An elliptic proof of the Lorentzian splitting theorems

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joint with T Beran, M Braun, M Calisti, N Gigli, A Ohanyan, F Rott, C Sämann (8 and 5)

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What is a splitting theorem?

Example (convex functions, not necessarily smooth)

If the graph of a convex function $u : \mathbf{R}^n \longrightarrow \mathbf{R}$ contains a full line, say $\frac{\partial^2 u}{\partial t^2}(t, 0, \ldots, 0) = 0$ for all $t \in \mathbf{R}$, then $u(x) = U(x_2, \ldots, x_n)$ for all $x \in \mathbf{R}^n$

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Example (Riemannian manifolds; Cheeger-Gromoll '71)

If a connected complete Ricci nonnegative Riemannian manifold (M^n, g_{ij}) contains an isometric copy of a line $(\mathbf{R}, dr^2)$, then M is a geometric product of $(\mathbf{R}, dr^2)$ with a submanifold $(\Sigma^{n-1}, h_{ij} = g_{ij}|_\Sigma)$: i.e.

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Toponogov '64: proved earlier assuming nonnegative sectional curvature

Gigli '14+: nonsmooth version for infinitesimally Hilbertian metric-measure spaces (M, d, m) satisfying curvature-dimension condition $RCD(0, N)$.

This talk: *Lorentzian* analogs relevant to Einstein's theory of gravity

Schematic proof of Cheeger-Gromoll splitting theorem:

Let $\gamma : \mathbf{R} \longrightarrow M^n$ be the isometrically embedded line.

Busemann '32: $b_r(x) := d(x, \gamma(r)) - d(\gamma(0), \gamma(r))$ and $\pm b^{\pm} := \lim_{r \to \pm\infty} b_r$

- note b_r is 1-Lipschitz and $|\nabla b_r| = 1 = |\nabla b^{\pm}|$ a.e.; for $r > 0$,
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General relativity: Einstein's gravity and field equation

• Einstein's gravity is formulated on smooth Lorentzian manifolds,
but often predicts such manifolds are geodesically incomplete (nonsmooth)

Gravity not a force, but rather a manifestation of curvature of spacetime
"Spacetime tells matter how to move" (along timelike/null geodesics...)

Field equation "Matter tells spacetime how to bend"

geometry	=	*physics*
curvature	=	flux of energy and momentum

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Field equation "Matter tells spacetime how to bend"

$$\begin{aligned} \textit{geometry} &= \textit{physics} \\ \text{curvature} &= \text{flux of energy and momentum} \\ \mathrm{Ric}_{ij} - \frac{1}{2} R g_{ij} &= 8\pi T_{ij} \qquad (\text{replaces } \Delta\phi = \rho \text{ and } F = -\nabla\phi) \end{aligned}$$

• just integrate this local conservation law for $T_{ij}(x)$ to find g_{ij}...

What if matter distribution is unknown?

Can look at initial value problem (nonlinear wave equation); linearization produces gravity waves; singularities propagate...

Elliptic vs hyperbolic geometry

ELLIPTIC: $\mathbf{R}^n$ equipped with Euclidean norm $|v|_E := (\sum v_i^2)^{1/2}$
- $|v+w|_E \le |v|_E + |w|_E$

HYPERBOLIC: Minkowski space $\mathbf{R}^n$ equipped with the *hyperbolic 'norm'*

$$|v|_F := \begin{cases} (v_1^2 - \sum_{i\ge 2} v_i^2)^{1/2} & v \in F := \left\{ v \in \mathbf{R}^n \mid v_1 \ge (\sum_{i \ge 2} v_i^2)^{1/2} \right\} \\ -\infty & \textit{else} \end{cases}$$

- $|v+w|_F \ge |v|_F + |w|_F$

the future $F \subset \mathbf{R}^n$ is a convex cone; $v \in F$ called *causal* or *future-directed*
- v is *timelike* if $v \in F \setminus \partial F$
- v is *lightlike (or null)* if $v \in \partial F \setminus \{0\}$

(• v is *spacelike* iff $\pm v \notin F$ and *past-directed* if $-v \in F$)
- smooth curves are called *timelike (etc.)* if all tangents are timelike (etc.)

A crash course in differential geometry: action principles

Manifold M^n with symmetric nondegenerate C^k-smooth tensor $g_{ij} = g_{ji}$

RIEMANNIAN: $(g_{ij}) > 0$ defines Euclidean norm on each tangent space

- its geometry is also encoded in the (symmetric) distance function

$$d(x,y)^q := \inf_{\sigma(0)=x,\sigma(1)=y} \int |\dot\sigma_t|^q_{E_g} dt \qquad q > 1$$

LORENTZIAN: $g \sim (+1, -1, \ldots, -1)$ defines hyperbolic norm on $T_x M$

- its asymmetric geometry is also encoded in the time-separation function

$$\ell(x,y)^q := \sup_{\sigma(0)=x,\sigma(1)=y} \int |\dot\sigma_t|^q_{F_g} dt \qquad 0 \neq q < 1$$

- $(-\infty)^q := -\infty$ so $\ell(x,y) = -\infty$ unless a causal curve links x to y
- extremizers are independent of q; they are called *geodesics*
- $\ell(x,z) \geq \ell(x,y) + \ell(y,z)$ (analog of the triangle inequality d satisfies)

The Riemann curvature tensor

Given (timelike) geodesics $(\sigma_s)_{s \in [0,1]}$ and $(\tau_t)_{t \in [0,1]}$ with $\sigma_0 = \tau_0$ and $\dot\sigma_0 - \dot\tau_0 \in F \setminus \partial F$,

$$\ell(\sigma_s, \tau_t)^2 = |s\dot\sigma_0 - t\dot\tau_0|^2_{F_g} - \frac{Sec}{6} s^2 t^2 + O((|s| + |t|)^5)$$

where sectional curvature $Sec = R(\dot\sigma_0, \dot\tau_0, \dot\sigma_0, \dot\tau_0)$ is quadratic in $\dot\sigma_0 \wedge \dot\tau_0$ and measures the leading order correction to Pythagoras

- polarization of this quadratic form gives the *Riemann* tensor $R(\cdot, \cdot, \cdot, \cdot)$

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- polarization of this quadratic form gives the *Riemann* tensor $R(\cdot,\cdot,\cdot,\cdot)$
- its trace $\mathrm{Ric}_{ik} = g^{jl}R_{ijkl}$ yields the *Ricci* tensor; $\mathrm{Ric}(v,v)$ measures the correction to Pythagoras averaged over all triangles including side v
- second trace $R = g^{ik}\mathrm{Ric}_{ik}$ yields the *scalar curvature*; in the elliptic case it gives leading order correction to the area of a sphere of radius r (and to the volume of a ball of radius r)
- $d\mathrm{vol}_g(x) = \sqrt{|\det(g)|}d^n x$ in coordinates; (in the Riemannian case it coincides with the n-dimensional Hausdorff measure associated to d)

Energy conditions and singularity theorems

WEC (weak energy condition): $T(v, v) \geq 0$ for all future $v \in F$ (physical)
SEC (strong energy condition): $\mathrm{Ric}(v, v) \geq 0$ for all future $v \in F$ (less ")
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[Cosmological constant (dark matter): $\geq (n-1)Kg(v,v)$]

Hawking '66 (big bang type) singularities are generic:
SEC + mean curvature bound $H_\Sigma \geq h > 0$ on a suitable hypersurface Σ implies finite-time singularities along all timelike geodesics through Σ

Cavalletti-Mondino '20+: also in $TCD(0,N)$ metric-measure spacetimes

Penrose '65 (stellar collapse type) singularities are generic
NEC + trapped codimension-2 compact surface S + suitable noncompact hypersurface Σ imply finite-time singularity along some null geodesic

Graf '20 holds for $g_{ij} \in C^1$;

Open: version for $TCD(0,N)$ metric-measure spacetimes?

Smooth Lorentzian splitting theorems

• *'spacetime'*: a connected Lorentzian manifold (M^n, g_{ij}) which admits a continuous choice of F_g (distinguishing future from past).
• *'strong energy condition' SEC:* $g(v, v) > 0$ implies $\mathrm{Ric}(v, v) \geq 0$
• *'line'*: doubly-infinite, maximizing, timelike geodesic
• *'timelike geodesically complete'*: all timelike geodesics admit doubly-infinite extensions (maximizing locally but not necessarily globally)

Theorem (conjectured by Yau '82; proved by Newman '90)

Let (M^n, g_{ij}) be a SEC spacetime containing a line. If M is (a) timelike geodesically complete, then M is a geometric product of **R** *with a (Ricci nonnegative, complete) Riemannian submanifold Σ^{n-1}*

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Galloway '84: proved assuming compact Cauchy surfaces exist

Beem, Ehrlich, Galloway, Markvorsen '85: proved under sectional curvature bounds assuming (b) global hyperbolicity; (nonsmooth: BOR & Solis '23)

Eschenburg '88: proved under (a) + (b)

Galloway '89: proved under (b) without (a) using Bartnik '88

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Elliptic proof of the Newman (and Galloway) theorems

- BGMOS quintet 24+ $g_{ij} \in C^\infty(M^n)$; in-progress $g_{ij} \in C^1(M^n)$

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Theorem (BBCGMORS 24+ 'nonsmooth p-d'Alembert comparison')

For $p < 1$, the operator $\Box_p u := -\nabla \cdot (|\nabla u|_F^{p-2} \nabla u)$ is nonuniformly elliptic and (SEC) implies $\Box_p b_r^+ \leq \frac{n-1}{\ell(\cdot,\gamma(r))}$ distributionally, i.e. $\forall\ 0 \leq \phi \in C_c^1(M)$

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$$\int_M g\left(\nabla\phi, \frac{\nabla b_r^+}{|\nabla b_r^+|_F^{2-p}}\right) d\mathrm{vol}_g \leq (n-1)\int_M \frac{\phi(\cdot)d\mathrm{vol}_g(\cdot)}{\ell(\cdot,\gamma(r))}.$$

- for the distributional limit $r \to \infty$, need $\nabla b_r^+ \longrightarrow \nabla b^+$ strongly
- need uniform ellipticity; must bound $\{\nabla b_r^+\}_{r \geq R}$ away from lightcone

Convex p-energy: trading linearity for ellipticity

Additional conditions may ensure $\ell \neq +\infty$ and extremizers exist
• complete or *proper* (boundedly compact) in the Riemannian case
• (b) *globally hyperbolic* in the Lorentzian case (i.e. compact diamonds, future F varies continuously over M, no closed future-directed curves)

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Convex Hamiltonian $H(w) = -\frac{1}{p}|w|_{F^*}^p$ and Lagrangian $L(v) = -\frac{1}{q}|v|_F^q$ satisfy $DH = (DL)^{-1}$ if $p^{-1} + q^{-1} = 1$ (here $p < 0$ iff $0 < q < 1$)
- note $L = H^*$ jumps to $+\infty$ across future cone boundary ∂F (but H diverges continuously at the boundary of the dual cone F^*)
- nonsmooth already on smooth Lorentzian manifolds

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Beran Braun Calisti Gigli M. Ohanyan Rott Sämann (octet): extremizers of p-Dirichlet energy $u \mapsto \int_M H(du) d\mathrm{vol}_g$ rel. to compactly supported perturbations satisfy a new nonuniformly elliptic nonlinear PDE
- trade linearity of d'Alembertian for ellipticity of p-d'Alembertian!

Nondivergence expression of (nonuniform) ellipticity

$$\Box_p b = \nabla_i \left(\frac{\partial H}{\partial w_i} \Big|_{db} \right) = H^{ij} \nabla_i \nabla_j b + \Gamma^i_{ij} \frac{\partial H}{\partial w_j} \Big|_{db}$$

$$H(w) = -\frac{1}{p} |w|_{F^*}$$

$$H^i := \frac{\partial H}{\partial w_i} = -|w|^{p-2} g^{ik} w_k$$

$$H^{ij} := \frac{\partial^2 H}{\partial w_i \partial w_j} = |w|^{p-2} \left[(2-p) g^{ik} g^{jl} \frac{w_k w_l}{|w|^2} - g^{ij} \right]$$

$$\sim |w|^{p-2} \begin{bmatrix} 2-p-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0 \quad \text{if } p < 1$$

choosing Fermi coordinates around γ in which $w = db$ is the time axis

Nomizu-Ozeki '61 give a complete Riemannian metric $\tilde{g}$ on (M, g).

Theorem (Eschenburg '88 ... Galloway-Horta '96)

Under (a) or (b), $\gamma(0)$ admits a neighbourhood X and constants R, C such that if $r \geq R$ then (i) a maximizing geodesic σ connects each $x \in X$ to $\gamma(r)$; (ii) each such geodesic satisfies $\tilde{g}(\sigma'(0), \sigma'(0)) \leq C g(\sigma'(0), \sigma'(0))$ hence $\{b_r^+\}$ is timelike and uniformly equiLipschitz on X.

- uniformizes ellipticity on X

Lemma (BGMOS quintet: equi-semiconcavity (one derivative better))

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Lemma (BGMOS quintet: equi-semiconcavity (one derivative better))

For another constant C', all $u \in \{b_r^+\}_{r \geq R}$ and $(v, x) \in TX$ satisfy

$$\lim_{t \to 0} \frac{u(\exp_x^{\tilde{g}} tv + u(\exp_x^{\tilde{g}} -tv) - 2u(x)}{\tilde{g}(v, v)} \leq C'$$

- gives $db_r^+ \to db^+$ pointwise a.e., hence $|db^{\pm}|_{F^*} = 1$ a.e. and
- $\pm b^{\pm}$ are distributionally p-superharmonic $\Box_p b^+ \leq 0 \leq \Box_p b^-$

- now strong maximum principle improves $b^+ \geq b^-$ to $b^+ = b^- \in C^{1,1}(X)$
- elliptic regularity (i.e. Schauder theory) gives $b := b^\pm \in C^\infty(X)$

Homogeneity $2p - 2 < 0$ variant on Bochner-Ohta '14, Mondino-Suhr '23: $|du|_{F^*} = 1$ and $\Box_p u = 0$ imply

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$$\begin{aligned} 0 &= \nabla_i (H^{ij}|_{du} \nabla_j (H|_{du})) - H^i \nabla_i (\nabla_j (H^j|_{du})) \\ &= H^{ij} u_{jk} H^{kl} u_{li} + R_{ij} H^i H^j \\ &= \mathrm{Tr}\left[\left(\sqrt{D^2 H} \nabla^2 u \sqrt{D^2 H}\right)^2\right] + \mathrm{Ric}(DH, DH) \end{aligned}$$

- timelike Ricci nonnegative gives Lorentzian $\mathrm{Hess}\, b = 0$ in X

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- timelike Ricci nonnegative gives Lorentzian $\mathrm{Hess}\, b = 0$ in X
- hence ∇b is a Killing vector field (its flow gives a local isometry on X)
- $\Sigma := \{x \in X \mid b(x) = 0\}$ is totally geodesic (its normal ∇b is parallel)
- along Σ, metric splits into tangent $g_{ij}dy^i dy^j$ and normal components dr^2
- simplify Eschenburg '88, Galloway '89, Newman '90 to get from X to M

THANK YOU