Fibonacci IDIS 303 Oct. 10, 2006

The Fibonacci sequence $\{u_n\}$ starts with 0 and 1, and then each term is obtained as the sum of the previous two:

$$u_n = u_{n-1} + u_{n-2}$$

The first fifty terms are tabulated at the right.

This is certainly the most famous "sequence" in mathematics—most high school students seem to know something about it, and it is also an object of continued study by mathematicians. In fact, it has spawned a professional journal, *The Fibonacci Quarterly*. Here we will devote a number of sections to a few quite different aspects of this sequence, all the way from arithmetic to pine cones.

Here, we look for arithmetic patterns in the numbers—an excellent activity for small group work. I simply hand the students the list of the first 50 numbers, and tell them to *Go nuts!*

Of course there's lots of patterns having to do with squares. Such as:

$$3^2 + 5^2 = 34$$
$$5^2 + 8^2 = 89$$

and

$$5^2 - 2^2 = 21$$
$$8^2 - 3^2 = 55$$

and

$$8^2 - 5^2 = 39 = 3\times13$$
$$13^2 - 8^2 = 105 = 5\times21$$

And here's one with products:

$$2\times5 + 3\times8 = 34$$
$$3\times8 + 5\times13 = 89$$

These become quite satisfying when we use the u_n notation, because we can see how the subscripts relate. Rather than write the formulae in general, I'll rewrite the above cases—it's easier to see the general pattern with specific subscripts.

$$u_4^2 + u_5^2 = u_9$$
$$u_5^2 + u_6^2 = u_{11}$$

and

$$u_5^2 - u_3^2 = u_8$$
$$u_6^2 - u_4^2 = u_{10}$$

and

$$u_6^2 - u_5^2 = u_4\times u_7$$
$$u_7^2 - u_6^2 = u_5\times u_8$$

$$u_3\times u_5 + u_4\times u_6 = u_9$$
$$u_4\times u_6 + u_5\times u_7 = u_{11}$$

and crazy stuff like:

$$u_5^3 + u_4^3 - u_3^3 = u_{12}$$
$$u_6^3 + u_5^3 - u_4^3 = u_{15}$$

u_1	1
u_2	1
u_3	2
u_4	3
u_5	5
u_6	8
u_7	13
u_8	21
u_9	34
u_{10}	55
u_{11}	89
u_{12}	144
u_{13}	233
u_{14}	377
u_{15}	610
u_{16}	987
u_{17}	1597
u_{18}	2584
u_{19}	4181
u_{20}	6765
u_{21}	10946
u_{22}	17711
u_{23}	28657
u_{24}	46368
u_{25}	75025
u_{26}	121393
u_{27}	196418
u_{28}	317811
u_{29}	514229
u_{30}	832040
u_{31}	1346269
u_{32}	2178309
u_{33}	3524578
u_{34}	5702887
u_{35}	9227465
u_{36}	14930352
u_{37}	24157817
u_{38}	39088169
u_{39}	63245986
u_{40}	102334155
u_{41}	165580141
u_{42}	267914296
u_{43}	433494437
u_{44}	701408733
u_{45}	1134903170
u_{46}	1836311903
u_{47}	2971215073
u_{48}	4807526976
u_{49}	7778742049
u_{50}	12586269025

One of my students recently started with the observation that the sequence 34,55,89 is enticingly related to the earlier sequence 3,5,8. Indeed: $\begin{Bmatrix} 34-1 \\ 55 \\ 89-1 \end{Bmatrix} = 11 \times \begin{Bmatrix} 3 \\ 5 \\ 8 \end{Bmatrix}$. I'll leave this one with you.

Sometimes I find that the students are interested in trying to find "proofs" of these, perhaps with some inductive thinking, but often they seem much happier just to explore. Occasionally a "physical" model of the Fibonacci numbers or just the right geometric picture (problem 2 below), can give us an unexpected argument for some of these identities. Those are my favorite proofs. But here we will look briefly at the inductive approach.

Mathematical Induction.

Mathematical induction provides one of the standard ways to establish formulae like those presented above. It works particularly naturally for Fibonacci number properties as the numbers themselves are generated inductively. Sometimes the inductive argument is straightforward, and sometimes it's not and requires some ingenuity, and that's always fun.

As an example of an inductive argument, I choose the pleasing observation that the square of any Fibonacci number differs by 1 from the product of the two adjacent numbers:

$$u_n^2 = u_{n-1} \times u_{n+1} \pm 1$$

Here the signs alternate, being 1 for n odd and –1 for n even. For example:

$$8^2 = 5 \times 13 - 1$$
$$13^2 = 8 \times 21 + 1$$

When I'm doing an inductive proof, I usually work with numbers instead of symbols, keeping in mind what the numbers mean. In this case, I will start with the expression $13^2 - 8 \times 21$ and I want to "show" that this is equal to +1 and along the way I want to use the fact that $8^2 - 5 \times 13$ is equal to –1. And the trick is to take the 21 (which plays the role of u_{n+1} which is the "highest" subscript to occur) and break it down following the Fibonacci rule. Thus:

$$13^2 - 8 \times 21 = 13^2 - 8 \times (8 + 13) = 13^2 - 8 \times 13 - 8^2 = 5 \times 13 - 8^2 = 1$$

and I'm done. I maneouvered the expression into a form in which I had an 8^2 to work with, and in the last step I used the fact that $8^2 - 5 \times 13 = 1$. What I've done, essentially, is to show that the expression $u_n^2 - u_{n-1}u_{n+1}$ changes sign when n is increased by 1. This will establish the formula for all n if we just "get it started" and for that we can check that $u_1^2 - u_0 u_2 = 1$, which certainly holds.

If I want to construct a rigorous proof using the u_n notation, I can now use the above calculation as a template. I start by assuming

$$u_n^2 = u_{n-1}u_{n+1} \pm 1$$

We then calculate:

$$u_{n+1}^2 - u_n u_{n+2} = u_{n+1}^2 - u_n(u_{n+1} + u_n) = u_{n+1}(u_{n+1} - u_n) - u_n^2 = \mp 1$$

And we are done.

Problems

1. What I often do, after I give the class the table of the first 50 Fibonacci numbers to pour over, is to say: who can tell me the sum of all the Fibonacci numbers on the page: $u_0+u_1+u_2+...+u_{50}$? I like this challenge because it recalls the famous question that Gauss was given at age 12 when he was asked to sum the first 100 natural numbers. In that case, the teacher was hoping to keep the class quiet for a hour, but Gauss came up with the answer in a few moments.

2. A formula that looks like it should be easy meat for induction is the sum of squares, for example,

$$5^2 + 8^2 = 89.$$

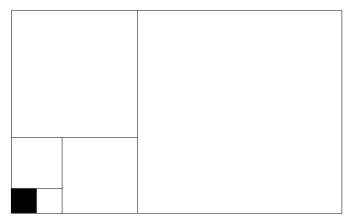
But I am having trouble with this one! Can you see how to do it?

3. The difference of squares formula, of which an example is $u_7^2 - u_6^2 = u_5 \times u_8$ is easy to prove directly. The left hand side is dying to be factored.

4.(a) Verify that

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8\times 13.$$

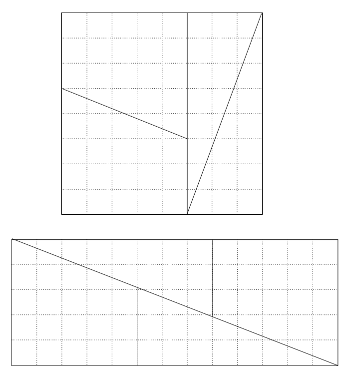
Does this generalize? Show that a geometric proof of this identity can be obtained from the picture at the right. Does this proof also generalize?



(b) This is a particularly simple formula to establish by induction. Provide the argument.

5. Is the relationship $\begin{Bmatrix} 34-1 \\ 55 \\ 89-1 \end{Bmatrix} = 11 \times \begin{Bmatrix} 3 \\ 5 \\ 8 \end{Bmatrix}$ part of a general pattern? [Hint: try to construct a similar relationship with 2,3,5 in the right hand bracket instead of 3,5,8.]

6. The identity $8^2 = 5\times 13 - 1$, and its generalizations discussed in the math induction example above, give rise to a famous geometrical paradox illustrated by the diagram at the right. The rectangle and the square are composed of the same 4 pieces, yet the rectangle has area 65 and the square has area 64. Go figure.



7. Take the Fibonacci sequence 1 1 2 3 5 8 13 etc. and divide the first term by 100, the second by 1000, the third by 10000, etc. and then add them all up (to infinity)—the sum is 1/89. Wow.
[Hint: The standard approach to finding the sum of an infinite geometric series is to multiply the series by something (r) and then subtract the two versions of the series, one from the other, and see what we get. The same type of trick might work here.]

8. Verify that $(3\times13)^2 + (2\times5\times8)^2 = 89^2$. Is this part of a general relationship? This is amusing because it gives us a family of Pythagorean triangles.

9. $1\times1 + 1\times2 + 2\times3 + 3\times5 + 5\times8 + 8\times13 = ?$ Generalize?

10. $\frac{1}{1\times2}+\frac{1}{1\times3}+\frac{1}{2\times5}+\frac{1}{3\times8}+\frac{1}{5\times13}+\frac{1}{8\times13}+\ldots=1$

Can you establish this?

11. Take any row of Pascal's triangle. Multiply the nth entry of the row by u_n, and add everything up. What do you get? For example, for the fifth row:

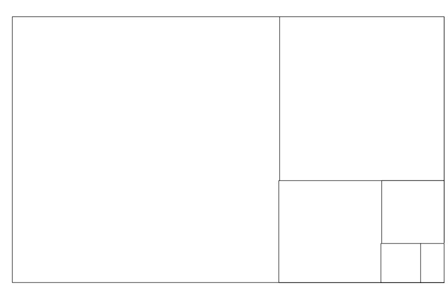
$$1\times1 + 5\times1 + 10\times2 + 10\times3 + 5\times5 + 1\times8 = ?$$

[You get a formula of Cesaro.]

$$\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1&&&&&&&&&& \\
&&&&&&&&&1&&1&&&&&&&&& \\
&&&&&&&&1&&2&&1&&&&&&&& \\
&&&&&&&1&&3&&3&&1&&&&&&& \\
&&&&&&1&&4&&6&&4&&1&&&&&& \\
&&&&&1&&5&&10&&10&&5&&1&&&&& \\
&&&&1&&6&&15&&20&&15&&6&&1&&&& \\
&&&1&&7&&21&&35&&35&&21&&7&&1&&& \\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&& \\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1& \\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1
\end{array}$$

12. Show that the quotients, 1/1, 2/1, 3/2, 5/3, 8/5, ... of successive Fibonacci numbers approach the golden ratio τ defined as the positive root of the equation $x^2 = x+1$.
[Find a recursive relationship for the quotients $r_n = u_n / u_{n-1}$].

13. The Greeks had the idea that the rectangle of the most pleasing proportions is the one with the property that if you cut out a square, what you are left with has the same shape as before. That means we can keep going removing squares forever, and never lose the shape. Show that the two sides of this rectangle are in ratio τ, the golden mean.



14. (A problem of Steinhaus) Consider the sequence beginning with $x = 1$ and $x = a$ which satisfies the recursion:

$$x_n = x_{n-2} - x_{n-1}.$$

Find all values of a for which the sequence has only positive terms. [Hint: solve #12 first, and look carefully at how the sequence in #12 approaches its limit.]