What's the best way to... ?

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Abstract

Dilemmas involving optimization, economy, and logistics pervade the natural and the man-made worlds. Mathematics provides a beautiful language for framing such problems, and sharp tools for resolving them. This talk examines one problem from the research frontier, which ties together geometry, economics, and analysis, yet remains accessible at a grade 12 level. It will be followed by a problem session and discussion, focused on the possibility of a problems-based mathematics curriculum and the role of optimality therein. This discussion is motivated by the question of whether mathematics is different from poetry: "Must technique displace art as the focus of our teaching efforts?"

The example problem we probe was formulated by Monge in 1781. Motivated by economics, the problem is described as follows: Given a density $f(x)$ of iron mines throughout the countryside, and a density $g(y)$ of factories which require iron ore, decide which mines should supply ore to each factory in order to minimize the total transportation costs. Taking the mines and factories to be distributed continuously throughout Euclidean space — or a curved landscape with obstacles — and the cost per ton of ore transported from the mine at x to the factory at y to be specified as a function of the distance, yields a problem with deep connections to geometry and non-linear partial differential equations.

For costs $c(x, y) = h(d(x, y))$ given by strictly convex or strictly concave increasing functions $h \geq 0$ of the distance, the solution takes the form of a measure-preserving map between the densities f and g. It is unique, and uniquely characterized by its geometry. Even on the line this mapping may be intricate. This talk explores some unexpected features of the solution for concave costs, including the emergence of a local / global hierarchy which seems as fascinating from the economic as the mathematical point of view.

EXPLORATORY PROBLEMS

These are intended to be exploratory problems, which means that everyone should be able to start them and make some progress, but few will be able to finish them.

1. **A Marriage Problem**: Locate N brides at points $x_1 \leq x_2 \leq \ldots \leq x_N$ along the real line, and N grooms at points $y_1 \leq y_2 \ldots \leq y_N$. The question is which groom should be paired with which bride in order to minimize the average distance squared that the grooms must commute to get to their brides.

a) Show that the optimal pairing is achieved if the groom at x_i is paired with the bride at y_i for each $i = 1, \ldots, N$. Do the case $N = 2$ first!

b) Show that the same pairing remains optimal if we try to minimize the average value of some other power $c(x, y) = |x - y|^p$ of the distance provided $p \geq 1$; (or more generally if $\partial^2 c/\partial x \partial y \leq 0$).

c) Give an example to show that the pairing may not be optimal if $p < 1$.

2. **Extremal Magic Matrices:** A set $C \subset \mathbf{R}^n$ is convex if it contains the line segment $\{(1-t)x + tx' \mid t \in [0, 1]\}$ whenever x and x' are distinct points in C. A point $e \in C$ is said to be *extreme* if it is NOT the midpoint of any segment with endpoints in C. Consider the space $P(n)$ of $n \times n$ *doubly stochastic matrices* — i.e., matrices with non-negative coefficients, for which the entries in each column and each row sum to 1.

(a) Prove $P(n)$ forms a convex subset of all $n \times n$ matrices; it can be visualized as a convex polytope in $\mathbf{R}^{n^2}$.

(b) Find the extreme points of $P(n)$. How many of them are there?

The analogous continuum question involves non-negative densities $f(x, y) \geq 0$ on the square which integrate to one along each horizontal or vertical segment:

$$F = \left\{ f(x, y) \geq 0 \;\middle|\; \int_0^1 f(x_o, y) dy = 1 = \int_0^1 f(x, y_o) dx \text{ for each } x_o, y_o \in [0, 1] \right\}$$

c) Show F is convex but has no extreme points.

d) However, if we close F in a suitable metric (allowing f to be a "non-negative measure" or "generalized function"), then $\overline{F}$ has extreme points E; indeed, $\overline{F}$ is the smallest closed convex set containing E. Finding a characterization of the extreme points analogous to (b) has remained an open problem through the 20th century.

3. **Parish Boundaries**. Distribute parishioners uniformly throughout a region $\Omega \subset \mathbf{R}^2$, served by only a few churches located at $y_1, \ldots, y_N \in \mathbf{R}^2$. Each church can accommodates a fraction $1/N$ of the parishioners. Assume the parishioners are assigned to parishes so as to minimize the average *distance* commuted to mass.

a) If $N = 2$, show the region Ω is divided into two parishes, and the dividing curve forms the arc of a hyperbola. Where are the churches relative to its foci?

b) If instead we minimize the average distance *squared*, show the dividing curve lies along straight line. How does its direction compare with a line through the two churches?

c) Redo part (b) with N churches. Show that each parish forms a *convex* polygon (or at least the intersection of such a polygon with Ω).

d) Explore analogs of (a–c) for non-uniform distributions of parishioners on Ω, and/or churches of varied sizes.

4. **Convex-Concave Min-Max:** Suppose $f(x, y)$ is a continuous function on $I \times J$, where $I, J \subset \mathbf{R}$ are two compact intervals. Then

$$\inf_{x \in I} \sup_{y \in J} f(x, y) \geq \sup_{y \in J} \inf_{x \in I} f(x, y).$$

Although equality will not hold in general, prove that it does hold provided both that $f(x, y_o)$ is a convex function of x for each $y_o \in J$ and $f(x_o, y)$ is a concave function of y for each $x_o \in I$.