# A geometric approach to regularity of optimal maps


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(with Brendle, Léger, Rankin '24 and Y-H Kim, M Warren '10)

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# Outline

# Minimal hypersurfaces in $\mathbf{R}^{n+1}$

$$u \in \arg \min_{u|_{\partial\Omega}=f} \int_\Omega \sqrt{1+|\nabla u|^2} d^n x \qquad \text{'minimizing'}$$

satisfies
$$0 = \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \qquad \text{'minimal'}$$

Blow-up: $u(0) = 0 = \nabla u(0)$ yields $u_0(x) = \lim_{r_k \to 0} {r_k}^{-2} u(r_k x)$ minimal on $\mathbf{R}^n$

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COUNTEREXAMPLES (Bombieri-deGiorgi-Giusti '68) whenever $n \geq 7$.

HIGHER CODIMENSION:

- (Federer '69) each algebraic curve (or analytic variety $p(z) = 0$ in $\mathbf{C}^n$) is minimal

- for analogous minimization in higher codimension, singularities have codimension $\geq 2$ (Almgren '00)

# Maximal spacelike hypersurfaces in Minkowski space $\mathbf{R}^{n,1}$

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CONS:

- $SO(n,1)$ is noncompact, unlike $SO(n+1)$.
- uniformity of ellipticity degenerates as $|\nabla u_0| \to 1$;
- orientation delicacies (associated e.g. with disconnectedness of $S^0$)

# What about spacelike $n$-volume maximizers in e.g. $\mathbf{R}^{n,m}$?

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THM 1 (Kim–M.–Warren '10):
graphs of optimal maps are spacelike maximizing (with $m = n$)

THM 2 (Brendle–Leger–M.–Rankin '24) A sign becomes favorable in the pseudo-Riemannian setting (relative to the Riemannian case) allowing us to give a new proof of Ma–Trudinger–Wang's (2005) regularity results.

# Submanifold Geometry

Let $\Sigma^n \subset \hat{M}^{n+m}$ be a maximal spacelike submanifold of a manifold $\hat{M}$ equipped with a signature $(n, m)$ metric $\hat{g}(\cdot, \cdot)$ and its associated Riemann tensor $\hat{R}(\cdot, \cdot, \cdot, \cdot)$. Here *spacelike* means $g := \hat{g}|_{(T\Sigma)^2} > 0$, *maximal* means zero mean curvature vector $H = \mathrm{tr}_M \mathrm{I\!I} = 0$ and $\mathrm{I\!I}_z : (T_z\Sigma)^2 \longrightarrow (T_z\Sigma)^\perp$ is the *second fundamental form*

$$\mathrm{I\!I}(X, Y) := \hat{D}_X Y - D_X Y,$$

i.e. the difference between the $\hat{g}$-covariant derivative $\hat{D}$ and $g$-covariant derivative $D$ on tangent fields $X, Y$ to $\Sigma$.

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i.e. the difference between the $\hat{g}$-covariant derivative $\hat{D}$ and $g$-covariant derivative $D$ on tangent fields $X, Y$ to $\Sigma$. Let $e_1, \ldots, e_n$ diagonalize $S$ and $\hat{E}_1, \ldots, \hat{E}_{n+m}$ be local $\hat{g}$-orthonormal frames on $\Sigma$ and $\hat{M}$ respectively.

## Lemma (Brendle–Leger–M.–Rankin '24)

*If $\hat{S}$ is any positive-definite symmetric (0,2)-tensor on $\hat{M}$ and $S = \hat{S}|_{(T\Sigma)^2}$, there is a constant $c = c(\|\hat{g}, \hat{g}^{-1}, \hat{S}\|_{C^2(\{z\})})$ independent of $\Sigma$ such that*

$$\frac{(\Delta_g S)(e_n, e_n)}{2S(e_n, e_n)} \geq \sum_{l=1}^{n} (\hat{R}(e_l, e_n, e_l, e_n) - cS(e_l, e_l))$$

Proof sketch: After a long computation exploiting maximality ($H = 0$),

$$\frac{\Delta S}{2}(e_n, e_n) = \sum_{l=1}^{n} \Big[ \frac{1}{2}(\hat{D}^2_{e_l,e_l}\hat{S})(e_n, e_n) + 2(\hat{D}_{e_l}\hat{S})(\mathrm{II}(e_l, e_n), e_n)$$

$$+ \hat{S}(\mathrm{II}(e_l, e_n), \mathrm{II}(e_l, e_n)) - \sum_{\alpha,\beta=1}^{n+m} \hat{g}^{\alpha\beta}\hat{R}(e_l, e_n, e_l, \hat{E}_\alpha)\hat{S}(\hat{E}_\beta, e_n)$$

$$+ S(e_n, e_n)\Big[\hat{R}(e_l, e_n, e_l, e_n) - \hat{g}(\mathrm{II}(e_l, e_n), \mathrm{II}(e_l, e_n))\Big]\Big]$$

$$\geq$$

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$$+\hat S(\mathrm{I\!I}(e_l,e_n),\mathrm{I\!I}(e_l,e_n)) - \sum_{\alpha,\beta=1}^{n+m}\hat g^{\alpha\beta}\hat R(e_l,e_n,e_l,\hat E_\alpha)\hat S(\hat E_\beta,e_n)$$

$$+S(e_n,e_n)\Big[\hat R(e_l,e_n,e_l,e_n) - \hat g(\mathrm{I\!I}(e_l,e_n),\mathrm{I\!I}(e_l,e_n))\Big]\Big]$$

$$\geq S(e_n,e_n)\sum_{i=1}^{n}[\hat R(e_l,e_n,e_l,e_n) - cS(e_l,e_l)]$$

□

**Corollary (A priori lower bound for the maximum of $S(e_n, e_n)$)**

*If $(z, e_n) \in T\Sigma$ maximize $S$ locally, then $0 \geq (\Delta_g S)(e_n, e_n)$ hence*

$$\mu_n := S(e_n,e_n) \geq \frac{1}{c}\sum_{l=1}^{n}\hat R(e_l,e_n,e_l,e_n) = \frac{1}{c}\mathrm{tr}_\Sigma \hat R(\cdot,e_n,\cdot,e_n).$$

# Optimal transport

$b(x, y)$ 'benefit' per unit mass transported from $x \in \Omega$ to $\bar{x} \in \bar{\Omega}$
$\Omega, \bar{\Omega} \subset\subset \mathbf{R}^n$ open and bounded (or oriented manifolds); 'landscapes';
n-forms $0 < \mu, \bar{\mu}$ on $\Omega, \bar{\Omega}$; normalized densities of supply and demand
$\mu(x) = \rho(x) dx^1 \wedge \cdots \wedge dx^n$ and $\bar{\mu}(\bar{x}) = \bar{\rho}(\bar{x}) d\bar{x}^1 \wedge \ldots d\bar{x}^n$

MONGE (1781): seek

$$\sup_{F_{\#}\mu = \bar{\mu}} \int_{\Omega} b(x, F(x)) \mu$$

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$$\sup_{F_{\#}\mu = \bar{\mu}} \int_\Omega b(x, F(x))\mu = \min_{b \le u \oplus \bar{u}} \int_\Omega u\mu + \int_{\bar{\Omega}} \bar{u}\bar{\mu}$$

KANTOROVICH (1942)

- $\det DF(x) = \pm \rho(x)/\bar{\rho}(F(x))$ if $F : \Omega \longrightarrow \bar{\Omega}$ is a diffeomorphism

# HYPOTHESES (Ma-Trudinger-Wang '05)

(A0) $b \in C^4(\mathrm{cl}(\Omega \times \bar{\Omega}))$ and for each $x \in \mathrm{cl}(\Omega)$:

(A1) $\bar{x} \in \mathrm{cl}(\bar{\Omega}) \mapsto D_x b(x, \bar{x}) := (\frac{\partial b}{\partial x^1}, \ldots, \frac{\partial b}{\partial x^n})$ is a diffeomorphism;

(A2) with convex range $\bar{\Omega}_x = D_x b(x, \mathrm{cl}(\bar{\Omega}))$

DEFN: $t \in [0, 1] \mapsto (x, \bar{x}_t) \in \mathrm{cl}(\Omega \times \bar{\Omega})$ is called a $b$-segment if

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$$\frac{d^2}{dt^2}[D_x b(x, \bar{x}_t)] = 0 \qquad \forall t \in [0,1]$$

Assume $b^*(\bar{x}, x) := b(x, \bar{x})$ also satisfies (A0)-(A2) and

$$(A3) \qquad \frac{\partial^2 b}{\partial s \partial t}\bigg|_{s=0=t}(x_s, \bar{x}_t) = 0 \quad \Longrightarrow \quad \frac{\partial^4 b}{\partial^2 s \partial^2 t}\bigg|_{s=0=t}(x_s, \bar{x}_t) > 0$$

whenever $(x_0, \bar{x}_t)$ is a $b$-segment (and $(x_s)_{s \in [0,1]} \in C^2$)

**Theorem (Gangbo '95, Levin '96; Gangbo–McCann '95–'96)**

*If (A0–A1) a unique minimizer $F_{\#}\mu = \bar{\mu}$ exists.*

**Theorem (Ma–Trudinger–Wang '05; interior regularity)**

*If also (A2–A3) and $\log\rho, \log\bar{\rho} \in C^{k,\alpha}$, for $k \geq 2$ and $0 < \alpha < 1$, then $F \in C^{k+1,\alpha}_{loc}(\Omega, \bar{\Omega})$.*

- first regularity result for an open class of costs $c = -b$
- subsequent improvements / related results by many authors
- Loeper '10: if $\overline{(A3)}$ fails $\exists \log\rho, \log\bar{\rho} \in C^\infty$ for which $F$ discontinuous

# A geometric view (Kim–M. '10)

RMK: Kantorovich $\gamma = (id \times F)_{\#}\mu$ satisfies $\Delta \geq 0$ on $\Sigma \times \Sigma := (\mathrm{spt}\gamma)^2$, where

$$\begin{aligned}\Delta(x, \bar{x}; x_0, \bar{x}_0) &= b(x, \bar{x}) + b(x_0, \bar{x}_0) - b(x, \bar{x}_0) - b(x_0, \bar{x}) \\ &=: \Delta_0(x, \bar{x}).\end{aligned}$$

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Fix $(x_0, \bar{x}_0) \in \hat{M} := \Omega \times \bar{\Omega}$. Taylor expanding $\Delta_0(x, \bar{x})$ around $(x_0, \bar{x}_0)$ yields

$$\Delta_0(x_0 + \delta x, \bar{x}_0 + \delta \bar{x}) \quad =$$

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$$\Delta_0(x_0 + \delta x, \bar{x}_0 + \delta \bar{x}) = \frac{1}{2}(\delta x, \delta \bar{x})\, Hess\, \Delta_0 \begin{pmatrix} \delta x \\ \delta \bar{x} \end{pmatrix} + O(|\delta x| + |\delta \bar{x}|)^3$$

$$= \sum_{i,j=1}^{n} \delta x^i \delta \bar{x}^j \frac{\partial^2 b}{\partial x^j \partial \bar{x}^j} + O(|\delta x| + |\delta \bar{x}|)^3$$

- $\hat{h} := Hess_{(x_0, \bar{x}_0)} \Delta_0$ is pseudo-Riemannian since $\det \frac{\partial^2 b}{\partial x^i \partial \bar{x}^j} \neq 0$ by (A1)
- its signature is $(n, n)$ since $(\delta x, \pm \delta \bar{x})$ flips the sign of the sum above
- $\Sigma :=$spt$\gamma$ is *rectifiably nontimelike*, since $h = \hat{h}|_{T\Sigma^2} \geq 0$ by RMK above.

- e.g. for $b(x, y) = x \cdot y$,

$$\begin{aligned} \Delta_0(x, y) &:= b(x, y) + b(x_0, y_0) - b(x, y_0) - b(x_0, y) \\ &= (x - x_0) \cdot (y - y_0) \\ &\text{and} \\ Hess_{(x_0, y_0)} \Delta_0 &= \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \end{aligned}$$

- more generally,

$$\begin{aligned} \hat{h} := Hess_{(x_0, y_0)} \Delta_0 &= \begin{bmatrix} 0 & D^2_{x^i y^j} b(x_0, y_0) \\ D^2_{x^i y^j} b(x_0, y_0)^T & 0 \end{bmatrix} \\ &\text{so} \\ \Delta_0(x_0 + \delta x, y_0 + \delta y) &= -\Delta_0(x_0 + \delta x, y_0 - \delta y) + l.o.t. \end{aligned}$$

- thus $\hat{h}$ has signature $(n, n)$, depends only on $b$
- (Kim-M. '10) (A2) $\Leftrightarrow$ geodesic convexity of each $\{x\} \times \bar{\Omega}$ in $(\Omega \times \bar{\Omega}, \hat{h})$
- note $\{x\} \times \bar{\Omega}$ and similarly $\Omega \times \{\bar{x}\}$ are both $\hat{h}$-null

# Conformal and calibrated geometries

THM (Kim–M. '10) If (A0)-(A2) then (A3) $\Leftrightarrow$
$\hat{R}(p \oplus 0, 0 \oplus \bar{p}, p \oplus 0, 0 \oplus \bar{p}) > 0$ whenever $\hat{h}(p \oplus 0, 0 \oplus \bar{p}) = 0$.

**Theorem (Kim–M.–Warren '10 spacelike maximizing)**

*b-optimality of $\gamma$ implies $\Sigma =$spt$(\gamma)$ is volume maximizing (wrt compactly supported perturbations) for a conformally equivalent metric $\hat{g} = \chi \hat{h}$, with conformal factor $\chi(x, \bar{x}) > 0$ chosen so that the volume form $\mathrm{vol}_{\hat{g}} = \pm \mu \wedge \bar{\mu}$, (i.e. has Lebesgue density $\rho(x)\bar{\rho}(\bar{x})$ on $\hat{M} = \Omega \times \bar{\Omega}$).*

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- In particular $\Sigma$ has zero mean curvature wrt the metric $\hat{g}$.
- above characterizations of (A2) and (A3) also work with $\hat{g}$ in place of $\hat{h}$.

Proof sketch: $\Phi = \frac{1}{2}(\mu + \bar{\mu})$ is a calibration of $\Sigma$; i.e. $d\Phi = 0$ and $\Phi_z(\wedge_{i=1}^n v_i) \geq \| \wedge_{i=1}^n v_i \|_{\hat{g}}$ on the $n$-Grassmannian$^+$ of $\hat{M}$ with equality a.e. on $\wedge^n T_z \Sigma$, so for $\Sigma - \Sigma' = \partial \Lambda$

$$\mathop{\mathrm{vol}}_g \Sigma = \int_\Sigma \Phi = \int_{\Sigma'} \Phi \geq \mathrm{vol}_g \Sigma'.$$

Fix $(s_{ij}) > 0$ (say Euclidean) on $\Omega$. Then the induced Riemannian metric

$$\hat{S} := \sum_{i,j=1}^{n} \left( s_{ij} dx^i \otimes dx^j + \chi^2 \sum_{k,l=1}^{n} s^{ij} \frac{\partial^2 b}{\partial x^i \partial \bar{x}^k} \frac{\partial^2 b}{\partial x^j \partial \bar{x}^l} d\bar{x}^k \otimes d\bar{x}^l \right)$$

satisfies $\mathrm{vol}_{\hat{S}} = \mu \wedge \bar{\mu}$ on $\hat{M} = \Omega \times \bar{\Omega}$. (A0–A3) yields $\kappa > 0$ such that

$$\hat{R}_{\hat{g}}(p \oplus 0, 0 \oplus \bar{p}, p \oplus 0, 0 \oplus \bar{p}) \geq \kappa |p \wedge \bar{p}|^2_{\hat{S}} \qquad \forall \ \mathrm{null} \ (z, p \oplus \bar{p}) \in T\hat{M}.$$

**Theorem (Brendle–Leger–M.–Rankin '24 apriori spacelike estimate)**

*If $0 \leq \hat{\phi} \in C_c^\infty(\Omega \times \bar{\Omega})$ and $F_\# \mu = \bar{\mu}$ is a smooth b-optimal diffeomorphism then*

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**Theorem (Brendle–Leger–M.–Rankin '24 apriori spacelike estimate)**

*If $0 \leq \hat{\phi} \in C_c^\infty(\Omega \times \bar{\Omega})$ and $F_\# \mu = \bar{\mu}$ is a smooth b-optimal diffeomorphism then*

$$0 < (\kappa \phi^2)^{n-1} S \leq cg$$

*on $\Sigma =$ Graph$(F) \subset \Omega \times \bar{\Omega} = \hat{M}$, where $(\phi, S, g) = (\hat{\phi}, \hat{S}, \hat{g})|_\Sigma$ and $c = c(\|\hat{g}, \hat{g}^{-1}, \hat{S}, \hat{\phi}\|_{C^2(\mathrm{spt}\,\hat{\phi})}, \|\log \frac{\mu}{\mathrm{vol}_s}\|_{C^0})$ is independent of $\mu, \bar{\mu}$.*

- after this regularity follows by local replacement using continuity method

Proof sketch: Kantorovich dual potentials satisfy

$$u(x) + \bar{u}(\bar{x}) - b(x, \bar{x}) \geq 0$$

on $\hat{M}$ with equality on $\Sigma =$Graph$(F)$. Thus

$$\begin{aligned} Du(x) - D_x b(x, F(x)) &= 0 \qquad (FOC) \\ D^2 u(x) - D^2_{xx} b(x, F(x)) &\geq 0. \qquad (SOC) \end{aligned}$$

Differentiating (FOC) yields

$$D^2 u - D^2_{xx} b(x, F(x)) = D^2_{x\bar{x}} b(x, F(x)) DF(x)$$

whose determinant

$$\log \det[D^2 u - D^2_{xx} b(x, F(x))] = \log \left| \frac{\rho}{\bar{\rho}} \det D^2_{x\bar{x}} b \right|_{\bar{x} = F(x)} \in L^\infty$$

is bounded by the asserted constants. At least (SOC) becomes strict.

- If uniform, the PDE is uniformly elliptic and Schauder theory applies.

If $z = (x, F(x))$ maximizes the largest eigenvalue of $\phi^{2(n-1)} S$ relative to $g$, we can extend the Euclidean coordinates $(x^1, \ldots, x^n)$ which diagonalize $\Lambda := (D^2 u - D^2_{xx} b(x_0, F(x_0)) \chi > 0$ to Riemannian normal coordinates for $\hat{S}$. Taking $p_i = \frac{\partial}{\partial x^i}$ to be the eigenvector of $\Lambda$ with eigenvalue $\lambda_i$, we can build a $g$-orthonormal basis $e_i = \frac{1}{\sqrt{2}}(\frac{p_i}{\sqrt{\lambda_i}} \oplus \sqrt{\lambda_i} \bar{p}_i)$ for $T_z \Sigma$ where

$$\bar{p}_i = \lambda_i^{-1} \sum_{k=1}^{n} \frac{\partial F^k}{\partial x^i} \frac{\partial}{\partial \bar{x}^k}.$$

Moreover $S(e_i, e_j) = \mu_i \delta_{ij}$ with $\mu_i = \frac{\lambda_i + \lambda_i^{-1}}{2}$. Ordering the eigenvalues so $\mu_i \leq \mu_n$, multilinearity and the special structure of the Riemann tensor $\hat{R}_{\hat{g}}$ yield

If $z = (x, F(x))$ maximizes the largest eigenvalue of $\phi^{2(n-1)}S$ relative to $g$, we can extend the Euclidean coordinates $(x^1, \ldots, x^n)$ which diagonalize $\Lambda := (D^2 u - D^2_{xx} b(x_0, F(x_0)))\chi > 0$ to Riemannian normal coordinates for $\hat{S}$. Taking $p_i = \frac{\partial}{\partial x^i}$ to be the eigenvector of $\Lambda$ with eigenvalue $\lambda_i$, we can build a $g$-orthonormal basis $e_i = \frac{1}{\sqrt{2}}(\frac{p_i}{\sqrt{\lambda_i}} \oplus \sqrt{\lambda_i}\bar{p}_i)$ for $T_z\Sigma$ where

$$\bar{p}_i = \lambda_i^{-1} \sum_{k=1}^{n} \frac{\partial F^k}{\partial x^i} \frac{\partial}{\partial \bar{x}^k}.$$

Moreover $S(e_i, e_j) = \mu_i \delta_{ij}$ with $\mu_i = \frac{\lambda_i + \lambda_i^{-1}}{2}$. Ordering the eigenvalues so $\mu_i \leq \mu_n$, multilinearity and the special structure of the Riemann tensor $\hat{R}_{\hat{g}}$ yield

$$\hat{R}(e_i, e_n, e_i, e_n) \geq \frac{\kappa}{4}(\frac{\lambda_n}{\lambda_i} + \frac{\lambda_i}{\lambda_n}) - c(\mu_i + \mu_n + 1).$$

The arithmetic-geometric mean $\neq$ and determinant $\prod_{i=1}^n \lambda_i$ bounds give

$$\sum_{i=1}^{n-1} \hat{R}_{\hat{g}}(e_i, e_n, e_i, e_n) \quad \geq \frac{\kappa}{c} \mu_n{}^{\frac{n}{n-1}} - c\mu_n.$$

But our Corollary bounds this sum $\leq c\phi^{-2}\mu_n$, hence $\mu_n$ is bounded! □

# References: www.math.toronto.edu/mccann/publications


S Brendle, F Léger, RJ McCann, C Rankin.
A geometric approach to apriori estimates for optimal maps.
*J. Reine Angew. Math.* **187** (2024) 251–266.

Y-H Kim and RJ McCann.
Continuity, curvature, and the general covariance of optimal transportation.
*J. Eur. Math. Soc. (JEMS)* **12** (2010) 1009–1040.

Y-H Kim, RJ McCann and M Warren.
Pseudo-Riemannian geometry calibrates optimal transportation.
*Math. Res. Lett.* **17** (2010) 1183–1197.

RJ McCann.
A glimpse into the differential topology and geometry of optimal transport.
*Discrete Contin. Dyn. Syst.* **34** (2014) 1605–1621.

Thank you!