Mode Stability for Charged Scalar Fields on sub-extremal Reissner–Nordström black holes

Flavio Rossetti, Maxime Van de Moortel

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Abstract

We prove mode stability for massless charged scalar fields on Reissner–Nordström black hole exterior regions in the full sub-extremal range, without a smallness assumption on the charge coupling. This complements a previous result of Gajic on the extremal and near-extremal cases.

Formal verification and AI statement. The proof was developed through interactions with AI models (ChatGPT 5.6 and Claude Opus 5.4), which were prompted to adapt the mode-stability approach introduced by P. Hintz in *Mode stability for the scalar wave equation on subextremal Kerr–de Sitter spacetimes* [1]. The argument presented here closely follows Hintz's approach. The result answers a question of interest to the general relativity community. The key ideas underlying the adaptation were generated by the AI models; the authors have subsequently worked through the argument in detail and rewritten the proof, which will also be appended to our forthcoming proof of decay for charged scalar fields on sub-extremal Reissner–Nordström black holes using Fourier-based methods [2]. They take full responsibility for the correctness of the mathematical statements and proofs. The present proof of mode stability has also been formally verified in Lean 4. The formalization is available here.

Introduction

We consider the massless charged wave equation with a fixed electromagnetic potential

$$g^{\mu\nu} D_\mu D_\nu \phi = 0, \quad D_\mu = \nabla_\mu + i q_0 A_\mu, \qquad A = \frac{e}{r}\, dt \tag{1}$$

on the exterior of a sub-extremal Reissner–Nordström black hole,

$$g = -\Omega^2 dt^2 + \Omega^{-2} dr^2 + r^2 d\omega^2_{\mathbb{S}^2}, \qquad \Omega^2 = 1 - \frac{2M}{r} + \frac{e^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2}, \tag{2}$$

where $M > 0$, $0 < |e| < M$, $r_\pm = M \pm \sqrt{M^2 - e^2}$, and $r > r_+$. Here ∇ is the Levi–Civita connection of g, ϕ is complex-valued, and $q_0 \in \mathbb{R} \setminus \{0\}$ is its charge coupling. Both the metric g and the electromagnetic potential A are fixed.

The purpose of this note is to prove mode stability for every sub-extremal Reissner–Nordström exterior, every angular frequency, and every charge coupling. More precisely, we prove that any solution to (1) of the form

$$\phi(t, r, \theta, \varphi) = e^{-i\omega t} \frac{u(r)}{r} Y_L(\theta, \varphi), \tag{3}$$

with $\Im\omega \geq 0$, $\omega \neq 0$, satisfying the standard outgoing regularity conditions at the event horizon and at infinity (see next paragraphs), has to be the trivial solution. Here Y_L denotes any nonzero

spherical harmonic of degree L on the sphere, satisfying

$$-\Delta_{\mathbb{S}^2} Y_L = L(L+1)Y_L, \qquad L \in \mathbb{N}_0.$$

The proof is an adaptation of Hintz's argument for the scalar wave equation on Kerr–de Sitter [1]. In that argument, two currents are constructed, one on each side of a frequency-dependent radius r_0 between the event horizon and the cosmological horizon. Each current has monotonicity properties and is associated to the solution conjugated by an oscillatory phase. Evaluated at r_0, the two currents give opposite-sign bounds on the same quantity, which forces the solution to vanish. This proves mode stability for real superradiant frequencies, which is the central part of Hintz's work.

In the charged scalar field context considered here, the approach needs to be minimally modified to account for the asymptotically flat end. A key aspect, which does not appear in the cosmological setting, is a specific identity satisfied on the Reissner–Nordström black hole exterior, see already Lemma 1.5. For sufficiently small $|q_0 e|$, mode stability for a charged scalar field follows by perturbing the non-charged equation; see [12, Section 4] and [13, Section 1.1.7]. Our result removes this smallness assumption throughout the sub-extremal range.

To introduce the notation of the main statements, we consider the (t, r, θ, φ) coordinates of (2). We write $dx/dr = \Omega^{-2}$ and notice that the radial equation for u (coming from (1)) becomes

$$\partial_x^2 u + \left[\left(\omega - \frac{q_0 e}{r}\right)^2 - \Omega^2 \left(\frac{L(L+1)}{r^2} + \frac{\partial_r \Omega^2}{r}\right)\right] u = 0. \tag{4}$$

Definition 1 (Mode solution)**.** *We say that a nonzero solution ϕ to* (1) *is a mode solution if*

$$\phi(t, r, \theta, \varphi) = e^{-i\omega t} \frac{u(r)}{r} Y_L(\theta, \varphi),$$

with $\omega \neq 0$, where $u(r)$ is such that

$$u = B_H e^{-i\left(\omega - \frac{q_0 e}{r_+}\right)x} \left(1 + o(1)\right), \qquad \text{as } x \to -\infty, \tag{5}$$

$$u = B_I e^{i\omega x} r^{-iq_0 e} \left(1 + O(r^{-1})\right), \qquad \text{as } x \to +\infty, \tag{6}$$

for some non-zero coefficients B_H and B_I. We say that ϕ is an exponentially growing mode solution if the above is satisfied in the case $\Im(\omega) > 0$. We say that ϕ is a real mode solution if the above is satisfied with $\omega \in \mathbb{R} \setminus \{0\}$.

The case $q_0 e = 0$ corresponds to the uncharged wave equation. We henceforth assume $q_0 e > 0$; the result for the opposite sign follows by sending $(\omega, q_0 e)$ to $(-\overline{\omega}, -q_0 e)$. Set

$$\omega_{\text{res}} := \frac{q_0 e}{r_+}, \qquad \lambda := q_0 e, \qquad \sigma := \frac{\omega}{\omega_{\text{res}}}, \qquad \kappa := \frac{r_+ - r_-}{r_+}.$$

In the proof below, we divide r and x by r_+ and reuse the same notation for the rescaled coordinates. This gives the forthcoming (1.4).

Definition 2. *Let $0 < \kappa \leq 1$, $\lambda > 0$, $\Im\sigma \geq 0$, $\sigma \neq 0$, $L \in \mathbb{N}_0$. Let u_H and u_I be the unique*[1] *solutions to* (4) *satisfying, respectively,* (5) *and* (6)*, with the normalizations $B_H = B_I = 1$. Then, we say that* mode stability *holds for* (1) *if the Wronskian*

$$\mathcal{W}_{\kappa,\lambda}(\sigma, L) := \mathcal{W}(u_H, u_I) = (\partial_x u_H)\, u_I - u_H\, (\partial_x u_I)$$

satisfies

$$\mathcal{W}_{\kappa,\lambda}(\sigma, L) \neq 0.$$

[1]The existence and uniqueness statement is an application of results on ODE singularities, see e.g. [3, 2].

Our theorem 1.2 states that mode stability holds for (1), i.e. that equation (1) does not admit any mode solution with $\Im\omega \geq 0$ in the sense of Definition 1.

The main obstruction to proving mode stability is *superradiance*. In particular, suppose that ϕ is a real mode solution. Then, (4) implies that $\partial_x \Im(u'\overline{u}) = 0$ (see already section 1.9). Therefore, the boundary conditions (5) and (6) lead to the relation

$$\omega|B_I|^2 + (\omega - \omega_{\text{res}})|B_H|^2 = 0. \tag{7}$$

We define the superradiant interval

$$0 < \omega < \omega_{\text{res}} = \frac{q_0 e}{r_+}, \qquad \text{equivalently} \qquad 0 < \sigma < 1. \tag{8}$$

Outside the superradiant interval, (7) gives a contradiction, showing that no real mode solutions exist for such frequencies. In the superradiant interval (8), however, the two terms in (7) have opposite signs and no contradiction arises in such a range. Indeed, the flux of the conserved stationary energy into the horizon has the sign of $\omega(\omega - \omega_{\text{res}})$. The intuition behind this mechanism is that, although the metric is static, the interaction with the electric field allows the scalar field to extract energy from the black hole [4]; see also the numerical study [7]. Charged superradiance is related to the Klein phenomenon for electrostatic potentials, where amplification is compatible with an indefinite conserved energy; see Bachelot [5] for a rigorous scattering theory and a conjecture on mode stability for charged scalar fields. In particular, a simple positivity argument does not suffice to prove mode stability for frequencies in the range (8) for charged scalar fields, unlike in the uncharged case.

Likewise, for $\Im\omega > 0$, integrating $\partial_x \Im(u'\overline{u})$ in $\mathbb{R}$ confines a hypothetical growing mode to $0 < \Re\omega < \omega_{\text{res}}$; however it does not exclude it (see already lemma 1.3).

The distinction between fixing and evolving the electromagnetic field is relevant here. In the Maxwell–charged-scalar-field system the metric is fixed but A evolves together with ϕ through the Maxwell equations,

$$\nabla^\mu F_{\mu\nu} = \frac{iq_0}{2}(\phi\overline{D_\nu\phi} - \overline{\phi}D_\nu\phi), \qquad F = dA.$$

This system is nonlinear, and the sum of the Maxwell and scalar stress-energy tensors is conserved. In [11], boundedness, integrated local energy decay, and inverse-polynomial decay were proved for the above nonlinear system in spherical symmetry on fixed sub-extremal Reissner–Nordström and Schwarzschild exteriors, under smallness assumptions on the Maxwell charge and the initial scalar field energy.

In the Einstein–Maxwell–charged-scalar-field system, the metric also evolves. The equation studied here is the scalar equation in the linearization of that system around Reissner–Nordström with $\phi = 0$: the scalar stress-energy tensor and Maxwell current are quadratic in ϕ, so scalar perturbations decouple at first order. The decay of charged fields in the exterior is also relevant to the interior problem and to Strong Cosmic Censorship. In spherical symmetry, the results of [18] establish stability of the Cauchy horizon and curvature blow-up under quantitative decay assumptions on the event horizon. The work [19] showed that, when this decay is slow, its oscillatory character can determine whether the metric and scalar field extend continuously across the Cauchy horizon. On Reissner–Nordström–de Sitter, exponential local energy decay holds for massive charged Klein–Gordon fields when the charge product is sufficiently small at fixed positive mass [8]; for sufficiently small positive cosmological constant, exponentially growing modes occur in a regime of small charge and mass that includes the massless case [9]. Prescribing such a decay rate in the

interior region allows one to construct H^1 metric extensions beyond the Cauchy horizon of solutions to the spherically symmetric Einstein-Maxwell–charged scalar field system, when considering smooth initial data [10]

For the linear equation with fixed electromagnetic potential in the asymptotically flat case, Hod and Piran [6] predicted inverse-power tails, with oscillations along the event horizon. Gajic [12, 13] recently proved energy boundedness, integrated energy decay, and precise late-time asymptotics on extremal and sufficiently near-extremal Reissner–Nordström black holes, without a smallness assumption on $|q_0 e|$. Away from extremality, their extension to arbitrary coupling was conditional on real-frequency mode stability. Related late-time asymptotics for inverse-square potentials were obtained by Gajic [20] on Schwarzschild and by Gajic and the second author [21] on Minkowski; the latter work also treats a charged wave equation with a fixed Coulomb field.

Superradiance is also the principal obstruction to the elementary mode-stability argument for neutral waves on Kerr, where it is caused by rotation. Whiting [15] famously overcame this obstruction for exponentially growing modes by an integral transformation of the radial equation, producing an equation to which a positive energy argument applies. Shlapentokh-Rothman [16] proved real-axis mode stability and quantitative estimates using this transformation. See also Andersson, Ma, Paganini, and Whiting [17] for real-axis mode stability of the Teukolsky equation on sub-extremal Kerr. Its adaptation to sub-extremal charged Reissner–Nordström does not remove superradiance completely: the transformed horizon frequency is a positive multiple of $\omega - q_0 e/(r_+ + r_-)$, leaving $0 < \omega < q_0 e/(r_+ + r_-)$ untreated; see [12, Appendix B].

In the extremal case, Gajic [12, Theorem 7.14] instead adapts the Whiting-type integral transformation introduced by Teixeira da Costa [22] for extremal Kerr. The resulting real-frequency Wronskian bounds, away from $\omega = 0$ and $\omega = \omega_{\mathrm{res}}$, persist near extremality by an ODE perturbation argument. Separate estimates treat the threshold regimes. The remaining mode-stability problem at arbitrary coupling away from extremality is the one addressed here. On compact parameter and frequency sets away from the thresholds, the required positive lower bound for the Wronskian follows from nonvanishing and continuous dependence of the normalized outgoing solutions.

Our proof follows Hintz's conjugation and current argument [1]; the key steps are as follows:

- Standard energy and flux identities reduce the analysis of nonzero-frequency modes with $\Im\sigma \geq 0$ to $0 < \Re\sigma < 1$.
- We compactify the black hole exterior, introducing the coordinate $y = 1 - \frac{r_+}{r} \in (0, 1)$.
- We remove the outgoing phases by setting $v_- = e^{i\Phi} u$ near the horizon $\{y = 0\}$ and $v_+ = e^{-i\Phi} u$ near infinity $\{y = 1\}$, with the two conjugations meeting at the transition point $y_0 = 1 - \Re\sigma \in (0, 1)$.
- The conjugated $v_\pm$ admit currents which agree at $y = y_0$. For Reissner–Nordström, the sign needed to compare these currents follows from the explicit factorization in Lemma 1.5.
- The angular contribution becomes the constant $L(L + 1)$; retaining it in the current gives additional nonnegative boundary terms.
- This proves mode stability for every L without a restriction on λ. The same identity also excludes frequencies in the open upper half-plane.
- The case $\sigma = 0$ is treated (see Proposition 1.10) by constructing solutions with prescribed asymptotics near the regular singular point at infinity of the radial ODE; this is a separate argument, independent from the $\omega \neq 0$ analysis.

1 Mode stability for charged scalar fields

While the key argument in this section is strongly inspired by the construction in Hintz [1], the factorization used below is specific to the Reissner–Nordström coefficients considered here.

1.1 Reissner–Nordström coefficients and outgoing solutions

Fix $\lambda > 0$ and $L \in \mathbb{N}_0$, the latter corresponding to the eigenvalue $L(L+1)$ for an L-th spherical harmonic. We normalize the event horizon radius

$$r_+ = 1,$$

and define $0 < \kappa \leq 1$ as twice the surface gravity of the event horizon of a Reissner-Nordström black hole. Then, we set:

$$\Omega^2(r) := \frac{(r-1)(r-1+\kappa)}{r^2}, \qquad r \in (1, \infty), \tag{1.1}$$

which coincides with the function Ω^2 in (2), under our normalization assumptions. The tortoise coordinate x is characterized, up to an additive constant, by

$$\frac{dr}{dx} = \Omega^2(r). \tag{1.2}$$

Thus $x \to -\infty$ at the event horizon $\{r = 1\}$ and $x \to +\infty$ at infinity $\{r = +\infty\}$. We also set

$$K(r) := \frac{2-\kappa}{2r^3} - \frac{1-\kappa}{r^4} > 0. \tag{1.3}$$

In the following, we study

$$\partial_x^2 u + \left[\lambda^2 \left(\sigma - \frac{1}{r}\right)^2 - 2\Omega^2(r)K(r) - L(L+1)\frac{\Omega^2(r)}{r^2}\right] u = 0. \tag{1.4}$$

We note that (1.4) coincides with (4), since the black hole mass and charge are related to κ by the relations $2M = 2 - \kappa$ and $e^2 = 1 - \kappa$.

We then define the energy current $Q[u] := \Im(\overline{u}\,\partial_x u)$: differentiating $Q[u]$ using (1.4) then gives

$$Q[u]'(x) = -2\lambda^2 \Im\sigma \left(\Re\sigma - \frac{1}{r}\right) |u|^2. \tag{1.5}$$

For $\Im\sigma \geq 0$ we define the outgoing solution at the event horizon as the unique solution u_H to (1.4) satisfying

$$e^{i\lambda(\sigma-1)x} u_H = 1 + O(r-1), \qquad \partial_x\Big(e^{i\lambda(\sigma-1)x} u_H\Big) = O(r-1), \quad \text{as } r \to 1. \tag{1.6}$$

For $\sigma \neq 0$, the outgoing solution at infinity is defined using the long-range part of the potential in (1.4). Let Φ be defined by

$$\partial_r \Phi = \frac{\lambda(\sigma - r^{-1})}{\Omega^2(r)} \tag{1.7}$$

Then

$$\Phi(r) = \lambda\sigma r + \lambda\big((2-\kappa)\sigma - 1\big)\log r + O(1) \text{ as } r \to +\infty. \tag{1.8}$$

$$\Phi(r) = \lambda(\sigma - 1)x(r) + C_H + O(e^{\kappa x(r)}) \text{ as } r \to 1. \tag{1.9}$$

Here $C_H \in \mathbb{C}$ depends on the additive constant in the phase. The outgoing solution at infinity is defined as the unique solution u_I to (1.4) satisfying

$$e^{-i\Phi}u_I = 1 + O(r^{-1}), \qquad \partial_r\Big(e^{-i\Phi}u_I\Big) = O(r^{-2}), \quad \text{as } r \to +\infty. \tag{1.10}$$

This coincides with (6), by exploiting the relation between the r and x coordinates in (1.2).

Definition 1.1 (Outgoing at both ends)**.** *We say that a nonzero solution u to* (1.4) *is outgoing at both ends, or, equivalently, corresponds to a mode solution, if both* (1.6) *and* (1.10) *hold with the leading constants* 1 *replaced by independent nonzero amplitudes B_H, B_I.*

Using the Wronskian convention

$$W(f,h) := (\partial_x f)h - (\partial_x h)f,$$

define

$$\mathcal{W}_{\kappa,\lambda}(\sigma, L) := W(u_H, u_I). \tag{1.11}$$

The Wronskian is independent of x (see (1.4)) and its nonvanishing does not depend on the nonzero normalizations of u_H and u_I.

Theorem 1.2 (Mode stability for the charged scalar field)**.** *For every $0 < \kappa \leq 1$, $\lambda > 0$, and $L \in \mathbb{N}_0$, the radial equation* (1.4) *has no nonzero solution outgoing at both ends for any $\sigma \neq 0$ with $\Im\sigma \geq 0$. Equivalently,*

$$\mathcal{W}_{\kappa,\lambda}(\sigma, L) \neq 0 \qquad (\sigma \neq 0,\ \Im\sigma \geq 0). \tag{1.12}$$

At $\sigma = 0$, the same conclusion holds with a specific convention for the outgoing solution at infinity[2] *(see already Proposition 1.10).*

Remark 1.1. *During the preparation of this paper, we learned that Z. Li, S. Ma, and L. Zhang independently obtained the same mode stability result [23]. After a private communication between the two groups, we have jointly decided to make both papers public simultaneously.*

In the proof, we also use (1.6) and (1.10) with the leading constants 1 replaced by arbitrary amplitudes B_H, B_I, retaining the derivative bounds; at $\sigma = 0$, we likewise allow $B_I = 0$ in (1.51). The term outgoing continues to require a nonzero amplitude.

1.2 Localization of a hypothetical mode in the open upper half-plane

Lemma 1.3 (Localization of the real part)**.** *Suppose that $\Im\sigma > 0$ and that a nonzero solution u of* (1.4) *is outgoing at both ends. Then $0 < \Re\sigma < 1$.*

Proof. The outgoing asymptotics (1.6) and (1.10) imply

$$\begin{aligned} |u| + |\partial_x u| &= O\Big(e^{\lambda\Im\sigma\, x}\Big) \quad \text{as } x \to -\infty, \\ |u| + |\partial_x u| &= O\Big(e^{-\lambda\Im\sigma\, x}\Big) \quad \text{as } x \to +\infty, \end{aligned} \tag{1.13}$$

where we used the fact that (1.7) and (1.2) imply $\partial_x\Im\Phi = \lambda\Im\sigma$.

[2]The family of outgoing solutions we construct is not continuous in the limit $\sigma \to 0^+$.

In particular, $Q[u] := \Im(\overline{u}\,\partial_x u)$ tends to zero as $x \to \pm\infty$. Integrating (1.5) on $\mathbb{R}$ gives

$$2\lambda^2 \Im\sigma \int_{-\infty}^{+\infty} \left(\Re\sigma - \frac{1}{r}\right) |u|^2 \, dx = 0 \tag{1.14}$$

If $\Re\sigma \leq 0$ or $\Re\sigma \geq 1$, the integrand has a fixed sign, and since $0 < r(x)^{-1} < 1$, continuity therefore gives $u = 0$, a contradiction.

□

1.3 Compactification of the exterior

Set

$$y := 1 - \frac{1}{r}. \tag{1.15}$$

Then $0 < y < 1$, with $y = 0$ corresponding to the event horizon and $y = 1$ to infinity. Define

$$\mu(y) := \frac{dy}{dx} = \frac{\Omega^2(r)}{r^2} = y(1-y)^2[\kappa + (1-\kappa)y]. \tag{1.16}$$

From now on, primes denote derivatives with respect to y. In particular,

$$\partial_x = \mu\partial_y, \qquad \partial_x^2 u = \mu(\mu u')'. \tag{1.17}$$

Note that $\mu > 0$ for $0 < y < 1$. Dividing (1.4) by $-\mu$ gives

$$-(\mu u')' - \frac{\lambda^2(\sigma - 1 + y)^2}{\mu} u + [V + L(L+1)]u = 0, \tag{1.18}$$

where (see (1.3))

$$\begin{aligned} V(y) &:= \frac{2\Omega^2(r(y))K(r(y))}{\mu(y)} = 2r(y)^2 K(r(y)) \\ &= (2-\kappa)(1-y) - 2(1-\kappa)(1-y)^2 \\ &= (1-y)[\kappa + 2(1-\kappa)y]. \end{aligned} \tag{1.19}$$

Thus

$$\mu(y) > 0, \qquad V(y) > 0 \qquad 0 < y < 1, \tag{1.20}$$

and

$$\mu(0) = \mu(1) = 0, \qquad V(0) = \kappa, \qquad V(1) = 0. \tag{1.21}$$

1.4 Two-sided outgoing phase conjugation

For the current argument, assume

$$\Im\sigma \geq 0, \qquad 0 < \Re\sigma < 1, \qquad y_0 := 1 - \Re\sigma \in (0,1). \tag{1.22}$$

Note that y_0 corresponds to the location of the quadratic transition point in the potential, since when $\sigma \in \mathbb{R}$

$$\sigma - \frac{1}{r(y_0)} = 0. \tag{1.23}$$

Choose the additive constant in the phase in (1.7) so that

$$\Phi(y) = \lambda \int_{y_0}^{y} \frac{z+\sigma-1}{\mu(z)}\, dz. \tag{1.24}$$

Then

$$\Phi(y_0) = 0, \qquad \mu(y)\Phi'(y) = \lambda(y-y_0) + i\lambda \Im\sigma, \quad \forall\, 0 < y < 1. \tag{1.25}$$

Note that the presence of a second-order transition point in (1.18), due to the quadratic term $-\frac{\lambda^2(\sigma-1+y)^2}{\mu}u$, is the key difficulty. We address this difficulty by conjugating u with its natural phase near the event horizon $\{y=0\}$ and near infinity $\{y=1\}$ respectively, namely we define

$$v_-(y) := e^{i\Phi(y)}u(y) \quad \text{for } 0 < y \le y_0, \qquad v_+(y) := e^{-i\Phi(y)}u(y) \quad \text{for } y_0 \le y < 1. \tag{1.26}$$

The corresponding phase defined as $\Phi(y)1_{\{0<y\le y_0\}} - \Phi(y)1_{\{y_0\le y<1\}}$ is continuous at y_0 but not C^2; it is C^1 at y_0 if and only if $\Im(\sigma) = 0$. We define $y \mapsto \varepsilon(y)$ to be $\varepsilon = -1$ in $(0, y_0]$ and $\varepsilon = +1$ in $(y_0, 1)$, and, with a small abuse of notation, we write $u = e^{i\varepsilon\Phi}v_\varepsilon$ to denote either of the above functions. For a solution u satisfying (1.6) and (1.10), these conjugations remove the endpoint phases.

Lemma 1.4 (Conjugated equation)**.** *The functions v_ε, $\varepsilon = \pm 1$, satisfy*

$$-(\mu v_\varepsilon')' + [V + L(L+1)]v_\varepsilon - i\varepsilon\lambda\big[2(y-y_0)v_\varepsilon' + v_\varepsilon\big] + 2\varepsilon\lambda\Im\sigma\, v_\varepsilon' = 0. \tag{1.27}$$

Proof. Differentiating $u = e^{i\varepsilon\Phi}v_\varepsilon$ gives

$$\mu u' = e^{i\varepsilon\Phi}\big[\mu v_\varepsilon' + i\varepsilon\lambda(y+\sigma-1)v_\varepsilon\big]. \tag{1.28}$$

$$(\mu u')' = e^{i\varepsilon\Phi}\left[(\mu v_\varepsilon')' + 2i\varepsilon\lambda(y+\sigma-1)v_\varepsilon' + i\varepsilon\lambda v_\varepsilon - \frac{\lambda^2(y+\sigma-1)^2}{\mu}v_\varepsilon\right].$$

Substitute this into (1.18). The quadratic terms $\frac{\lambda^2(\sigma-1+y)^2}{\mu}u$ cancel, and writing $y+\sigma-1 = (y-y_0) + i\Im\sigma$ gives (1.27). □

Combining $\partial_y = r^2\partial_r$ with (1.6) and (1.10), we find the following asymptotics: there are constants B_H, B_I such that

$$v_-(y) = B_H + O(y), \qquad v_-'(y) = O(1) \qquad \text{as } y \to 0, \tag{1.29}$$

and

$$v_+(y) = B_I + O(1-y), \qquad v_+'(y) = O(1) \qquad \text{as } y \to 1. \tag{1.30}$$

1.5 The Reissner–Nordström positivity identity

Define

$$H(y) := 2\mu(y) - (y-y_0)\mu'(y). \tag{1.31}$$

We will derive the following useful identity.

Lemma 1.5. *For $0 < \kappa \le 1$ and $0 \le y, y_0 \le 1$,*

$$H(y) - (y-y_0)^2 V(y) = (1-y)(1-y_0)\big[\kappa(y+y_0-2yy_0) + 2yy_0\big]. \tag{1.32}$$

In particular, if $0 < y_0 < 1$, then

$$H(y) > (y-y_0)^2 V(y) \ge 0 \qquad \text{for } 0 < y < 1. \tag{1.33}$$

Proof. Expanding (1.16), (1.19) and using the fact that

$$\mu'(y) = (1-y)\left(-4y^2(1-\kappa) - 3\kappa y + 2y(1-\kappa) + \kappa\right)$$

gives (1.32). The right-hand side of (1.32) is nonnegative for $0 < \kappa \leq 1$ and $0 \leq y, y_0 \leq 1$. For $0 < y, y_0 < 1$, the fact that

$$y + y_0 - 2yy_0 = y(1-y_0) + y_0(1-y) > 0$$

proves (1.33). □

1.6 The modified Hintz current

Define, on each side of the transition point y_0, for $\varepsilon = \pm 1$:

$$\begin{aligned}\mathcal{J}_\varepsilon := & -\mu(y-y_0)|v_\varepsilon'|^2 - \mu\Re(v_\varepsilon'\overline{v_\varepsilon}) \\ & + (y-y_0)[V + L(L+1)]|v_\varepsilon|^2 + \varepsilon\lambda\Im\sigma\,|v_\varepsilon|^2. \end{aligned} \tag{1.34}$$

Lemma 1.6 (Differential current identity)**.** *For every $\varepsilon = \pm 1$, on its corresponding half-interval, we have:*

$$\mathcal{J}_\varepsilon' = -\left[H + 4\lambda\Im\sigma\,|y-y_0|\right]|v_\varepsilon'|^2 + (y-y_0)V'|v_\varepsilon|^2. \tag{1.35}$$

Proof. Multiply (1.27) by $2(y-y_0)\overline{v_\varepsilon'} + \overline{v_\varepsilon}$ and take real parts. The phase term contributes zero, since

$$\Re\left(-i\varepsilon\lambda\,\left|2(y-y_0)v_\varepsilon' + v_\varepsilon\right|^2\right) = 0.$$

The remaining terms, after applying the product rule, give

$$0 = \mathcal{J}_\varepsilon' + \left[2\mu - (y-y_0)\mu' + 4\varepsilon\lambda\Im\sigma\,(y-y_0)\right]|v_\varepsilon'|^2 - (y-y_0)V'|v_\varepsilon|^2.$$

Now use that $\varepsilon(y-y_0) = |y-y_0|$ on the entire interval $(0,1)$. □

Proposition 1.7 (Positive integrated identity)**.** *Suppose $0 < y_1 < y_2 < 1$ and either $y_1 < y_2 \leq y_0$ or $y_0 \leq y_1 < y_2$. Take $\varepsilon = -1$ if $y_2 \leq y_0$ and $\varepsilon = +1$ if $y_1 \geq y_0$. Then:*

$$\begin{aligned}&\mathcal{J}_\varepsilon(y_2) - \mathcal{J}_\varepsilon(y_1) - \left[(y-y_0)V|v_\varepsilon|^2\right]_{y_1}^{y_2} \\ &\quad = -\int_{y_1}^{y_2}\left[H - (y-y_0)^2V + 4\lambda\Im\sigma\,|y-y_0|\right]|v_\varepsilon'|^2\,dy \\ &\qquad - \int_{y_1}^{y_2} V\left|v_\varepsilon + (y-y_0)v_\varepsilon'\right|^2\,dy. \end{aligned} \tag{1.36}$$

The RHS of (1.36) *vanishes only if v_ε vanishes on $[y_1, y_2]$. For outgoing solutions, the identity also holds at $y_1 = 0$ or $y_2 = 1$, with endpoint values understood as limits.*

Proof. We integrate by parts using V, keeping the constant angular term in $\mathcal{J}_\varepsilon$. This gives

$$\int_{y_1}^{y_2}(y-y_0)V'|v_\varepsilon|^2\,dy = \left[(y-y_0)V|v_\varepsilon|^2\right]_{y_1}^{y_2} - \int_{y_1}^{y_2}\left[V|v_\varepsilon|^2 + 2(y-y_0)V\Re(v_\varepsilon'\overline{v_\varepsilon})\right]\,dy. \tag{1.37}$$

We then complete the square as follows:

$$\begin{aligned}&\left[H + 4\lambda\Im\sigma\,|y-y_0|\right]|v_\varepsilon'|^2 + 2(y-y_0)V\Re(v_\varepsilon'\overline{v_\varepsilon}) + V|v_\varepsilon|^2 \\ &\quad = \left[H - (y-y_0)^2V + 4\lambda\Im\sigma\,|y-y_0|\right]|v_\varepsilon'|^2 + V\left|v_\varepsilon + (y-y_0)v_\varepsilon'\right|^2. \end{aligned} \tag{1.38}$$

Combining these two identities with (1.35) proves (1.36).

By (1.33), (1.20), $\Im\sigma \geq 0$ and $0 < \Re\sigma < 1$, if the sum of the integrals vanishes, continuity gives $v_\varepsilon' = 0$ and $v_\varepsilon + (y - y_0)v_\varepsilon' = 0$ throughout the interval; hence $v_\varepsilon = 0$.

Finally, the coefficients extend continuously to $y = 0, 1$, and (1.29)–(1.30) give bounded v_ε and v_ε' at the relevant endpoint. Thus the integrals converge and the identity extends to the endpoints by taking limits. □

1.7 Evaluation of the current at endpoints $y \in \{0, 1\}$ and the transition point y_0

At the transition point, $\Phi(y_0) = 0$ by definition, so

$$v_\varepsilon(y_0) = u(y_0), \qquad v_\varepsilon'(y_0) = u'(y_0) + \varepsilon \frac{\lambda \Im\sigma}{\mu(y_0)} u(y_0). \tag{1.39}$$

Substitution into (1.34) gives

$$\mathcal{J}_-(y_0) = \mathcal{J}_+(y_0) = -\mu(y_0)\Re\big(u'(y_0)\overline{u(y_0)}\big). \tag{1.40}$$

Using (1.29)–(1.30) and (1.21), we obtain

$$\begin{aligned} \mathcal{J}_-(0) &= -\big[y_0(\kappa + L(L+1)) + \lambda\Im\sigma\big]|B_H|^2, \\ \mathcal{J}_+(1) &= \big[(1 - y_0)L(L+1) + \lambda\Im\sigma\big]|B_I|^2. \end{aligned} \tag{1.41}$$

We now apply (1.36) over the intervals $[0, y_0]$ (with $\varepsilon = -1$) and $[y_0, 1]$ (with $\varepsilon = 1$). Let $\mathcal{E}_-$ and $\mathcal{E}_+$ denote the sums of the two nonnegative integrals in (1.36) over $[0, y_0]$ and $[y_0, 1]$, respectively. In the case of the interval $[0, y_0]$, the boundary term is

$$\Big[(y - y_0)V|v_-|^2\Big]_0^{y_0} = \kappa y_0 |B_H|^2.$$

It cancels the corresponding term in $\mathcal{J}_-(0)$, and hence

$$\mathcal{J}_-(y_0) = -\big[y_0 L(L+1) + \lambda\Im\sigma\big]|B_H|^2 - \mathcal{E}_-. \tag{1.42}$$

When considering the interval $[y_0, 1]$ in (1.36), the boundary term vanishes at both ends, so

$$\mathcal{J}_+(y_0) = \big[(1 - y_0)L(L+1) + \lambda\Im\sigma\big]|B_I|^2 + \mathcal{E}_+. \tag{1.43}$$

Since the two currents coincide at y_0 (equation (1.40)), we have:

$$\begin{aligned} 0 = {} & \lambda\Im\sigma\big(|B_H|^2 + |B_I|^2\big) \\ & + L(L+1)\big[y_0|B_H|^2 + (1 - y_0)|B_I|^2\big] + \mathcal{E}_- + \mathcal{E}_+. \end{aligned} \tag{1.44}$$

1.8 Exclusion of exponentially growing modes and real superradiant modes

Proposition 1.8 (No exponentially growing modes)**.** *There is no nonzero solution of* (1.4) *outgoing at both ends when* $\Im\sigma > 0$.

Proof. If such a solution existed, Lemma 1.3 would give $0 < \Re\sigma < 1$, so $y_0 \in (0, 1)$ (see (1.22)) and the results from section 1.4 onwards apply.

All terms in (1.44) are nonnegative, so $\mathcal{E}_- = \mathcal{E}_+ = 0$. Proposition 1.7 then gives $v_- = v_+ = 0$, hence $u = 0$, a contradiction.

□

Proposition 1.9 (No real modes in the superradiant interval)**.** *There is no nonzero solution of* (1.4) *outgoing at both ends when* $\sigma \in (0,1)$.

Proof. Now $\Im\sigma = 0$ and $0 < y_0 < 1$. Equation (1.44) gives

$$L(L+1)\big[y_0|B_H|^2 + (1-y_0)|B_I|^2\big] + \mathcal{E}_- + \mathcal{E}_+ = 0.$$

All terms are nonnegative, so $\mathcal{E}_- = \mathcal{E}_+ = 0$. By the strict positivity in Proposition 1.7, $v_- = 0$ on $(0, y_0)$ and $v_+ = 0$ on $(y_0, 1)$. Thus $u = 0$, a contradiction. □

1.9 Real frequencies outside the superradiant interval

For real σ, the coefficients in (1.4) are real-valued. Hence the energy current

$$Q[u] := \Im(\overline{u}\,\partial_x u) \tag{1.45}$$

is independent of x. If $\sigma \neq 0$ and a solution satisfies both endpoint asymptotic conditions (1.6) and (1.10), with amplitudes B_H, B_I possibly zero, then

$$Q[u](x) \to \lambda(1-\sigma)|B_H|^2 \text{ as } x \to -\infty, \qquad Q[u](x) \to \lambda\sigma|B_I|^2 \text{ as } x \to +\infty. \tag{1.46}$$

For $\sigma < 0$ or $\sigma > 1$, this forces $B_H = B_I = 0$. At $\sigma = 1$, u_H is bounded and satisfies

$$\partial_x u_H \longrightarrow 0 \qquad x \to -\infty.$$

Thus, $Q[u_H](x) \to 0$ as $x \to -\infty$, so equality with the $\lambda|B_I|^2$ term forces $B_I = 0$. This also excludes a common outgoing solution. Together with Proposition 1.9, we conclude that

$$\mathcal{W}_{\kappa,\lambda}(\sigma, L) \neq 0 \qquad \text{for every real } \sigma \neq 0. \tag{1.47}$$

1.10 The case $\sigma = 0$

At $\sigma = 0$, the ODE (1.4) becomes

$$\partial_x^2 u + \left[\frac{\lambda^2 - L(L+1)}{r^2} + O(r^{-3})\right] u = 0 \text{ as} \qquad r \to +\infty. \tag{1.48}$$

Equivalently, in the r coordinate,

$$\partial_r^2 u + O(r^{-2})\partial_r u + \left[\frac{\lambda^2 - L(L+1)}{r^2} + O(r^{-3})\right] u = 0. \tag{1.49}$$

The coefficients of the exact equation (1.4) at $\sigma = 0$, expressed in r, are analytic in $1/r$ near zero after the displayed powers are factored out. This ODE has a regular singular point at $r = +\infty$ (see e.g. [14]), so there exists a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ so that

$$u(r) = r^s \sum_{n=0}^{+\infty} a_n r^{-n}$$

is a solution to (1.4) at $\sigma = 0$ in $r > r_+$, where s is one of the solutions to

$$s(s-1) + \lambda^2 - L(L+1) = 0. \tag{1.50}$$

In the following, we make a choice for such an s, see already Remark 1.2.

Proposition 1.10 (Non-zero Wronskian when $\sigma = 0$)**.** *At $\sigma = 0$, assume $B_I \neq 0$ and define the solution u_I to* (1.4) *as*

$$u_I(r;\lambda,0) = \begin{cases} B_I r^{\frac{1}{2}-\sqrt{(L+\frac{1}{2})^2-\lambda^2}}\big(1+O(r^{-1})\big), & 0<\lambda<L+\frac{1}{2}, \\ B_I r^{1/2}\big(1+O(r^{-1})\big), & \lambda = L+\frac{1}{2}, \\ B_I r^{\frac{1}{2}-i\sqrt{\lambda^2-(L+\frac{1}{2})^2}}\big(1+O(r^{-1})\big), & \lambda > L+\frac{1}{2}. \end{cases} \tag{1.51}$$

Here, the $O(r^{-1})$ term can be differentiated in r, and its ∂_r-derivative is $O(r^{-2})$. With this definition, we have

$$\mathcal{W}_{\kappa,\lambda}(0,L) \neq 0 \qquad (\lambda > 0). \tag{1.52}$$

Proof. At $\sigma = 0$, a nonzero solution $u(x)$ satisfying the outgoing condition (1.6) at the event horizon obeys the following asymptotics

$$u(x) = B_H e^{i\lambda x}\big(1+O(r(x)-1)\big), \text{ as } x \to -\infty, \tag{1.53}$$

with the corresponding differentiated asymptotics. Its energy current is therefore

$$Q[u] = \lambda |B_H|^2 > 0. \tag{1.54}$$

For $0 < \lambda \leq L+\frac{1}{2}$, write $u_I = B_I r^s(1+\rho)$, where $s = \frac{1}{2} - \sqrt{(L+\frac{1}{2})^2-\lambda^2} \leq \frac{1}{2}$, $\rho = O(r^{-1})$ and $\partial_r \rho = O(r^{-2})$. Since s is real,

$$Q[u_I] = \Omega^2 |B_I|^2 r^{2s} \Im\big((1+\overline{\rho})\partial_r \rho\big) = O(r^{2s-2}) \longrightarrow 0.$$

By energy conservation, $Q[u_I] = 0$, including when $B_I = 0$. For $\lambda > L+\frac{1}{2}$, the differentiable asymptotics in (1.51), together with $\partial_x = \Omega^2 \partial_r$ and $\Omega^2 \to 1$, give

$$Q[u_I] = -\sqrt{\lambda^2 - (L+\frac{1}{2})^2}\,|B_I|^2 \leq 0. \tag{1.55}$$

Thus a nonzero common outgoing solution would have positive flux at the horizon and nonpositive flux at infinity, contradicting conservation of Q. This proves (1.52). □

Remark 1.2 (Meaning of the threshold convention.)**.** *For $\lambda > L+\frac{1}{2}$, the powers $r^{1/2\pm i\sqrt{\lambda^2-(L+1/2)^2}}$ have the same modulus. The sign in* (1.51) *is therefore part of the definition, not a consequence of selecting the slower-growing solution. It selects the branch with $Q[u_I] < 0$; the selected branches for $0 < \lambda \leq L+\frac{1}{2}$ have $Q[u_I] = 0$. The value $\mathcal{W}_{\kappa,\lambda}(0,L)$ is defined separately by this convention.*

Proof of Theorem 1.2. Proposition 1.8 excludes all frequencies with $\Im\sigma > 0$. The real superradiant interval is covered by Proposition 1.9, and the flux argument excludes all other nonzero real frequencies. Proposition 1.10 treats $\sigma = 0$ under the stated threshold convention.

More precisely, the preceding arguments give $B_I = 0$ for $\sigma \neq 0$ and $B_H = 0$ for $\sigma = 0$ whenever a solution satisfies both asymptotic conditions (1.6) and (1.10), with (1.51) replacing the latter when $\sigma = 0$. To deduce the Wronskian conclusion directly, suppose $W(u_H, u_I) = 0$. Since $W(u_H, u_I)$ does not depend on x, it vanishes for every x. If $\sigma \neq 0$, the horizon outgoing condition (1.6) makes u_H nonvanishing sufficiently near the horizon, and

$$\partial_x\Big(\frac{u_I}{u_H}\Big) = \frac{W(u_I,u_H)}{u_H^2} = 0.$$

Thus $u_I = cu_H$ there for some $c \in \mathbb{C}$. The solution u_I therefore satisfies the horizon outgoing condition (1.6) with amplitude cB_H, while its amplitude B_I at $r = +\infty$ is nonzero, contradicting $B_I = 0$. If $\sigma = 0$, use instead that u_I is nonvanishing sufficiently near infinity. The same quotient argument gives $u_H = cu_I$ there, so u_H satisfies the threshold condition with amplitude cB_I. This contradicts $B_H = 0$, since its horizon amplitude is nonzero.

□

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