Lipschitz free boundaries in the monopolist's problem

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arXiv:2301.07660 and arXiv:2303.04937 and more in progress

29 May 2024

Outline

1. Monopolist's problem
2. Examples and History
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4. Results
5. New duality certifying solutions
6. A free boundary problem hidden in Rochet-Choné's square example
7. The bunching regions have Lipschitz free boundary

Monopolist's problem

Given compact sets $X \subset \mathbf{R}^m$, $Y \subset \mathbf{R}^n$, and 'direct utility'
$b(x, y)$ = value of product $y \in Y$ to buyer $x \in X$
$c(y)$ = monopolist's cost to produce $y \in Y$
$d\mu(x)$ = relative frequency of buyer $x \in X$ (as compared to $x' \in X$)

Monopolist's problem: choose price menu $v : Y \longrightarrow Z$ to maximize profits

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$$\tilde{\Pi}(v) := \int_X [v(y_v(x)) - c(y_v(x))] d\mu(x), \qquad \text{where}$$

Agent x's problem: choose $y_v(x)$ to maximize

$$y_v(x) \in \arg\max_{y \in Y} b(x, y) - v(y)$$

Constraints: v lower semicontinuous, $0 \in Y$ and $v(0) = 0$.

Examples

- airline ticket pricing
- insurance
- educational signaling
- optimal taxation: replace profit maximization with a budget constraint for providing services

Some history:

Mirrlees '71, Spence '73 ($n = 1 = m$): $\frac{\partial^2 b}{\partial x \partial y} > 0$ implies $\frac{dy_v}{dx} \geq 0$

Rochet-Choné '98 ($n = m > 1$): $b(x, y) = x \cdot y$ *bilinear* implies $y_v(x) = Dv^*(x)$ convex gradient; bunching

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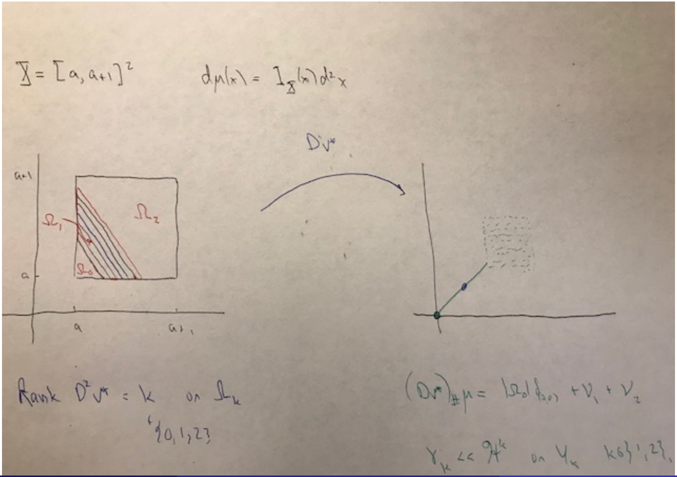
$y_v(x) = Dv^*(x)$ convex gradient; bunching for $c(y) = \frac{1}{2}|y|^2$



Carlier-Lachand-Robert '03: b bilinear gives $v^* \in C^1(\operatorname{spt}\mu)$;

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is called the 'indirect utility' to shopper x

Figalli-Kim-M. '11:
convexity of principal's problem under strengthening of (MTW) on $b(x,y)$

M.-Rankin-Zhang '23+:
$u = v^* \in C^{1,1}_{loc}((\mathrm{spt}\,\mu)^0)$ under same strengthening

Noldeke-Samuelson (ECTA '18), Zhang (ET '19) M.-Zhang (CPAM '19):
generalize to preferences $G(x,y,z) \neq b(x,y) - z$ and profits
$\pi(x,y,z) \neq z - c(y)$ nonlinear in price $z \in \mathbf{R}$

Rochet-Choné $b(x, y) = x \cdot y$ in terms of buyers' utilities u

$$u(x) := v^*(x) := \max_{y \in Y}[x \cdot y - v(y)] \tag{1}$$

implies

$$Du(x) = D_x b(x, y_v(x)) = y_v(x)$$

so we identify

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so we identify

$$y_v(x) = Du(x)$$

and maximize

$$\begin{aligned}\tilde{\Pi}(v) &= \int_X (v - c)(Du(x))d\mu(x) \\ &= \int_X [b(x,y) - u(x) - c(y)]_{y=Du(x)} d\mu(x) =: -L(u)\end{aligned}$$

among u of form (1) (i.e. among convex $u(\cdot) \geq 0$ with $Du \in Y$)

A new duality for bilinear preferences

Following Rochet-Choné '98 choose $b(x,y) = x \cdot y$ and $X, Y \subset \mathbf{R}^n$ convex so profit

$$-L(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)$$

with

$$u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)$$
$$\in \mathcal{U} := \{u : X \longrightarrow [0,\infty] \ \mathit{convex} \mid Du(X) \subset Y\}$$

THM (M.-Zhang arXiv:2301.07660 Y a convex cone; c.f. Kolesnikov-Sandomirskiy-Tsyvinski-Zimin 22+ on Beckmann auctions):

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$$\max_{u \in \mathcal{U}} -L(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)$$

where

$$\mathcal{S} := \bigcap_{u \in \mathcal{U}} \left\{ S : X \longrightarrow \mathbf{R}^n \mid \int_X [(x - S(x)) \cdot Du - u(x)] d\mu(x) \leq 0 \right\}$$

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where

$$\mathcal{S} := \bigcap_{u\in\mathcal{U}} \{S : X \longrightarrow \mathbf{R}^n \mid \langle x \cdot Du(x) - u(x)\rangle_\mu \leq \langle S(x) \cdot Du(x)\rangle_\mu\}$$

In words: the monopolists maximum profit coincides with the net value of a co-op able to offer its members good $y \in Y$ at price=cost $c(y)$, minimized over possible distributions $S_{\#}\mu$ of co-op memberships satisfying

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In words: the monopolists maximum profit coincides with the net value of a co-op able to offer its members good $y \in Y$ at price=cost $c(y)$, minimized over possible distributions $S_{\#}\mu$ of co-op memberships satisfying the strange constraint that when members whose true type is $S(x)$ irrationally display the behaviour of x facing each monopolist price menu, the expected gross value of the resulting assignment $Du(x)$ to those co-op members dominates the monopolist's expected gross revenue $\langle x \cdot Du(x) - u(x) \rangle_\mu$.

Proof: Rockafellar-Fenchel duality; $(\leq)$: $S \in \mathcal{S}$, $u \in \mathcal{U}$ and definition of c^*

$$-L(u) = \langle x \cdot Du(x) - u - c(Du(x)) \rangle_\mu \leq \cdots \leq \langle c^* \circ S \rangle_\mu$$

□

Partition into convex bunches of different dimension

$$u \in \underset{\text{convex } u \geq 0}{\arg\max} -L(u)$$

minimizes net loss

$$L(u) := \int_{[a,a+1]^2} \left(\frac{1}{2}|Du(x) - x|^2 + u - \frac{1}{2}|x|^2 \right) d\mu(x)$$

(Convex) isoproduct bunch (= equivalence class = contact set = leaf)

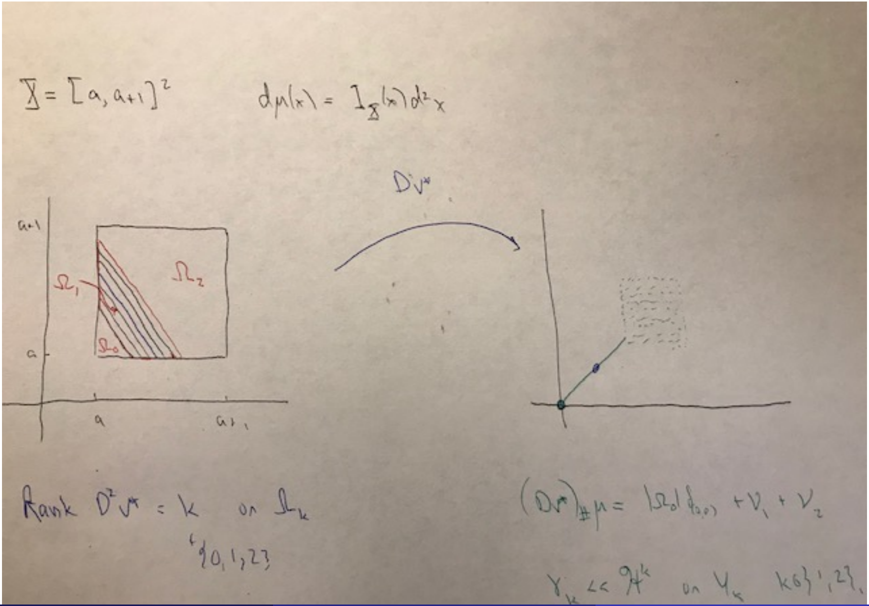
$$\tilde{x} := \{x' \in X \mid Du(x') = Du(x)\} \subset X$$

foliate interior of $\Omega_{n-i} := \{x \in X \mid \dim(\tilde{x}) = i\}$.

Lemma (Leaves reach boundary; any normal distortion is outward)

(o) $\Omega_0 = \{x \in X \mid u = 0\}$ *interior non-empty,* * *foliated by a single leaf.*
(i) if $x \in \Omega_1 \cup \cdots \Omega_{n-1}$ *there exists* $x' \in \tilde{x} \cap \partial X$
(ii) if $x \in \Omega_{n-1}$ *(or* X *is strictly* * *convex) then* $\hat{n}(x') \cdot (Du(x') - x') \geq 0$.
(iii) Ω_n *is relatively open in* X*, foliated by points, i.e.* u *is strictly convex.*

Rochet-Choné's square example revisited; $c(y) = \frac{1}{2}|y|^2$



Variational calculus for obstacle problem plus convexity

Setting $u_i := u$ on $\Omega_i := \{x \in X \mid \mathrm{Dim}(\tilde{x}) = n - i\}$ (now $n = 2$) gives

- on Ω_0 exclusion: $u_0 = 0$ (c.f. Armstrong '94)
- on Ω_1, Euler-Lagrange ODE: if $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$ then

$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$

subject to boundary conditions $u_1 = u_0$ and $Du_1 = Du_0$ at lower boundary.

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- on Ω_2 Euler-Lagrange PDE: $\Delta u_2 = 3$ subject to boundary conditions

$$(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial X \cap \bar{\Omega}_2$$

$$(Du_2 - Du_1) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1 \quad \text{(Neumann)}$$

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OVERDETERMINED!

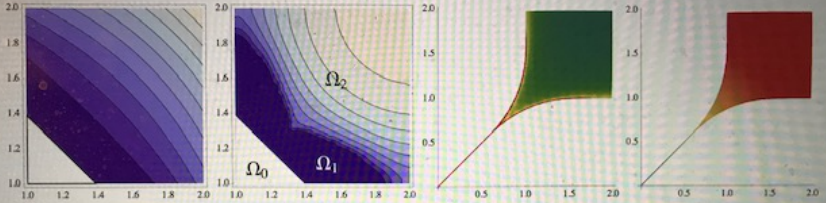
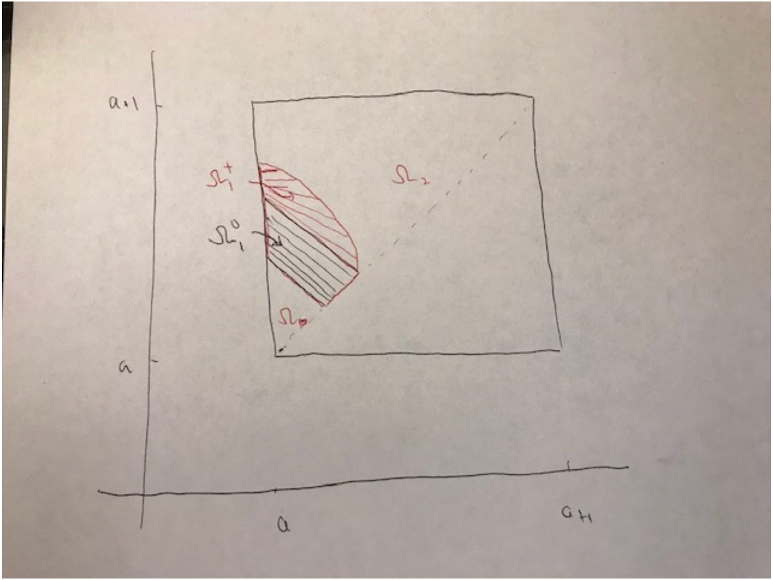


Fig. 1 Numerical approximation U of the solution of the classical Monopolist's problem (1), computed on a 50×50 grid. *Left* level sets of U, with $U = 0$ in *white*. *Center left* level sets of $\det(\nabla^2 U)$ (with again $U = 0$ in *white*); note the degenerate region Ω_1 where $\det(\nabla^2 U) = 0$. *Center right* distribution of products sold by the monopolist. *Right* profit margin of the monopolist for each type of product (margins are low on the one dimensional part of the product line, at the *bottom left*). *Color scales* on Fig. 10 (color figure online)

Springer

J.-M. Mirebeau (2016)



c.f. Boerma-Tsyvinski-Zimin 22+ blunt Ω_1^0 vs targeted Ω_1^+ bunching

Free boundary problem

$u = u_i$ on Ω_i where

- on Ω_0 exclusion: $u_0 = 0$

- on Ω_1^0, Rochet-Choné's ODE: $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$ where
$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$
subject to boundary conditions $k = 0$ and $k' = 0$ at lower boundary.

- on Ω_1^+, $u_1 = u_1^+$ given by a NEW system of ODE (for height $h(\cdot)$ and length $R(\cdot)$ of isochoice segments together with profile of $u_1^+(\cdot)$ along them), with boundary conditions

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- on Ω_2, PDE: $\Delta u_2 = 3$ with Rochet-Choné's overdetermined conditions

$$\begin{aligned}
(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 &\quad \text{on} \quad \partial X \cap \bar{\Omega}_2 \text{ and on } \{x_1 = x_2\} \\
(Du_2 - Du_1^+) \cdot \hat{n}_{\Omega_2}(x) = 0 &\quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad \text{(Neumann)} \\
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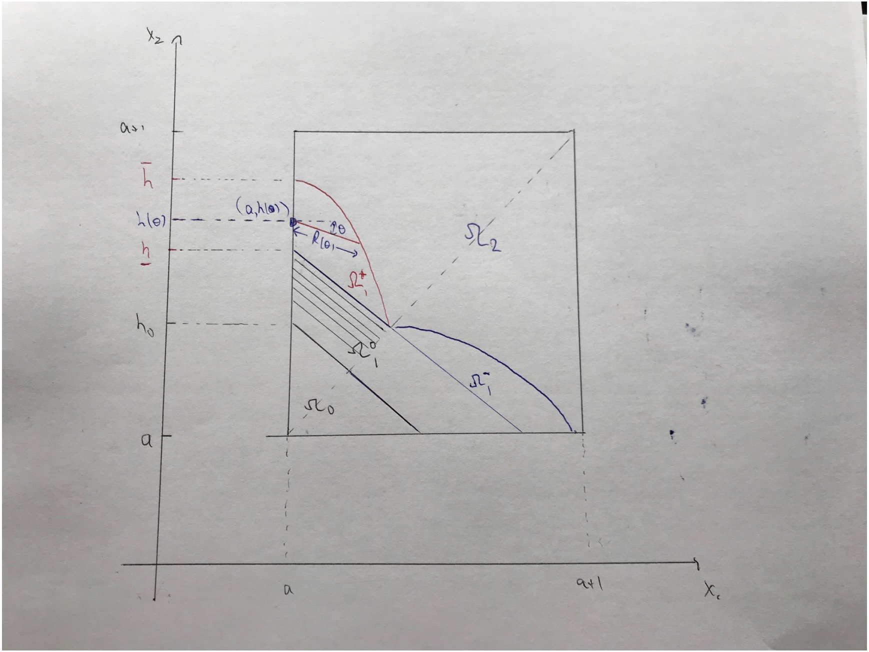

x_2
$a+1$
$\bar{h}$
$h(\theta)$
$\underline{h}$
h_0
a
$(a,h(\theta))$
2θ
$R(\theta)$
Ω_2
Ω_1^+
Ω_1^0
Ω_1^-
Ω_0
a
$a+1$
x_1

Precise Euler-Lagrange equation in the 'missing' region Ω_1^+

Index each isochoice segment in Ω_1^+ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta)r + b(\theta).$$

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For $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ with $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$, solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta)\sin\theta - m(\theta)\cos\theta + a) = \frac{3}{2}R^2(\theta)\cos\theta \quad (2)$$

$$m(-\tfrac{\pi}{4}) = 0, \qquad m'(-\tfrac{\pi}{4}) = \tfrac{1}{\sqrt{2}}k'(a + \underline{h}).$$

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$$h(t) = \underline{h} + \frac{1}{3}\int_{-\pi/4}^{t} (m''(\theta) + m(\theta) - 2R(\theta))\frac{d\theta}{\cos\theta}, \quad (4)$$

$$b(t) = \frac{1}{2}k(a + \underline{h}) + \int_{-\pi/4}^{t} (m'(\theta)\cos\theta + m(\theta)\sin\theta)h'(\theta)d\theta. \quad (5)$$

- for $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ Lipschitz (say) and $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ we can solve (2)–(12) to find Ω_1^+ and u_+^1.
- we can then solve the resulting Neumann problem for $\Delta u_2 = 3$ on Ω_2
- what is work-in-progress is that some choice of $\underline{h}$ and $R(\cdot)$ also yields $u_1 - u_2 = const$ on $\partial\Omega_2 \setminus \partial X$,

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WHY IS IT NATURAL FOR SUCH A CHOICE TO EXIST?

- for $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ Lipschitz (say) and $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ we can solve (2)–(12) to find Ω_1^+ and u_+^1.

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- a unique optimizer $\bar{u} \in \mathcal{U}$ is known to exist (Rochet-Choné) and $\bar{u} \in C^{1,1}_{loc}(X^0)$ (Caffarelli-Lions); if the sets Ω_i where its Hessian is rank i are smooth enough, and Ω_1 has the expected 3 components, then (2)–(12) and the overdetermined Poisson problem $\Delta u_2 = 3$ must be satisfied

- but maybe Ω_i are not smooth enough, or Ω_1 is not (simply) connected and/or has more than three components (some too small for the numerics to resolve); this is excluded by our work-in-progress...

The bunching regions have Lipschitz free boundary

Recall: Caffarelli-Lion's '06+ assert $u \in C^{1,1}_{loc}(X^0)$.

• sharp: examples for $n = 1 = m$ show $u \notin C^2_{loc}(X^0)$

• if we can quantify $u \notin C^2$ along free boundary, Clarke's Lipschitz implicit function theorem applied to the normal derivative $\frac{\partial u}{\partial r}$ will allow us to write the free boundary separting Ω_1 from Ω_2 as a Lipschitz graph over θ

• on Ω_2 side have $\Delta u = 3$.

• on Ω_1 return to variational analysis of $\min\{L(u) \mid 0 \leq u \text{ convex}\}$ where

$$L(u) = \frac{1}{2} \int_{[a,a+1]^2} \left(|Du - x|^2 + u - \frac{|x|^2}{2} \right) d\mathcal{H}^2(x)$$

Rochet-Choné characterized minimizer by $L(u + w) \geq L(u)$ for all convex $w \geq 0$.

Equivalently $w \geq 0$ convex implies $\int w d\sigma \geq 0$ for variational derivative:

$$d\sigma = \frac{\delta L}{\delta u} = (3 - \Delta u) d\mathcal{H}^2|_X + (Du - x) \cdot \hat{n} d\mathcal{H}^1|_{\partial X}.$$

Thus positive and negative parts of σ in convex order! $\sigma^-(w) \leq \sigma^+(w)$

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Use the equivalence relation $x \sim x' \Leftrightarrow Du(x) = Du(x')$ given by product selected to disintegrate σ, so $\tilde{\sigma} = (Du)_\#(\sigma^+)$ and $\forall \phi \in C([a, a+1]^2)$,

$$\int_{[a,a+1]^2} \phi(x) d\sigma(x) = \int_{[a,a+1]^2/Du} d\tilde{\sigma}(\tilde{x}) \int_{\tilde{x} \subset [a,a+1]^2} \phi(x) d\sigma_{\tilde{x}}(x),$$

Equivalently $w \geq 0$ convex implies $\int w d\sigma \geq 0$ for variational derivative:

$$d\sigma = \frac{\delta L}{\delta u} = (3 - \Delta u) d\mathcal{H}^2|_X + (Du - x) \cdot \hat{n} d\mathcal{H}^1|_{\partial X}.$$

Thus positive and negative parts of σ in convex order! $\sigma^-(w) \leq \sigma^+(w)$

Use the equivalence relation $x \sim x' \Leftrightarrow Du(x) = Du(x')$ given by product selected to disintegrate σ, so $\tilde{\sigma} = (Du)_\#(\sigma^+)$ and $\forall \phi \in C([a, a+1]^2)$,

$$\int_{[a,a+1]^2} \phi(x) d\sigma(x) = \int_{[a,a+1]^2/Du} d\tilde{\sigma}(\tilde{x}) \int_{\tilde{x} \subset [a,a+1]^2} \phi(x) d\sigma_{\tilde{x}}(x),$$

Rochet-Choné '98: convex order inherited by $\tilde{\sigma}$-a.e. conditional measure: $\sigma^-_{\tilde{x}}(w) \leq \sigma^+_{\tilde{x}}(w) \forall w$ convex. Thus $\sigma^\pm_{\tilde{x}}$ have the same mass & center of mass; get $\sigma^+_{\tilde{x}}$ from $\sigma^-_{\tilde{x}}$ by sweeping / balayage / mean-preserving spreads if $\tilde{x} \neq 0$ (Cartier-Fell-Meyer '56).

- In the region $x \in \Omega^0_1$, this tells uniform negativity of $d\sigma_{\tilde{x}}(r) \sim dr$ over the segment interior is balanced by positive Dirac masses at the endpoints.
- In the region $x \in \Omega^+_1$, it tells $d\sigma_{\tilde{x}}(r) \sim (3r - 2R)dr$ increases affinely in $0 < r < R(\theta)$, balancing a positive Dirac mass at $r = 0$.
- The resultant discontinuity in Δu at $r = R(\theta)$ implies $R(\theta)$ is Lipschitz!

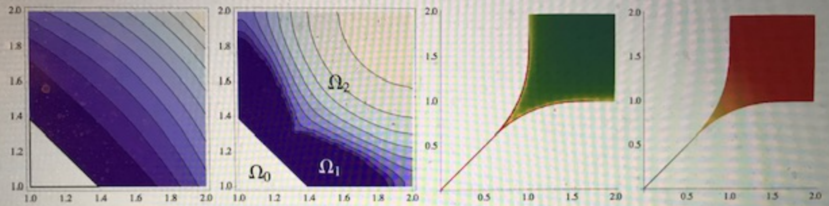

Fig. 1 Numerical approximation U of the solution of the classical Monopolist's problem (1), computed on a 50×50 grid. *Left* level sets of U, with $U = 0$ in *white*. *Center left* level sets of $\det(\nabla^2 U)$ (with again $U = 0$ in *white*); note the degenerate region Ω_1 where $\det(\nabla^2 U) = 0$. *Center right* distribution of products sold by the monopolist. *Right* profit margin of the monopolist for each type of product (margins are low on the one dimensional part of the product line, at the *bottom left*). *Color scales* on Fig. 10 (color figure online)

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Proof sketch (assuming (r, θ) are good coordinates):

Now $x(r,\theta) = (a, h(\theta)) + r(\cos\theta, \sin\theta)$ and $u_1^+(x) = m(\theta) r + b(\theta)$ yield

$$\text{Jacobians} \qquad d\mathcal{H}^2|_X = |h' \cos\theta + r| dr d\theta$$

$$d\mathcal{H}^1|_{\partial X} = |h'(\theta)| d\theta$$

$$\text{Laplacian} \qquad \Delta u = \frac{m'' + m}{h' \cos\theta + r}$$

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$$\text{so} \qquad -d\sigma = -\frac{\delta L}{\delta u} = (\Delta u - 3) d\mathcal{H}^2|_X - \hat{n} \cdot (Du - x) d\mathcal{H}^1|_{\partial X}.$$

factors into conditional measures given by

$$\mp d\sigma_{\tilde{x}} = [m'' + m - 3(h'\cos\theta + r) - \hat{n}(x) \cdot (Du - x) h'(\theta) \delta_0(r)] dr$$

- the last term represents a point mass where the segment $\tilde{x}$ intersects ∂X

$$\mp\frac{d\sigma_{\tilde{x}}}{dr} = m'' + m - 3(h'\cos\theta + r) - \hat{n}(x)\cdot(Du - x)h'(\theta)\delta_0(r)$$

Since $\sigma_{\tilde{x}}^- \preceq \sigma_{\tilde{x}}^+$ in convex order, $\int_0^R w d\sigma_{\tilde{x}} = 0$ for $\pm w(r) \in \{1, r\}$,

$$(m'' + m - 3h'\cos\theta)R - \frac{3}{2}R^2 = \hat{n}(x)\cdot(Du - x)h'(\theta) \quad (6)$$

$$(m'' + m - 3h'\cos\theta) = 2R \quad (7)$$

$$\mp \frac{d\sigma_{\tilde{x}}}{dr} = m'' + m - 3(h' \cos\theta + r) - \hat{n}(x) \cdot (Du - x) h'(\theta) \delta_0(r)$$

Since $\sigma^-_{\tilde{x}} \preceq \sigma^+_{\tilde{x}}$ in convex order, $\int_0^R w d\sigma_{\tilde{x}} = 0$ for $\pm w(r) \in \{1, r\}$,

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$$(m'' + m - 3h' \cos\theta) = 2R \tag{7}$$

Choosing $w(r)$ strictly convex shows $\sigma^+_{\tilde{x}}$ must be obtained from $\sigma^-_{\tilde{x}}$ by mean-preserving spread; hence the point mass is in $\sigma^+_{\tilde{x}}$ not $\sigma^-_{\tilde{x}}$. From (6)-(7),

$$0 \leq \frac{1}{2} R(\theta)^2 = \hat{n}(x) \cdot (Du - x) h'(\theta). \tag{8}$$

Now $\frac{d\mathcal{H}^1|_{\partial X}}{d\theta} = |h'(\theta)| = +h'(\theta) \geq 0$ hence normal distortion is outward; Also $R > 0$ implies point mass (8)$\neq 0$ hence $0 \neq \Delta u - 3 = \frac{2R - 3r}{h' \cos\theta + r}$.

Lemma (Leaves reach boundary; any normal distortion is outward)

(o) $\Omega_0 = \{x \in X \mid u = 0\}$ *interior non-empty,** *foliated by a single leaf.*
(i) if $x \in \Omega_1 \cup \cdots \Omega_{n-1}$ *there exists* $x' \in \tilde{x} \cap \partial X$
(ii) if $x \in \Omega_{n-1}$ *(or* X *is strictly** *convex) then* $\hat{n}(x') \cdot (Du(x') - x') \geq 0$.
(iii) Ω_n *is relatively open in* X, *foliated by points, i.e.* u *is strictly convex.*

Also $x(r,\theta) = (a, h(\theta)) + r(\cos\theta, \sin\theta)$ and $u_1^+(x) = m(\theta)r + b(\theta)$ yield

$$Du \equiv \begin{pmatrix} \frac{\partial u}{\partial x_1}(x(r,\theta)) \\ \frac{\partial u}{\partial x_2}(x(r,\theta)) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} m(\theta) \\ m'(\theta) \end{pmatrix}.$$

hence

$$e(\theta) := \frac{\partial u}{\partial x_2} = m' \cos\theta + m \sin\theta$$

$$f(\theta) := \hat{n} \cdot (Du - x) = (m' \sin\theta - m\cos\theta + a).$$

Using f in (8) to replace $h' = \frac{R^2}{2f}$ in the first moment condition (7) yields

$$m''(\theta) + m(\theta) - 2R(\theta) = \frac{3R^2(\theta)}{2f(\theta)} \cos\theta$$

Euler-Lagrange equation in overlooked region Ω_1^+

Index each isochoice segment in Ω_1^+ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta) r + b(\theta).$$

For $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ with $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$, solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta)\sin\theta - m(\theta)\cos\theta + a) = \frac{3}{2}R^2(\theta)\cos\theta \quad (9)$$

$$m(-\tfrac{\pi}{4}) = 0, \qquad m'(-\tfrac{\pi}{4}) = \tfrac{1}{\sqrt{2}}k'(a + \underline{h}). \qquad \text{Then set} \quad (10)$$

$$h(t) = \underline{h} + \frac{1}{3}\int_{-\pi/4}^{t} (m''(\theta) + m(\theta) - 2R(\theta))\frac{d\theta}{\cos\theta}, \quad (11)$$

$$b(t) = \frac{1}{2}k(a + \underline{h}) + \int_{-\pi/4}^{t} (m'(\theta)\cos\theta + m(\theta)\sin\theta)h'(\theta)d\theta. \quad (12)$$

THANK YOU!

A regularity result: Lipschitz product selection

Theorem (M.-Rankin-Zhang '23+)

If b and $\tilde{b}(y,x) = b(x,y)$ both satisfy (B0-B3), c satisfies (C0-C2) and $d\mu(x) = fdx$ with $\log f \in C^{0,1}$ then $u \in C^{1,1}_{loc}(X^0)$.

- extends Caffarelli-Lions '06+ to b & c non-quadratic
- improves Chen '13 from C^1_{loc} to $C^{1,1}_{loc}$
- sharp: examples for $n = 1 = m$ show $u \notin C^2_{loc}(X^0)$
- idea: use energetic comparison to pinch u between parabolas

Lemma (A geometric lemma)

Given $d > 0$, there exists $C_0, C_1, C_2 > 0$ such that if $u = u^{\tilde{b}b}$ is optimal and $d(x_0, \partial X) > d$ and $y_0 = \bar{y}_b(Du(x_0), x_0)$ then if $r < C_0$ and

$$h = \sup_{x \in B_r(x_0)} u(x) - [u(x_0) + b(x, y_0) - b(x_0, y_0)] > 0$$

then some $A(\cdot) = b(\cdot, y') + a'$ makes $S := \{x \in X \mid u < A\}$ a neighourhood of x_0 with

$$\sup_{x \in S} A(x) - u(x) \le h$$

and

$$\frac{1}{|S|} \int_S \Big[c(y) - b(x, y) \Big]_{y=y'}^{y=\bar{y}(Du(x),x)} f(x)dx \ge -C_1 h + C_2 \frac{h^2}{r^2}.$$

Proof: