SOAR Math Course Homework Eleven Winter, 2002

These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

1. Recall that the *center* $Z(G)$ of a group G is the set of elements that commute with every other element of G:

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}.$$

We found in class that the braid group B_3 can be written

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

(Here σ_k is "line k over line $k+1$" and group multiplication ab is joining the top of braid b to the bottom of braid a.)

(a) Prove that $g = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ (the 180° twist) is *not* in the center $Z(B_3)$. Hint: write down the pictures for the products $\sigma_1 g$ and $g\sigma_1$. Are they the same?

(b) Prove that g^2 (the "360° twist") *is* in the center $Z(B_3)$.

2. We will consider moves (as defined in class) as elements of the *cube group* of transformations of the Rubik's cube. (We will see that this is a group whose order is more than 4.3×10^{19}.) Elements of this group may have small order, as we'll see in this problem.

(a) What is the order of the element $\mathsf{BULU}^{-1}\mathsf{L}^{-1}\mathsf{B}^{-1}$? Hint: This causes the permutations

$$\mathsf{uf} \to \mathsf{ul} \to \mathsf{bu} \to \mathsf{uf}, \qquad \mathsf{ufr} \to \mathsf{rfu} \to \mathsf{flu}, \qquad \mathsf{ulb} \to \mathsf{rub} \to \mathsf{bul}.$$

The first is a 3-cycle (an element of order 3) and the other two are both 6-cycles (elements of order 6). The total move can be thought of as the product of these three (disjoint) permutations on the cubies. What is the order of this product?

Use the same idea to find the order of the moves...

(b) UR

(c) UR^{-1}

(d) UR^2

Notice that you're finding the smallest value of k so that, for example, $(\mathsf{UR})^k$ is the identity. If you start with a solved cube, this should be easy enough (though tedious) to check.

3. How big is the cube group? We said in the previous problem that it had more than 4.3×10^{19} elements, but Anton wants to know how big a number this is. Here we go...

   (a) Assume that you can make 10 turns per second. If you started at the birth of the universe, would you have been able to make 4.3×10^{19} turns by now? (Assume the age of the universe is 10 billion (10^{10}) years and that there are 365.25 days per year.)

   (b) A Canadian dime is roughly 1.22 mm thick (as compared to 1.35 mm for a US dime). Stack 4.3×10^{19} Canadian dimes. How high is this stack? Let's ask this question in a different way: the average distance from the earth to the sun is about $149,597,870$ km (this is called 1 AU, or Astronomical Unit). How many AU is our pile of dimes?

   (c) Continuing our previous thought: a Canadian dime has a mass of 1.75 g. Leda, the smallest of Jupiter's moons, has mass of 5.68×10^{15} kg. How many Ledas would it take to equal the mass of 4.3×10^{19} Canadian dimes?

   (d) Finally, suppose we actually *owned* our pile of 4.3×10^{19} Canadian dimes. Bill Gates, according to some web site, is currently worth roughly US$ 59.3 billion. Let us use this as our standard: 1 BG = US$ 5.93×10^{10}. At an exchange rate of $1 Canadian is $0.646 US, how many BG's is our pile of dimes worth?

4. [A repeat assignment!] Participate in the contest! See the course web page. Deadline for entries: Monday, January 6th at midnight.