Metric measure spacetimes: a nonsmooth approach to gravity

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Adopting a reference measure m

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p-d'Alembertian becomes elliptic(!) for homogeneity $p - 1 < 0$

9. nonsmooth splitting: extremizing timelike line yields product geometry

(currently open; possible future work)

A nonsmooth framework for gravity

- replace Lorentz manifold (M, g_{ij}) of relativity with *metric spacetime* M (variant on Kunzinger-Sämann's '18 Lorentzian prelength spaces; also Minguzzi–Suhr's '24 bounded Lorentzian metric spaces, [M.24], Müller...)
- $\ell : M^2 \longrightarrow \{-\infty\} \cup [0, \infty)$ is called a *time-separation* function if

$$\ell(x,y) + \ell(y,z) \leq \ell(x,z) \qquad \forall x,y,z \in M$$

- ℓ defines the *transitive* relations *causality* $\leq$ and *chronology* $\ll$ by:

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- ℓ defines the *transitive* relations *causality* $\le$ and *chronology* $\ll$ by:

$$\begin{array}{rll} & \le \, := \{\ell \ge 0\} & \ll \, := \{\ell > 0\} \\ \textit{future} & J^+(x) = \{y \in M \mid y \ge x\} & I^+(x) := \{y \in M \mid y \gg x\} \\ \textit{past} & J^-(z) := \{y \in M \mid y \le z\} & I^-(z) := \{y \in M \mid y \ll z\} \end{array}$$

- assume $\ell(y, y) = 0$ $\forall y \in M$, so (the preorder) $\le$ is reflexive
- *chronological topology:* the coarsest topology with $I^\pm(y)$ open $\forall y \in M$

- a topology is called *Polish* if it has a complete, separable metrization

Definition (Metric spacetime; time-reversal)

A time-separation function $\ell : M^2 \longrightarrow \{-\infty\} \cup [0, \infty)$ as above makes (M, ℓ) a *metric spacetime* if the chronological topology it induces is Polish. The *time-reversal* (M, ℓ^*) of (M, ℓ) refers to $\ell^*(y, x) = \ell(x, y)$.

- metrizability implies $\leq$ is partial-order: i.e. $(x \leq z \;\&\; z \leq x) \Rightarrow (x = z)$
- $\leq$ is *forward-complete* $\Leftrightarrow$ $\lim_{i\to\infty} x_i$ exists whenever $x_i \leq x_{i+1} \leq z (\forall i \in \mathbf{N})$

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Definition (Forward spacetime — our standing assumption)

A metric spacetime (M, ℓ) (with its causal and chronological relations $\leq$ and $\ll$ and Polish chronological topology) is called *forward* if the partial order $\leq$ is forward-complete and ℓ is upper semicontinuous.

- write (M, ℓ) is *backward* $\Leftrightarrow$ its time-reversal (M, ℓ^*) is forward
- let $J^+(X) := \cup_{x \in X} J^+(x)$ and $J^-(Z) := \cup_{z \in Z} J^-(z)$

Definition (Emeralds)

An *emerald* refers to $J(X, Z) := J^+(X) \cap J^-(Z)$ with $X, Z \subset M$ compact.

We say (M, ℓ) *has compact emeralds* if every emerald is compact.

Example (Manifolds)

- any globally hyperbolic Lorentzian length space is a forward spacetime
- [MiH19] a smooth Lorentzian manifold of dimension $n \geq 3$ is *globally hyperbolic* $\Leftrightarrow$ it has compact emeralds

Example (Manifolds with boundary)

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Example (Manifolds with boundary)

The closed interval $[-1, 1]$ with the time-separation

$$\ell(x,y) := \begin{cases} y - x & \text{if } y \geq x, \\ -\infty & \text{else,} \end{cases}$$

is a forward spacetime (not a Lorentzian length space nor a manifold, but its interior $(-1,1)$ is a smooth globally hyperbolic Lorentzian manifold)

Calculus of worldlines (i.e. nondecreasing curves)

Definition (Causal curve and ℓ-speed; c.f. Ambrosio '90 for (M, d))

$\sigma : [0,1] \longrightarrow M$ is *causal* $\Leftrightarrow \sigma_s := \sigma(s) \leq \sigma(t)$ for all $0 \leq s < t \leq 1$;
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Its *ℓ-speed* refers to the limit on $(0,1)$ (Lemma: it exists pointwise a.e.!)

$$|\dot{\sigma}(s)| := \lim_{h \downarrow 0} \frac{\ell(\sigma_{s+h}, \sigma_s)}{h}$$

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- in a metric (resp. forward) spacetime, discontinuities of a causal curve σ are countable (and σ may be taken left-continuous without loss, resp.)
- the set $LCC([0, 1]; M)$ of Left-Continuous Causal curves metrized by

$$D(\sigma, \tau) := d(\sigma_0, \tau_0) + \int_0^1 d(\sigma_s, \tau_s) ds$$

is Polish, if d makes the chronological topology Polish on (M, ℓ)
- Limit-curve theorem: $C \subset M$ compact makes $LCC([0, 1]; C)$ D-compact

'Concave' q-Lagrangian action and (rough) ℓ-geodesics

Definition (q-Lagrangian action, geodesics c.f. [Mi19,M.20,MoS23])

Given $0 \neq q < 1$, the *action* of a causal curve refers to

$$A_q[\sigma] := \frac{1}{q} \int_0^1 |\dot{\sigma}(s)|^q ds$$

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Causal curves *maximizing* this action (for given endpoints) are called *rough geodesics*; if $\sigma \in LCC([0,1]; M)$ then simply *geodesic*.

- recall twin paradox

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- maximizers are independent of q;
- the set of geodesics is denoted $CGeo(M)$;
- curves in $TGeo(M) := \{\sigma \in CGeo(M) \mid A_q[\sigma] \neq 0\}$ are called *timelike* or *ℓ-geodesics*.

- Caveat: cannot interpret ‘concavity’ without a tangent space (or cone)

Nonbranching conditions; characterizing geodesics

Lemma (Indpendence of q; affine parameterization)

A curve $\sigma : [0, 1] \longrightarrow M$ *is a rough (ℓ-)geodesic iff* $\forall 0 \leq s < t \leq 1$,

$$\ell(\sigma(s), \sigma(t)) = (t - s)\ell(\sigma(0), \sigma(1)) \geq 0 \quad (> 0).$$

Definition (Timelike nonbranching conditions; c.f. [CM24])

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Definition (Timelike nonbranching conditions; c.f. [CM24])

The metric spacetime is called *timelike nonbranching* if any pair of rough ℓ-geodesics that agree on $(\frac{1}{3}, \frac{2}{3})$ also agree on $[0,1]$.

Lemma (Regularity of geodesics; c.f. [M.24])

If a forward spacetime is timelike nonbranching, any rough ℓ-geodesic is left-continuous (but not necessarily right-continuous).

Fuzzy events: lifting the geometry from events to measures

Optimal transport on forward spacetimes: for $0 \neq q < 1$ the linear program

$$\ell_q(\mu,\nu) := \sup_{\gamma \in \Gamma_{\leq}(\mu,\nu)} \left(\int_{M^2} \ell(x,y)^q d\gamma(x,y) \right)^{1/q}$$

defines a time-separation (and q-independent causal relation [EMi17]) between Borel probability measures $\mu, \nu \in \mathcal{P}_{em}(M)$ on emeralds in M. Here

$$\Gamma_{\leq}(\mu,\nu) := \Big\{ \gamma \geq 0 \text{ on } M^2 \mid \gamma[\{\ell \geq 0\}] = 1, \quad \mu[Y] = \gamma[Y \times M] \\ \forall Y \subset M, \qquad \gamma[M \times Y] = \nu[Y] \Big\}$$

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- maximizers γ exist if $\Gamma_{\leq}(\mu,\nu) \neq \emptyset$ and are called q-optimal couplings
- thus ℓ_q-speed along any causal curve $(\mu_s)_{s \in [0,1]}$ of measures exists a.e.:

$$|\dot{\mu}_s|_q := \lim_{h \downarrow 0} \frac{\ell_q(\mu_s, \mu_{s+h})}{h}$$

Tangent fields; lifting curves $(\mu_t)_t$ to measures π on curves

Definition (Rough ℓ_q-geodesics can be defined like rough ℓ-geodesics)

Given $0 \neq q < 1$, the action of a causal curve $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}(M)$ is

$$\mathcal{A}_q[\mu] := \frac{1}{q}\int_0^1 |\dot\mu_t|_q^q dt \leq \frac{1}{q}\ell_q(\mu_0,\mu_1)^q < \infty \text{ if } \mu_0,\mu_1 \in \mathcal{P}_{em}(M).$$

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Theorem (Lifting curves of measures in forward spacetimes c.f.[Lis07])

Conversely, if $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}_{em}(M)$ is causal, narrowly left-continuous on $[0,1]$, and (M,ℓ) has compact emeralds (= or is locally causally convex) then it's induced by a plan $\pi \in \mathcal{P}(LCC([0,1];M))$ with expected action*

$$\int A_q[\sigma]d\pi(\sigma) = \mathcal{A}_q[\mu]$$

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$$\int A_q[\sigma]d\pi(\sigma) = \mathcal{A}_q[\mu] \quad (= \ell_q(\mu_0,\mu_1)^q/q \text{ if } \pi \text{ is "}q\text{-optimal"})$$

Consequences in forward spacetimes

- these measures π on curves (i.e. ‘plans’) represent tangent fields

Corollary (Optimal plans concentrate on geodesics)

If $\pi \in \mathcal{P}(LCC([0,1];M))$ is q-optimal, then $\pi[CGeo] = 1$.

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Corollary (Narrow forward-completeness in a forward spacetime)

If $\mu_i \leq \mu_{i+1} \leq \nu$ in $(\mathcal{P}(M), \ell_q)$, then $\lim_{i\to\infty} \mu_i$ converges narrowly.

- plays a crucial role in our eventual construction of q-optimal *'test'* plans; i.e. plans π with uniformly bounded instantaneous densities $(e_t)_{\#}\pi$

q-dualizability and narrow continuity of rough ℓ_q-geodesics

Definition (Strict timelike q-dualizability; c.f. [M.20] [CM24])

The pair $\mu, \nu \in \mathcal{P}_{em}(M)$ is *strictly timelike q-dualizable* iff every q-optimal coupling $\gamma \in \Gamma_{\leq}(\mu, \nu)$ vanishes outside $\{\ell > 0\}$.

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- local tightness can come from e.g., compact emeralds or narrow forward-completeness or

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- local tightness can come from e.g., compact emeralds or narrow forward-completeness or density bounds wrt a suitable reference measure m as in a metric-measure spacetime

(Exact, future-directed) cotangent fields; their magnitudes

Definition (Causal functions (nondecreasing); form a convex cone)

$u : M \longrightarrow [-\infty, \infty]$ is *causal* $\Leftrightarrow \ell(x, y) \geq 0$ implies $u(x) \leq u(y)$.

Definition (Metric-measure spacetime; test plan; maximal subslope)

Fix a Radon measure m on (M, ℓ) assigning finite mass to each emerald.

A plan $\pi \in \mathcal{P}(LCC([0,1]; M))$ is called (initially) *test* $\Leftrightarrow$

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$$u(\sigma_1) - u(\sigma_0) \geq \int_0^1 v(\sigma_t)|\dot{\sigma}_t| dt$$

for every test plan π and π-a.e. curve σ. They form a stable lattice. Each m-measurable causal u admits a *maximal weak subslope,* denoted $v = |du|$.

c.f. [AGS14]

Infinitesimal Minkowskianity

Lemma (Examples of weak subslopes; (equality m-a.e. for nice u, M))

Continuity of causal u and $\ell_+ = \max\{\ell, 0\}$ *imply m-a.e. y satisfies*

$$\liminf_{x \ll y} \frac{u(y) - u(x)}{\ell(x, y)} \leq |du(y)|, \qquad \liminf_{z \gg y} \frac{u(z) - u(y)}{\ell(y, z)} \leq |du(y)|.$$

Definition (c.f. infinitesimally Hilbertian [G15] rather than [AGS14d])

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$$|d(u+v)|^2 + |dv|^2 = 2|d(u+2v)|^2 + 2|du|^2 \qquad m\text{-a.e.}$$

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- equivalently, the following polarization is positively bilinear m-a.e.:

$$((du, dv)) := \frac{1}{2}|d(u+v)|^2 - \frac{1}{2}|du|^2 - \frac{1}{2}|dv|^2$$

- distinguishes Lorentz from Lorentz-Finsler metrics on e.g. $\mathbf{R}^n$ [BO24]

Convex analysis; *horizontal derivatives*; raising indices

Like smooth causal curves σ and functions u on a Lorentz manifold satisfy

$$\langle du, \dot\sigma \rangle \geq \frac{1}{p}\|du\|_*^p + \frac{1}{q}\|\dot\sigma\|^q \qquad \text{when } p^{-1} + q^{-1} = 1$$

with equality iff $\langle \dot\sigma, \cdot \rangle = \|du\|_*^{p-2} du(\cdot)$, i.e. iff $\dot\sigma = \|\nabla u\|^{p-2}\nabla u$ [M.20],

Theorem (Nonsmooth Fenchel-Young inequality for $0 < q < 1$)

If $(e_s)_\# \pi \to (e_0)_\# \pi$ narrowly, $|du|^p \in L^1(m)$, and π initially test then

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- limit on left called *horizontal* (*inner*, *Lagrangian*) derivative of u along π
- aims at bilinear pairing of π with u; (NB concave p-Dirichlet energy of u)

Definition (Identified tangent and cotangent fields; optimal transport)

If $\lim_{s\downarrow 0}$ exists and equality holds, we say π *represents the p-gradient* of u.
A nonlinear duality between some tangent and cotangent fields (π and u)

Perturbation & variational derivative of p-Dirichlet energy

• given m-measurable $E \subset M$, write $v \in Pert_p(u, E)$ if for all $\epsilon > 0$ small enough, $u + \epsilon v$ is causal and $|d(u + \epsilon v)|^p \in L^1(E, dm)$.

Theorem (Horizontal dominates vertical derivative; c.f. [G15])

If $u : M \longrightarrow \bar{\mathbf{R}}$ is causal, $v \in Pert_p(u, E)$, and π represents the p-gradient of u and is concentrated on curves remaining initially in E, then

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- last is direction v *vertical* (/ *outer* / *Eulerian*) derivative of p-energy at u
- nonlinear in u but becomes linear in v if two-sided limit in ϵ exists

Corollary (If (M, ℓ, m) is infinitesimally Minkowskian)

and if $-v, v \in Pert_p(u, E)$ then $\lim_{\epsilon \to 0}$ and $\lim_{s \downarrow 0}$ exist & equality holds above!

Curvature bounds via entropy

Given $N \in (1, \infty)$, define N-Renyi (or Boltzmann) entropy of $\mu \in \mathcal{P}(M)$ by

$$S_N(\mu) := -\int_M (\frac{d\mu}{dm})^{-1/N} d\mu \quad \text{(and } S_\infty(\mu) := \lim_{N\to\infty} N + NS_N(\mu))$$

• in the smooth globally hyperbolic setting, convexity properties of $t \in [0,1] \mapsto S_N(\mu_t)$ along ℓ_q-geodesics (or of $S_\infty(\mu_t)$) are well-known to characterize timelike lower Ricci curvature bounds [B23] [MoS23] [M.20]; c.f. [RS04][CM.S01][OV00][M.94] (or [EKS15])

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$TMCP^\pm$ (or $TMCP_e^+$): a nonsmooth timelike lower Ricci bound

• we impose only sublinearity of $S_N(\mu_t)$ along ℓ_q-geodesics starting or ending at a Dirac point mass — the *timelike measure contraction properties* $TMCP^\pm$ of [B23]; c.f. [CMo24] [LV09] [O07] [S06]

• if (μ_0, δ_z) are strictly timelike q-dualizable, $\mu_0 \in L^\infty(m)$ and (M, ℓ) timelike nonbranching has compact emeralds* precisely one ℓ_q-geodesic links μ_0 to δ_z; normalization $S_N(\delta_z) = 0$ but $S_\infty(\delta_z) := +\infty$.

A nonsmooth timelike lower Ricci curvature bound

Definition (Future timelike measure contraction property; c.f. [B23])

For $K \in \mathbf{R}$ write $(M, \ell, m) \in TMCP^+(K, N)$ if $\forall \mu_0 \in \mathcal{P}_{em}(M) \cap L^1(m)$ and each $z \in \operatorname{spt} m$ with $\mu_0[I^-(z)] = 1$, for some (hence all) $0 \neq q < 1$, there exists a (rough) ℓ_q-geodesic from μ_0 to $\mu_1 := \delta_z$ such that all $t \in [0, 1]$ and $N' \geq N$ satisfy

$$S_{N'}(\mu_t) \leq - \int \tau_{K,N'}^{(1-t)}(\ell(x,z)) \left(\frac{d\mu_0}{dm}(x)\right)^{-1/N'} d\mu_0(x).$$

Past version: $(M, \ell, m) \in TMCP^-(K, N) \Leftrightarrow (M, \ell^*, m) \in TMCP^+(K, N)$.

- $\tau_{0,N}^{(1-t)}(\ell) := 1 - t$ for $K = 0$;

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- $\tau_{0,N}^{(1-t)}(\ell) := 1 - t$ for $K = 0$; asserts sublinearity of $t \in [0, 1] \mapsto S_{N'}(\mu_t)$, and follows from the strong energy condition, a case of primary interest

- a smooth globally hyperbolic Lorentzian manifold $(M^n, g, d\mathrm{vol}_g)$ satisfies $TMCP^{\pm}(K, n) \Leftrightarrow Ric(v, v) \geq Kg(v, v)$ for all timelike $v \in TM$

Test plans: finding ℓ_q-geodesics having density bounds

Theorem (Initial test plans with Dirac targets; c.f. [B23,CMo17,R13])

Fix ($K \in \mathbf{R}$ or) $K = 0 \neq q < 1 < N < \infty$, a timelike nonbranching forward spacetime $(M, \ell, m) \in (TMCP^+ \cap TMCP^+_e)(K, N)$ and $z \in M$. If $\mu_0[I^-(z)] = 1$ for $\mu_0 \in L^\infty(m) \cap \mathcal{P}_{em}(M)$ then there exists a q-optimal plan π inducing (an ℓ_q-geodesic) $\mu_t := (e_t)_\# \pi$ from μ_0 to $\mu_1 := \delta_z$ such that $t \in [0,1] \mapsto S_{N'}(\mu_t)$ is (suitably) sublinear for each $N' \geq N$ and

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$$\|\frac{d\mu_t}{dm}\|_{L^\infty(m)} \leq \frac{c_{K,N,\ell}}{(1-t)^N} \|\frac{d\mu_0}{dm}\|_{L^\infty(m)}.$$

- $c_{0,N,\ell} = 1$ if $K = 0$ (else $c_{K,N,\ell} := \exp(t\|\ell\|_{L^\infty(\mu_0 \times \mu_1)} \sqrt{K_-(N-1)})$)
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- [B23] Boltzmann version $TMCP_e^+$ can be replaced by compact emeralds
- extends to non-Dirac targets if (μ_0, μ_1) strictly timelike q-dualizable

COROLLARY (Busemann and Lorentz distance functions have unit slope)
For $v(\cdot) = -\ell(\cdot, z)$, compact emeralds* imply $|dv| = 1$ m-a.e. on $I^-(z)$

When is the p-gradient of u represented by a test plan π?

$$u^{(q)}(z) := \sup_{x \in I^-(z)} u(x) + \frac{\ell(x,z)^q}{q} \qquad v_q(x) := \inf_{z \in I^+(x)}$$

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- write $u : M \longrightarrow \bar{\mathbf{R}}$ is $\frac{\ell^q}{q}$-*concave* if $u = v_q$ for some $v : M \longrightarrow \bar{\mathbf{R}}$; if $q < 0$
- then u is causal, upper semicontinuous, and $\partial_{\ell^q/q} u$ relatively closed in $\ll$

$$\partial_{\ell^q/q} u := \{x \ll z \mid u^{(q)}(z) = u(x) + \tfrac{\ell(x,z)^q}{q} \in \mathbf{R}\} \subset M^2, \text{ if } \ell_+ \in C(M)$$

Theorem (A metric Brenier-M. thm; cf.[MoS23,M.20,AGS14])

Fix $0 \neq q < 1$ and $p^{-1} + q^{-1} = 1$. Let (M, ℓ, m) timelike nonbranching have compact emeralds, ℓ_+ continuous and $u = (u^{(q)})_q$ with $|du|^p \in L^1(E, m)$.*

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$$|du|(\sigma_0) = \ell(\sigma_0, \sigma_1)^{q-1}.]$$

$$\mathrm{dom}\,\partial_{\ell^q/q} u := \{x \in M \mid \partial_{\ell^q/q} u \cap (\{x\} \times M) \neq \emptyset\}$$

Theorem (d'Alembert comparison theorem: $\Box_p u \leq N$ if $K = 0$)

Fix $0 \neq q < 1 = p^{-1} + q^{-1} < N < \infty$, a timelike nonbranching spacetime $(M, \ell, m) \in TMCP^+_{(e)}(K, N)$ with compact emeralds, $\ell_+ \in C(M)$, $K \in \mathbf{R}$ and $\frac{\ell^q}{q}$-concave $u = (u^{(q)})_q$.*

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$$\int_M d^+ v(\nabla u)|du|^{p-2} dm \leq \int_M \tilde{\tau}_{K,N}(|du|^{p-1}) v\, dm$$

$$\tilde{\tau}_{K,N}(r) := N \frac{\partial \tau^t_{K,N}(r)}{\partial t|_{t=1}} = \begin{cases} N & \text{if } K = 0 \\ 1 + r\sqrt{(N-1)|K|}\cot(r\sqrt{\frac{K}{N-1}}) & \text{else.} \end{cases}$$

e.g.

$$u(x) = \begin{cases} -\ell(x,z)^q/q & \forall x \in I^-(z), \\ +\infty & \text{else.} \end{cases}$$

Corollary

Same nonsmooth sense and setting with e.g. $K = 0$, a chain rule yields

$$\Box_p(-\ell(\cdot, z)) \leq \frac{N-1}{\ell(\cdot, z)} \qquad \text{on } I^-(z)$$

- Analogous results also hold true in backward spacetimes and $K \neq 0$.

After time-reversing them, the forward $(M, \ell, m) \in TMCP^-(K, N)$ satisfies

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• [BGM.OS25+] extends his timelike splitting theorem to $g_{ij} \in C^1(M)$

Defining the p-d'Alembertian

• thus even on smooth globally hyperbolic manifolds we obtain new results

• functional analysis: $\Box_q u$ is a measure, nonunique unless infinitesimally Minkowskian, $TMCP^{\pm}_{(e)}(K, N)$, and $Pert_p(u, E)$ is dense; c.f. [G15]

• localization: [B24+] establishes many fundamental properties of $\Box_p u$ by developing an approach based on needle decompositions; c.f. [CMo20]

Proof: Fix $s_N(r) = -r^{(N-1)/N}$, $u(\cdot) = -\ell^q(\cdot, z)/q$, $v \in Pert_p(u, I^-(z))$ and $d\mu_0 = \rho_0 dm$ with $\rho_0 = (c_0\, v)^{N/(N-1)} \in L^\infty$ cptly supported in $I^-(z)$. Our metric Brenier-M. theorem yields an initial test plan π representing the p-gradient of u with $\mu_1 = \delta_z$ and $\mu_t := (e_t)_\# \pi$ satisfying sublinearity:

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$$\begin{aligned}
\liminf_{t \downarrow 0} \frac{S_N(\mu_t) - S_N(\mu_0)}{t} &\geq \liminf_{t \downarrow 0} \int \frac{s_N' \circ \rho_0(\gamma_t) - s_N' \circ \rho_0(\gamma_0)}{t} d\pi(\gamma) \\
&\geq \liminf_{\varepsilon \downarrow 0} \int_M \frac{|d(u + \varepsilon\, s_N' \circ \rho_0)|^p - |d u|^p}{p\varepsilon} d\mu_0 \\
&=: \int_M d^+(s_N' \circ \rho_0)(\nabla u)\, |du|^{p-2} d\mu_0 \\
&= \frac{c_0}{N} \int_M d^+ v(\nabla u)\, |du|^{p-2} dm,
\end{aligned}$$

where the last identity follows from a suitable chain rule and the specific choice of ρ_0. Cancelling c_0 gives $\Box_p u \leq N$ distributionally! $\square$

Selected references (apologies for omissions/oversights):

Beran-Braun-Calisto-Gigli-M.-Ohanyan-Rott-Sämann 2408.15968 (2507.06836)

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[AGS14,AGS14d] Ambrosio–Gigli–Savaré (Inventiones, Duke)

[B23,B23,B24+] Braun (Nonlinear Analysis, JMPA, arXiv:2408.16525)

[BO24,BM.23+] Braun–Ohta (TAMS), Braun-M. (arXiv:2312.17158)

[CMo17,CMo20,CMo24] Cavalletti–Mondino (CCM, APDE, Camb J Math)

[EKS15] Erbar–Kuwada–Sturm (Inventiones)

[EMi17] Eckstein–Miller (AIHP)

[G15] Gigli (MAMS)

[Lis07] Lisini (CVPDE)

[KS18] Kunzinger–Sämann (Ann Global Anal Geom)

[M.94,M.20,M.24] M. (PhD, Cambridge J Math, CMP)

[Mi19,MiH19,MiS24] Minguzzi, Hounnonkpe-Minguzzi, Minguzzi-Suhr

[MoS23] Mondino–Suhr (J Euro Math Soc)

[O07,O14] Ohta (CM Helvetica, Anal Geom Metric Spaces)

[OV00,LV09] Otto–Villani (J. Funct. Anal.), Lott–Villani (Annals)

[R12,R13] Rajala (JFA, DCDS)

[RS04,S06] von Renesse–Sturm (CPAM), Sturm (Acta x 2)

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THANK YOU!