^2(v_* dx)$; what is established below is self-adjointness on the Gaussian-Sobolev space $H^1_{v_*}$.) Notice that for every $\psi \in \mathcal{D}(\mathcal{H}_E)$ we have the integration by parts formula

$$\int v_* \nabla \xi \cdot \nabla \psi \, dx \;=\; -\int \xi \nabla \cdot (v_* \nabla \psi)\; dx \quad \text{for all } \xi \in H^1_{v_*}, \tag{29}$$

which can be easily seen via approximation with $C_b^\infty(\operatorname{spt} v_*)$ functions. In the case $m > 1$, this implies the asymptotic no-flux condition $\nu \cdot v_* \nabla \psi = 0$ on ∂B_1 which in the case $\psi \in C^1(B_1)$ simply becomes

$$\lim_{|x|\uparrow 1} v_*(x) \nabla \psi(x) \cdot \frac{x}{|x|} = 0,$$

cf. [45, Remark 1].

Letting $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers, we recall the following theorem:

Theorem 2 (Porous medium spectrum from [45]; heat spectrum from Appendix below)**.** *The operator $\mathcal{H}_E : \mathcal{D}(\mathcal{H}_E) \to H^1_{v_*}$ is self-adjoint. Its spectrum is purely discrete and given by the eigenvalues*

$$\lambda_{\ell k} \;=\; \ell + 2k + 2k(k + \ell + \frac{N}{2} - 1)(m-1),$$

for $(\ell, k) \in \mathbf{N}_0 \times \mathbf{N}_0 \setminus \{(0,0)\}$ if $N \geq 2$ and $(\ell, k) \in \{0,1\} \times \mathbf{N}_0 \setminus \{(0,0)\}$ if $N = 1$. The corresponding eigenfunctions form an orthonormal basis for $H^1_{v_*}$. In the case $m = 1$, they are given by the Hermite polynomials, while if $m > 1$, the eigenfunctions are given by polynomials

$$\psi_{\ell n k}(x) = F(-k, \frac{1}{m-1} + \ell + \frac{N}{2} - 1 + k; \ell + \frac{N}{2}; |x|^2) Y_{\ell n}\left(\frac{x}{|x|}\right) |x|^\ell,$$

of degree $\ell + 2k$, where $n \in \{1, \ldots, N_\ell\}$ with $N_\ell = 1$ if $\ell = 0$ or $\ell = N = 1$ and $N_\ell = \frac{(N+\ell-3)!(N+2\ell-2)}{\ell!(N-2)!}$ else, where $F(a, b; c; z)$ is a hypergeometric function and $Y_{\ell n}$ is a spherical harmonic if $N \geq 2$, corresponding to the eigenvalue $\ell(\ell + N - 2)$ of $-\Delta_{\mathbb{S}^{N-1}}$ with multiplicity N_ℓ. Otherwise, if $N = 1$ it is $Y_{\ell n}(\pm 1) = (\pm 1)^\ell$.

Hermite polynomials can be computed recursively from

$$\psi_0(z) = 1, \quad \psi_{n+1}(z) = z\psi_n(z) - \psi_n'(z) \quad \text{for } n \in \mathbf{N},$$

where $z \in \mathbf{R}$, and then in higher dimensions for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}_0^N$ via $\psi_\alpha(x) = \psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_N}(x_N)$, cf., e.g., [48, 46]. Hypergeometric functions $F(a, b; c, z)$ are power series of the form

$$F(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!} z^j, \tag{30}$$

where $a, b, c, z \in \mathbf{R}$ and c is not a non-positive integer. The definition involves the Pochhammer symbols

$$(s)_j = s(s+1)\cdots(s+j-1), \quad \text{for } j \geq 1, \quad \text{and} \quad (s)_0 = 1.$$

Since the hypergeometric functions reduce to polynomials of degree k in the case $a = -k$, the eigenfunctions $\psi_{\ell n k}$ are polynomials of degree $\ell + 2k$ and are harmonic if $k = 0$. The literature on hypergeometric functions and spherical harmonics is vast, see, e.g., [1, 6, 29].

Armed with this understanding of the displacement Hessian Hess $E(v_*)$, the formulas (26)–(27) obtained in the preceding section identify the concrete representation of the displacement Hess $I_{\gamma^2/\alpha}(v_*)$ associated with the confined fourth-order equation (11) to be

$$\mathcal{H}_I := f(\mathcal{H}_E) = \frac{\mathcal{H}_E^2 + N(m-1)\mathcal{H}_E}{1 + N(m-1)}. \tag{31}$$

More explicitly, the eigenvector basis $\{\psi_{\ell nk}\}$ for $H^1_{v_*}$ yields

$$\begin{aligned} D(\mathcal{H}_E) &= \left\{ w = \sum_{\ell,n,k} c_{\ell k}\psi_{\ell nk} \in H^1_{v_*} : \sum_{\ell,n,k} \lambda^2_{\ell k}|c_{\ell nk}|^2 < \infty \right\}, \\ \mathcal{H}_E w &= \sum_{\ell,n,k} \lambda_{\ell k} c_{\ell nk}\psi_{\ell nk} \quad \text{if} \quad w = \sum_{\ell,n,k} c_{\ell nk}\psi_{\ell nk} \in D(\mathcal{H}_E). \end{aligned}$$

Noting that the eigenvalues are all bounded away from zero, we define $\mathcal{H}_I$ on

$$D(\mathcal{H}_I) := \left\{ w = \sum_{\ell,n,k} c_{\ell nk}\psi_{\ell nk} \in H^1_{v_*} : \sum_{\ell,n,k} (\lambda^2_{\ell k} + N(m-1)\lambda_{\ell k})^2 |c_{\ell nk}|^2 < \infty \right\},$$

by

$$\mathcal{H}_I w := \sum f(\lambda_{\ell k}) c_{\ell nk}\psi_{\ell nk} = \frac{1}{1+N(m-1)} \sum_{\ell,n,k} (\lambda^2_{\ell k} + N(m-1)\lambda_{\ell k}) c_{\ell nk}\psi_{\ell nk}.$$

With this definition, the following result is an immediate corollary of Theorem 2:

Corollary 3 (Spectral theory of our fourth-order flows)**.** *The operator $\mathcal{H}_I : \mathcal{D}(\mathcal{H}_I) \to H^1_{v_*}$ is self-adjoint and positive definite, and has purely discrete spectrum given by the eigenvalues*

$$\mu_{\ell k} := f(\lambda_{\ell k}) = \frac{\lambda^2_{\ell k} + N(m-1)\lambda_{\ell k}}{1+N(m-1)}.$$

The range of indices (ℓ, k), values $\lambda_{\ell k}$, and eigenvectors $\psi_{\ell nk}$ are as in Theorem 2, and $\mathcal{H}_I \psi_{\ell nk} = \mu_{\ell k}\psi_{\ell nk}$.

Remark 4 (Spectral gap and lowest modes)**.** *Note that the eigenvalues $\lambda_{\ell k}$ and hence $\mu_{\ell k}$ depend monotonically on ℓ, k, m and N. This makes it easy to identify the lowest lying modes to be $\lambda_{10} = 1$ and $\lambda_{20} = 2$. Indeed, the only modes below $\lambda_{40} = 4$ must correspond to polynomial eigenfunctions of degree $\ell + 2k < 4$, namely $\lambda_{30} = 3$ and perhaps $2 + N(m-1) = \lambda_{01} < \lambda_{11} = 3 + (N+2)(m-1)$, depending on the values of m and N. Thus the only modes below $\mu_{40} = \frac{4(4+N(m-1))}{1+N(m-1)}$ are $1 = \mu_{10} < \mu_{20} < \mu_{30}$ (corresponding to homogenous harmonic polynomials $\psi_{\ell nk}$ of degrees 1, 2 and 3 respectively) plus possibly $2(2+N(m-1)) = \mu_{01} < \mu_{11} = \frac{(3+(N+2)(m-1))(3+2(N+1)(m-1))}{1+N(m-1)}$.*

Although $\mathcal{H}_I$ is defined by its spectral decomposition, it is useful to have a representation of it as a partial differential operator. This is supplied by the following Proposition.

Proposition 5 (Hessian of the generalized information as a partial differential operator)**.** *The action of $(1+N(m-1))H_I$ on finite linear combinations of eigenfunctions coincides with that of $\mathcal{H}_E \circ \mathcal{H}_E + N(m-1)\mathcal{H}_E$, where $\mathcal{H}_E$ is the partial differential operator* (28)*. The domain of self-adjointmess of $\mathcal{H}_I$ coincides with*

$$\mathcal{D}(\mathcal{H}_I) \; = \; \left\{ \psi \in H^5_{\rm loc}(\mathrm{spt}(v_*)) : \; \psi, \, \mathcal{H}_E\psi, \, \mathcal{H}^2_E\psi \in H^1_{v_*} \right\}.$$

Proof. We compute:

$$\begin{aligned}
D(\mathcal{H}_E^2 + N(m-1)\mathcal{H}_E) &= \left\{ w = \sum_{\ell,n,k} c_{\ell nk}\psi_{\ell nk} \in H^1_{v_*} \,:\, \sum_{\ell,n,k} (\lambda_{\ell k}^2 + N(m-1)\lambda_{\ell k})^2 |c_{\ell nk}|^2 < \infty \right\} \\
&= \left\{ w = \sum_{\ell,n,k} c_{\ell nk}\psi_{\ell nk} \in H^1_{v_*} \,:\, \sum_{\ell,n,k} (\lambda_{\ell k}^2)^2 |c_{\ell nk}|^2 < \infty \right\} \quad (\text{since } \lambda_{\ell k} \geq 1) \\
&= \left\{ w = \sum_{\ell,n,k} c_{\ell nk}\psi_{\ell nk} \in D(\mathcal{H}_E) \,:\, \sum_{\ell,n,k} \lambda_{\ell k}^2 |\lambda_{\ell k} c_{\ell nk}|^2 < \infty \right\} \\
&= \left\{ w = \sum_{\ell,n,k} c_{\ell nk}\psi_{\ell nk} \in D(\mathcal{H}_E) \,:\, \mathcal{H}_E\Big(\mathcal{H}_E w\Big) \in H^1_{v_*} \right\} \\
&= \left\{ w = \sum_{\ell,n,k} c_{\ell nk}\psi_{\ell nk} \in H^1_{v_*} \,:\, \mathcal{H}_E w \in H^1_{v_*},\, \mathcal{H}_E\Big(\mathcal{H}_E w\Big) \in H^1_{v_*} \right\}.
\end{aligned}$$

In the interior of $\Omega = \operatorname{spt} v_*$, $\mathcal{H}_E$ is a second order elliptic operator, and $\mathcal{H}_E(\mathcal{H}_E w) \in H^1_{loc}(\Omega)$ then implies $w \in H^5_{loc}(\Omega)$ (by classical theory of elliptic regularity). □

7 Dynamical and analytic implications

The knowledge of the complete spectrum of the displacement Hessian operator is a promising starting point for a full asymptotic expansion of solutions to the fourth-order equation (1)–(3) around the self-similar solution. In view of the discreteness of the spectrum, all modes are in principle accessible. The eigenvalues of $\mathcal{H}_I$ are nonlinear functions of m and eigenvalue crossings occur throughout the spectrum. Moreover, the eigenvalues are increasing functions of the eigenvalues of the porous medium equation $\lambda_{\ell k}$, and thus, we expect the same ordering of eigenmodes with respect to the rate of convergence for both equations. For any value of N and m, the dynamics are translation-governed as the smallest eigenvalue $\mu_{10} = 1$ corresponds to a spatial translation in direction of the n-th coordinate axis e_n, for $n \in \{1, \ldots, N_1 = N\}$. The role of the eigenfunctions is best understood by considering geodesics in the Wasserstein space, which are given by displacement interpolants (cf. [36]) via $v_*(x) = \det\big(I + s\nabla^2\psi(x)\big)\, v_s(x + s\nabla\psi(x))$, that is, push-forwards of v_* under the map $\mathrm{id} + s\nabla\psi$. Observe that v_s generates tangent fields in the Wasserstein gradient flow interpretation of the dynamics, cf. Section 4, since $\partial_s|_{s=0}\, v_s = -\nabla\cdot(v_*\nabla\psi)$. The eigenfunctions ψ_{1n0} are affine functions for which the corresponding Lagrangian perturbations are translations $x + sc_n e_n$ with $c_n \in \mathbf{R}$. A fully rigorous justification of the translation-governed dynamics was obtained by Carrillo & Toscani [15] for $(N, m) = 1, 3/2)$ and by Matthes, Savaré, and the first author [35] for general N and $1 \leq m \leq 3/2$ who prove

$$d_2(v(t), v_*) \lesssim e^{-t}, \tag{32}$$

where d_2 denotes the 2-Wasserstein distance. This decay rate is sharp for translations. The second smallest eigenvalue μ_{20} corresponds to an affine shear in Lagrangian variables of the form $x + sA_{2n}x$ for some symmetric and trace-free matrix A_{2n}. Exact solutions of this form were studied in detail by Betelú and King for $m = 3/2$ [12] and Denzler and the first author [21] more generally.

A serious difficulty, however involves the potential lack of C^2 smoothness of $I_{\gamma^2/\alpha}(v)$ required to justify the approximation (25). What is really required is a function space metric or topology in which the dynamics near v_* depend differentiably on their initial conditions. However, at present this seems far out of reach: if such a topology existed, it would imply uniqueness and well-posedness for the fourth-order flow near v_*, or at least for GMM solutions thereto.

For the porous medium dynamics $m > 1$ on the other hand, such a framework was identified by Angenent [3] in one dimension, and in higher dimensions by Koch's habilitation thesis [34]. In the latter context however, there is a mismatch between this function space topology and the weighted Sobolev induced d_2 geometry for which the displacement Hessian has been diagonalized [45]. In the fast diffusion regime, the analogous mismatch was resolved by Denzler, Koch and McCann [19], which allowed these authors to obtain rigorous higher-order asymptotics of solutions to the fast diffusion equation around the self-similar solution based on the spectral analysis of Denzler and McCann. However, their expansion is naturally limited to a finite number of modes due to the occurrence of continuous spectrum. A complete picture of the $N = 1$ porous medium asympotics was obtained by Angenent however, whose diagonalization was set in the same little Hölder spaces where he established analyticity of the flow; it displays no continuous spectrum.

In spite of the above mentioned difficulty, let us proceed by advancing the conjectures suggested by our analysis. Due to the nonlinear corrections hiding in the error term of (25), or rather (34), near an attracting fixed point a smooth flow can potentially produce exponential decay not only at eigenvalue rates, but also at positive integer combinations of these. Moreover, when such an integer combination coincides with an eigenvalue, resonance can lead to the exponential decay being modulated by polynomial growth. This yields the following conjecture, which is a fourth-order analog of the results of Angenent [3], Denzler, Koch and McCann [19], and the conjecture appearing in [20, 45].

Recall from McCann's version [36, Remark 4.5] of Brenier's theorem [13], any measure $v(t, x) \geq 0$ with the same mass as v_* satisfies

$$v_*(x) = \det\left(I + D^2\psi(t, x)\right) v(t, x + \nabla\psi(t, x)) \tag{33}$$

v_*-a.e., where $\frac{1}{2}|x|^2 + \psi(t, x)$ is convex in x and uniquely determined up to an additive constant. Moreover, from [13] it is clear that (13) is achieved by

$$d_2^2(v_*, v(t)) = \int_{\mathbf{R}^N} v_*(x)|\nabla\psi(t, x)|^2 dx,$$

which also shows for $v(t) \in \mathcal{M}$ that $\psi(t, \cdot) \in H^1_{v_*}$. Alternately, from the point of view of differential geometry, identity (33) can be read as $v(t) = \exp_{v_*} \psi(t)$, where $\exp_{v_*} : T_{v_*}\mathcal{M} \to \mathcal{M}$ is the Riemannian exponential map, which introduces local coordinates on $\mathcal{M}$ given by Wasserstein geodesics through v_*.

On an abstract level, the linearization of the dynamical sysmtem $\frac{d}{dt}v = V(v)$ on the manifold $\mathcal{M}$ around the (attracting) fixed point $V(v_*) = 0$,

$$\frac{d}{dt}(v(t) - v_*) = DV(v_*)(v(t) - v_*) + O\left(\|v(t) - v_*\|^2\right), \tag{34}$$

expressed in new coordinates $\psi = G(v)$ takes the form

$$\begin{aligned}
\frac{d}{dt}(\psi(t) - \psi_*) &= DG(v(t))\frac{d}{dt}(v(t) - v_*) \\
&= DG(v_*)\frac{d}{dt}(v(t) - v_*) + O\left(\|v(t) - v_*\|^2\right) \\
&= DG(v_*)DV(v_*)(v(t) - v_*) + O\left(\|v(t) - v_*\|^2\right) \\
&= DG(v_*)DV(v_*)DG(v_*)^{-1}(\psi(t) - \psi_*) + O\left(\|\psi(t) - \psi_*\|^2\right).
\end{aligned}$$

That is, the generator $DV(v_*)$ of the linearized dynamics in one coordinate system is related to the generator $DG(v_*)DV(v_*)DG(v_*)^{-1}$ of the linearized dynamics in any other coordinate system by a similarity transformation given by a linear operator $DG(v_*)$ (which takes any tangent vector expressed in the first set of coordinates to its representation with respect to the second coordinates).

For the particular choice $\psi(t) = G(v(t))$, where $\psi(t)$ is given by (33), i.e., $G = \exp_{v_*}^{-1}$, it holds that $DG(v_*) = \mathcal{L}_{v_*}$, with $\mathcal{L}_{v_*}$ being the operator from (20). This identity is a consequence of $s\psi(t) = G(v(s, t))$ for geodesics $s \mapsto v(s, t)$. A short computation shows that $DG(v_*)DV(v_*)DG(v_*)^{-1} = \mathcal{H}_I$ — which motivates the term "displacement Hessian" and gives rise to our first conjecture:

Conjecture 6 (Complete asymptotics in Riemannian normal coordinates)**.** *Given a GMM solution $v(t,x)$ to* (11) *with the same mass as v_*, we conjecture an asymptotic expansion*

$$\psi(t,x) \sim \sum_{\vec{\beta}} e^{-\vec{\beta}\cdot\vec{\mu}t} \sum_{i=1}^{I(\beta)} c_i^{\vec{\beta}} \Psi_i^{\vec{\beta}}(x) t^i, \tag{35}$$

for the potential $\psi(t,x)$ of (33)*, where $\vec{\mu} = \{\mu_{\ell nk}\}$ represents the sequence of eigenvalues and the first summation takes place over multi-indices $\vec{\beta} = \{\beta_{\ell nk}\}$ with $\beta_{\ell nk} \in \mathbf{N}_0$. Here*

$$\vec{\beta}\cdot\vec{\mu} = \sum_{\ell,k\in\mathbf{N}_0} \sum_{n=1}^{N_\ell} \beta_{\ell nk}\mu_{\ell nk},$$

$\Psi_i^{\vec{\beta}}(x)$ is a polynomial determined by i, β, m and N which vanishes for all but $I(\beta) < \infty$ many i, and the constants $c_i^{\vec{\beta}}$ are determined by the initial condition $v(0,x)$. If β has length $1 = |\vec{\beta}| = \sum \beta_{\ell nk}$, hence represents an eigenvalue, then $\Psi_0^{\vec{\beta}}(x)$ is the corresponding eigenvector. By asymptotic expansion, we mean for each $\Lambda < \infty$ there exists a metric d_Λ for functions on $\mathbf{R}^N$ such that

$$d_\Lambda(\psi(t,\cdot), \psi_\Lambda(t,\cdot)) \le o(e^{-\Lambda t}) \tag{36}$$

where ψ_Λ represents the restriction of the sum (35) *to values of $\vec{\beta}$ for which $\vec{\beta}\cdot\vec{\mu} \le \Lambda$.*

Remark 7. *Equality in* (35) *seems likely to violate convexity of $\frac{1}{2}|x|^2 + \psi(t,x)$, suggesting the best we can hope for is an asymptotic statement such as* (36)*. For the porous medium dynamics in one-dimension [3], and fast diffusion dynamics in all dimensions [19], asymptotic expansions were obtained by* Angenent*, and* Denzler*,* Koch *and* McCann*, respectively. The present challenge is to identify a metric d_Λ which is strong enough for the conjecture to have useful consequences yet weak enough for the conjecture to be true.*

The fact that $\mathcal{H}_E\Psi_0^{\vec{\beta}} = \vec{\beta}\cdot\vec{\lambda}\Psi_0^{\vec{\beta}}$ is an eigenvector when $|\vec{\beta}| = 1$ yields more concrete predictions if, as in [19], we are willing to limit the accuracy of our approximation to detect only the lowest lying (= most persistent) modes. These modes are enumerated in Remark 4. As long as we stay within factor two of the lowest mode to be excited, resonances will not appear. Since no other modes lie within a factor of two of the ground state $\mu_{01} = 1$, we arrive at the corollary:

Corollary 8 (First correction to leading-order rate)**.** *For any GMM solution $v(t,x)$ to* (11) *with the same mass as $v_*(x)$ and $\epsilon > 0$,*

$$v(t,x) - *(x) = \frac{1}{m} v_*^{2-m}(x)[b\cdot xe^{-t} + o(e^{-(2-\epsilon)t})] \tag{37}$$

holds for some $b \in \mathbf{R}^N$ determined by the initial condition $v(0,\cdot)$ and all $x \in \mathrm{int}(\mathrm{spt}(v_*))$. Moreover, if $m > 1$, $\mathrm{spt}(v(t,\cdot)) \subset B_{r(t)}(0)$ where

$$r(t) \le 1 + |b|e^{-t} + o(e^{-(2-\epsilon)t}). \tag{38}$$

Proof. Since $\lambda_{\ell k} \ge \ell + 2k \ge 2$ for all $\ell, k \in \mathbf{N}_0$ except $\ell \le 1$ and $k = 0$, we see $\mu_{\ell k} \ge \lambda_{\ell k} > 2$ unless $(\ell,k) = (1,0)$. Thus there are no modes within factor 2 of $\mu_{10} = 1$. The corresponding eigenfunctions $\psi_{1n0}(x)$ depend linearly on x, so we find $\nabla\psi(t,x) = be^{-t} + o(e^{-(2-\epsilon)t})$ for some $b \in \mathbf{R}^N$. The map

$$y = x + \nabla\psi(t,x)$$

can then be inverted to yield

$$x = y - be^{-t} + o(e^{-(2-\epsilon)t}).$$

Noting (12) implies

$$mv_*^{m-2}\nabla v_*(y) = \begin{cases} -y & \text{if} \quad y \in \mathrm{int}(\mathrm{spt}\, v_*) \\ 0 & \text{if} \quad y \notin \mathrm{spt}\, v_* \end{cases} \tag{39}$$

from (33) we deduce

$$\begin{aligned}
v(t,y) &= v_*(y - be^{-t} + o(e^{-(2-\epsilon)t}))\det[I + o(e^{-(2-\epsilon)t})] \\
&= v_*(y) - e^{-t} b \cdot \nabla v_*(y) + o(e^{-(2-\epsilon)t}) \\
&= v_*(y) + \frac{1}{m} e^{-t} b \cdot y v_*^{2-m} + o(e^{-(2-\epsilon)t})
\end{aligned}$$

for $x \in \mathrm{int}(\mathrm{spt}(v_*))$, i.e., (37) holds. Also, in the case $m > 1$, we infer from (33) that $\mathrm{spt}(v(t,\cdot)) \subset B_{r(t)}(0)$ where $r(t) \le 1 + |b|e^{-t} + o(e^{-(2-\epsilon)t})$. $\square$

Finally, since (1) is translation invariant, it costs no generality to suppose the initial data $v(0,x)$ has center of mass at the origin. Lemma 1 ensures this condition is preserved by weak solutions (16) which vary continuously in $(\mathcal{M}, d_2)$, and by Generalized Minimizing Movements in particular. This rules out the excitation of the μ_{10} mode and forces $b = 0$ above. In this case, the lowest potentially excitable mode $\mu_{20} = \frac{2(2+N(m-1))}{1+N(m-1)}$ corresponds to shearing, and the only other modes which have the potential to lie within factor two of this are μ_{30} and μ_{01} (since $\mu_{11} > 2\mu_{20}$). This leads to a second corollary of our conjecture:

Corollary 9 (Improvements by centering and exchange of stability)**.** *If the center of mass of $v(0,\cdot)$ vanishes, then for any $\Lambda < 2\mu_{20}$,*

$$v(t,x) = v_*(x)(1 - cNe^{-\mu_{01}t}) + \frac{1}{m} v_*^{2-m}(x)[x \cdot Axe^{-\mu_{20}t} + 3h_3(x)e^{-\mu_{30}t} + c|x|^2 e^{-\mu_{01}t} + o(e^{-\Lambda t})] \tag{40}$$

for $x \in \mathrm{int}(\mathrm{spt}(v_*))$, where the trace-free matrix A, constant $c \in \mathbf{R}$, and homogeneous harmonic polynomial $h_3(x)$ of degree three are determined by the initial data. can be improved to

$$r(t) \le 1 + \|A\|e^{-\mu_{20}t} + ce^{-\mu_{01}t} + \sup_{|x|<1} |\nabla h_3(x)|e^{-\mu_{30}t} + o(e^{-\Lambda t}) \tag{41}$$

Proof. The strategy of proof is the same as for the preceding proposition. The eigenfunctions corresponding to μ_{20} and μ_{30} are homogeneous harmonic polynomials $h_2(x) = x \cdot Ax/2$ and $h_3(x)$ of degree 2 and 3 respectively, while the eigenfunction corresponding to μ_{01} is $\psi_{011}(x) = 1 - (N(m-1)\alpha)^{-1}|x|^2$. Here $A = A^*$ is symmetric without loss of generality, and trace-free. Since both μ_{30} and μ_{01} exceed $\mu_{20} > \Lambda/2$, inverting the map

$$\begin{aligned}
y &= x + \nabla\psi(t,x) \\
&= x + Axe^{-\mu_{20}t} + cxe^{-\mu_{01}t} + \nabla h_3(x)e^{-\mu_{30}t} + o(e^{-\Lambda t})
\end{aligned}$$

yields

$$x = y - Aye^{-\mu_{20}t} - cye^{-\mu_{01}t} - \nabla h_3(y)e^{-\mu_{30}t} + o(e^{-\Lambda t}).$$

Now (12), (33) and homogeneity and harmonicity $\mathrm{tr}A = 0 = \mathrm{tr}D^2 h_3(x)$ imply

$$\begin{aligned}
v(t,y) &= v_*(x)\det[(1 - ce^{-\mu_{01}t})I - Ae^{-\mu_{20}t} - D^2 h_3(x)e^{-\mu_{30}t} + o(e^{-\Lambda t})] \\
&= v_*(y)(1 - cNe^{-\mu_{01}t}) - (Aye^{-\mu_{20}t} + cye^{-\mu_{01}t} + \nabla h_3(y)e^{-\mu_{30}t}) \cdot \nabla v_*(y) + o(e^{-\Lambda t}) \\
&= v_*(y)(1 - cNe^{-\mu_{01}t}) + \frac{1}{m}(Aye^{-\mu_{20}t} + c|y|^2 e^{-\mu_{01}t} + \nabla h_3(y)e^{-\mu_{30}t}) \cdot yv_*^{2-m}(y) + o(e^{-\Lambda t})
\end{aligned}$$

for $y \in \mathrm{int}(\mathrm{spt}\, v_*)$. Moreover, for $x \in \mathrm{spt}\, v_*$ we see $r(t) = |y|$ satisfies (41). $\square$

Note that after the second eigenvalue μ_{20}, a first level crossing occurs and it involves two of the eigenvalues appearing in Corollary 9: $\mu_{30} \ge \mu_{01}$ precisely if $1 \ge N(m-1)$. This determines which of these two corrections to the shear rate μ_{20} in (40) will be dominant. These eigenvalues correspond to pear-shaped deformations (with order 3 symmetry) and dilations, respectively. The geometric complexity of the higher modes is increasing with the degree of the polynomials, and so we do not attempt to extend this discussion to larger values of $\mu_{\ell k}$.

Remark 10. *As an alternative approach to deriving such corollaries, it would be possible to develop an asymptotic expansion analogous to* (35) *for* $v(t,x)-v_*(x)$ *directly based on the operator* $\mathcal{L}_{v_*}^{-1}\mathcal{H}_I\mathcal{L}_{v_*} = v_*^{2-m}\circ \mathcal{H}_I \circ v_*^{m-2}$ *whose eigenfunctions are* v_*^{2-m} *and* $v_*^{2-m}(x)\hat{\psi}_{\ell nk}(x)$, *where* $\hat{\psi}_{\ell nk} = \psi_{\ell nk} + const$ *for each* (ℓ, n, k). *Here we have used the fact that the linearization of* (33) *near* $(v_*, 0)$ *yields the elliptic boundary value problem* (20) *relating the different representations* $\psi = \mathcal{L}_{v_*}(\delta v)$ *and* δv *of a tangent vector at* v_*. *Thus* $\delta v = \mathcal{L}_{v_*}^{-1}\psi$, *and* $m\mathcal{L}_{v_*}^{-1} = v_*^{2-m}\circ\mathcal{H}_E$ *from* (28), *while* $\mathcal{H}_E$ *commutes with* $\mathcal{H}_I = f(\mathcal{H}_E)$. *This observation helps to explain both the factors of* v_*^{2-m} *and some of the constants which appear in* (37)–(40), *analogously to [19].*

Finally, we close with some comments on the conjecture and its corollaries. We have stated these conjectures for Generalized Minimizing Movements rather than solutions to (1)–(3), since they are based on the same gradient flow intuition as the GMM construction, and it is not known whether all solutions to the free-boundary problem (1)–(3) can arise from the GMM construction, or not. In addition, it is the GMM solutions which are known [35] to becomes self-similar asymptotically (10) as $t \to \infty$. Corollary 8 already proposes a substantial improvement consistent with the established rate (32) for these solutions, and Corollary 9 and the conjecture go further still. What is absent from all three results however, is the specification of a sense in which the series (35) is supposed to approximate the solutions, or the error terms in (37)–(40) are supposed to be small. One might imagine (35) refers to the $H^1_{v_*}$ norm, yet past experience with second-order analogs suggests this is not the right metric for controlling the extent to which linearization approximates the nonlinear flow [3] [19]. On the other, hand, the corollaries themselves give some clues regarding the missing metric(s), at least when $m > 1$. In that case the right hand sides of (37)–(40) vanish outside the ball while the left hand sides need not (as for translations $v(x,0) = v_*(x-z)$.) This demonstrates clearly that if we hope to improve the latter expansions from pointwise to function-space statements, the desired function-space metric must be relatively insensitive to information near and outside the boundary of the support of the Barenblatt profile, which in our case is the unit ball. (Both the 2-Wasserstein distance and the framework advanced in Koch's habilitation have this property.) It is also the reason that pointwise statements (37)–(40) can be formulated only away from $\partial(\operatorname{spt} v_*)$. The uniform parabolicity away from this boundary yields some hope for establishing the desired smoothness of the flow if a suitable framework can be identified.

Appendix: Spectrum of the Ornstein–Uhlenbeck operator on $H^1_{v_*}$.

In this appendix, we compute the spectrum and the corresponding eigenvalues of the Ornstein–Uhlenbeck operator $\mathcal{H}_E : \mathcal{D}(\mathcal{H}_E) \to H^1_{v_*}$ where $m = 1$ and thus $\mathcal{H}_E\psi(x) = -\Delta\psi(x) + x\cdot\nabla\psi(x)$. Spectral properties of the differential operator $-\Delta + x\cdot\nabla$ are well-known to the stochastics community because of its role in stochastic processes, and also in the mathematical physics community because $\mathcal{H}_E$ is conjugate to the harmonic oscillator. The only difference between the operator studied here and the "classical" Ornstein–Uhlenbeck operator is the choice of the underlying Hilbert space. While the standard choice is the Gauss space $L^2(e^{-|x|^2/2}dx)$, we consider its Sobolev variant $\dot{H}^1(e^{-|x|^2/2}dx) = H^1_{v_*}$. We will see, however, that apart from the presence or absence of a zero eigenvalue reflecting the presence or absence of constant functions in the space, the spectrum of both operators is identical. More precisely, we have the following

Theorem 11. *The operator* $\mathcal{H}_E : \mathcal{D}(\mathcal{H}_E) \to H^1_{v_*}$ *is self-adjoint. Its spectrum* $\sigma(\mathcal{H}_E)$ *is purely discrete and given*

$$\sigma(\mathcal{H}_E) = \mathbf{N}.$$

The corresponding eigenfunctions are Hermite polynomials.

For the convenience of the reader, we will sketch the proof of this result in the sequel. It is based on the following two Propositions:

Proposition 12. *The operator* $\mathcal{H}_E : \mathcal{D}(\mathcal{H}_E) \to H^1_{v_*}$ *is nonnegative , self-adjoint, and has a bounded inverse.*

Proposition 13. *The operator* $\mathcal{H}_E : \mathcal{D}(\mathcal{H}_E) \to H^1_{v_*}$ *has a purely discrete spectrum.*

Proof of Theorem 11. We immediately deduce from Propositions 12 and 13 that the spectrum of $\mathcal{H}_E$ is a discrete subset of $(0,\infty)$, and thus it is enough to solve the eigenvalue problem for $\mathcal{H}_E$. On the one hand, one can easily show that $H^1_{v_*}$ embeds continuously into the Gauss space $L^2(e^{-|x|^2/2}dx) = L^2(v_* dx)$, cf. Lemma 15 below, and thus every eigenvalue must be an eigenvalue of the "classical" operator defined on the Hilbert space $L^2(v_* dx)$. It is well-known that the eigenvalues of the Ornstein–Uhlenbeck operator on $L^2(v_* dx)$ are all positive integers and the corresponding eigenfunctions are Hermite polynomials, cf. [46]. It is easily checked that every polynomial lies in the domain of $\mathcal{H}_E$. Therefore, we conclude that Hermite polynomials are eigenfunctions of $\mathcal{H}_E$ and thus $\sigma(\mathcal{H}_E) = \mathbf{N}$. □

Before turning to the proofs of Propositions 12 and 13, we derive some auxiliary results:

Lemma 14. *$C_b^\infty(\mathbf{R}^N)$ is dense in $H^1_{v_*}$.*

This is a fairly standard result and we therefore only sketch its proof.

Proof. We first observe that $L^\infty(\mathbf{R}^N) \cap H^1_{v_*}$ is dense in $H^1_{v_*}$, which can be easily seen by considering the truncated functions $\psi_M = \max\{-M, \min\{M, \psi\}\}$ for $M > 0$. It holds that

$$\lim_{M\uparrow\infty} \int v_* |\nabla\psi - \nabla\psi_M|^2\, dx \;=\; \lim_{M\uparrow\infty} \int_{|\psi|\geq M} v_* |\nabla\psi|^2\, dx \;=\; 0$$

by the dominated convergence theorem. The density of $C_b^\infty(\mathbf{R}^N)$ in $H^1_{v_*}$ then follows by a standard mollification argument, see, e.g., [45, Lemma 2]. □

Lemma 15. *There exists a constant $C > 0$ dependent only on the space dimension N such that for all $\psi \in H^1_{v_*}$*

$$\inf_{c\in\mathbf{R}} \int (1+|x|^2) v_* (\psi - c)^2\, dx \;\leq\; C \int v_* |\nabla\psi|^2\, dx \tag{42}$$

holds.

Proof. In the following $C > 0$ will always denote a universal constant (possibly dependent on N) whose value may change from line to line. Thanks to the density of smooth functions provided in the previous lemma, it is enough to prove the statement for $\psi \in C_b^\infty(\mathbf{R}^N)$. We first show that

$$\int (1+|x|^2) e^{-|x|^2/2} \psi^2\, dx \;\leq\; C \left(\int e^{-|x|^2/2}\psi^2\, dx + \int e^{-|x|^2/2} |\nabla\psi|^2\, dx \right). \tag{43}$$

Indeed, because $\nabla \cdot \left(x e^{-|x|^2/2} \right) = \left(N - |x|^2\right) e^{-|x|^2/2}$, we have that

$$\begin{aligned} \int |x|^2 e^{-|x|^2/2} \psi^2\, dx &= -\int \nabla\cdot \left(x e^{-|x|^2/2} \right) \psi^2\, dx + N \int e^{-|x|^2/2} \psi^2\, dx \\ &= 2 \int x e^{-|x|^2/2} \psi \cdot \nabla\psi\, dx + N \int e^{-|x|^2/2}\psi^2\, dx, \end{aligned}$$

and we have integrated by parts in the second identity. We apply Young's inequality $2ab \leq a^2 + b^2$ to deduce

$$\int |x|^2 e^{-|x|^2/2} \psi^2\, dx \;\leq\; C \left(\int e^{-|x|^2/2} \psi^2\, dx + \int e^{-|x|^2/2} |\nabla\psi|^2\, dx \right).$$

From this, (43) follows upon adding $\int e^{-|x|^2/2}\psi^2\, dx$ on both sides of the inequality.

We now turn to the proof of (42). We prove a slightly stronger statement by choosing $c = \fint_{B_R(0)} \psi\, dx$ for some $R > 0$ that has to be fixed later. Equivalently, we may assume that

$$\int_{B_R(0)} \psi\, dx \;=\; 0.$$

Then the Poincaré estimate on the ball $B_R(0)$ reads

$$\int_{B_R(0)} \psi^2\, dx \;\leq\; CR^2 \int_{B_R(0)} |\nabla\psi|^2\, dx,$$

and thus

$$\int_{B_R(0)} e^{-|x|^2/2}\psi^2\,dx \;\le\; \int_{B_R(0)} \psi^2\,dx \;\le\; Ce^{R^2/2}R^2 \int e^{-|x|^2/2}|\nabla\psi|^2\,dx.$$

On the other hand, we also have that

$$\int_{\mathbf{R}^N\setminus B_R(0)} e^{-|x|^2/2}\psi^2\,dx \;\le\; \frac{1}{R^2}\int |x|^2 e^{-|x|^2/2}\psi^2\,dx.$$

Consequently, combining the last two estimates with (43) yields

$$\begin{aligned}&\int (1+|x|^2)e^{-|x|^2/2}\psi^2\,dx\\ &\le\; C\left(\frac{1}{R^2}\int |x|^2 e^{-|x|^2/2}\psi^2\,dx + e^{R^2/2}R^2\int e^{-|x|^2/2}|\nabla\psi|^2\,dx\right).\end{aligned}$$

Choosing R sufficiently large (uniformly in ψ), we see that the first term on the right can be absorbed into the left-hand side of the inequality, which yields the statement of the lemma by the definition of v_*. □

Lemma 16. *The embedding of $H^1_{v_*}$ in $L^2(v_*dx)$ is compact.*

Proof. We deduce the statement of this lemma from the standard Rellich compactness lemma for classical Sobolev functions on bounded domains and from estimate (42). Let $\{\psi_n\}_{n\in\mathbf{N}}$ denote a bounded sequence in $H^1_{v_*}$. It is convenient to assume that $\int(1+|x|^2)v_*\psi\,dx = 0$, because then (42) holds with $c=0$. Since $H^1_{v_*}$ is a Hilbert space, there exists a $\psi \in H^1_{v_*}$ and a subsequence which converges to ψ weakly in $H^1_{v_*}$. By the continuous embedding provided by (42), this weak convergence also holds in $L^2((1+|x|^2)v_*dx)$. Moreover, since v_* is bounded away from zero on every compact subset of $\mathbf{R}^N$, it holds that $\{\psi_n\}_{n\in\mathbf{N}}$ is bounded in $H^1(B_k(0))$ for every $k\in\mathbf{N}$. Therefore, by the standard Rellich compactness lemma and since $v_*\lesssim 1$, we can extract a further subsequence $\{\psi_{n_k}\}_{k\in\mathbf{N}}$ such that

$$\int_{B_k(0)} v_*(\psi-\psi_{n_k})^2\,dx \;\le\; \frac{1}{k} \tag{44}$$

for every $k\in\mathbf{N}$. We now have

$$\begin{aligned}\int v_*(\psi-\psi_{n_k})^2\,dx \;&=\; \int_{B_k(0)} v_*(\psi-\psi_{n_k})^2\,dx + \int_{\mathbf{R}^N\setminus B_k(0)} v_*(\psi-\psi_{n_k})^2\,dx\\ &\overset{(44)}{\le}\; \frac{1}{k}+\frac{1}{k^2}\int(1+|x|^2)v_*(\psi-\psi_{n_k})^2\,dx.\end{aligned}$$

Since the integral on the right-hand side of the above inequality is bounded by the embedding (42), we deduce that

$$\int v_*(\psi-\psi_{n_k})^2\,dx \;\le\; \frac{1}{k}+\frac{C}{k^2}$$

for some uniform constant $C>0$. We let k converge to infinity to obtain the desired result. □

Lemma 17. *For every $u\in L^2(v_*^{-1}dx)$ with $\int u\,dx = 0$, there exists a unique $\psi\in H^2_{\mathrm{loc}}\cap H^1_{v_*}$ such that*

$$\int v_*\nabla\psi\cdot\nabla\varphi\,dx \;=\; \int u\varphi\,dx \tag{45}$$

for all $\varphi\in H^1_{v_*}$.

Proof. We first observe that (45) are the Euler–Lagrange equations for the convex energy functional

$$\mathcal{F}(\psi) = \frac{1}{2}\int v_*|\nabla\psi|^2\,dx - \int u\psi\,dx$$

defined for $\psi\in H^1_{v_*}$. Existence and uniqueness of a minimizers follows from soft methods based on the continuous embedding $H^1_{v_*}\subset L^2(v_*dx)$ established in Lemma 15, whose details we omit as they are fairly standard. □

We are now in the position to prove Propositions 12 and 13. As both statements are established very similarly to the analogous statements Propositions 1 and 2 from [45], we again omit most of the details. We start with the

Proof of Proposition 12. The proof of this proposition is very close to the one of [45, Prop. 1]. By Lemma 14, the operator $\mathcal{H}_E : \mathcal{D}(\mathcal{H}_E) \to H^1_{v_*}$ is densely defined. A simple integration-by-parts argument shows that $\mathcal{H}_E$ is nonnegative and symmetric. Moreover, via Lemmas 15 and 17 it can be shown that $\mathcal{H}_E$ is onto, which in turn implies that $\mathcal{H}_E$ is self-adjoint and has a bounded inverse via arguments from Functional Analysis (see [43, Theorem 13.11]). □

It remains to provide the

Proof of Proposition 13. By Proposition 12, $\mathcal{H}_E$ is invertible and has a bounded inverse. To prove the discreteness of the spectrum, we have to show that the resolvent $\mathcal{H}_E^{-1} : H^1_{v_*} \to H^1_{v_*}$ is compact, cf. [44, Prop. 2.11]. This, however, is a consequence of the Rellich compactness established in Lemma 16. We omit details and refer the interested reader to [45, Prop. 2] for a similar argument.

□

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