

These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

One simple way to write down a group is to use what's called a *presentation*. For example, we write D_n as

$$D_n = \langle r, m \mid r^n = 1, m^2 = 1, rm = mr^{n-1} \rangle.$$

The terms r and m are called the *generators* while $r^n = 1$, $m^2 = 1$, and $rm = mr^{n-1}$ are called the *relations*.

1. Write down a presentation for:

(a) \mathbf{Z}_n , and

(b) T (the group of symmetries of a tetrahedron).

2. Presentations are nice, but occasionally misleading, as in the following example. Consider the group

$$G = \langle a, b \mid ab = b^2a, ba = a^3b \rangle.$$

(a) Show that $aba^{-1}b^{-1} = b$ and $bab^{-1}a^{-1} = a^2$.

(b) Conclude from (a) that $b = a^{-2}$.

(c) Use this “new relation” to identify G as a familiar (and very simple) group.

3. A contest! Find a presentation of a familiar group that obscures the group as much as possible. You must include a demonstration that your group is familiar to the judge (me). The decision of the judge is final...
4. We heard in class that there are two groups of order 21. One of these is, of course, $\mathbf{Z}_{21} \cong C_{21}$. The other is non-abelian, and looks like:

$$G_k = \langle a, b \mid a^3 = 1, b^7 = 1, ba = ab^k \rangle.$$

The question is: what is k ? (Of course, G_k is supposed to be non-abelian, so $k \neq 1$. Also, $b^7 = 1$, so you may assume $1 < k < 7$.) Does this give us five different non-abelian groups of order 21, does this only work for some k , or are all the different G_k the same (isomorphic)?

5. We conjectured in class today that, for a positive integer n ,

$$\sum_{\substack{m \mid n \\ 1 \leq m \leq n}} \phi(m) = n.$$

Prove this for $n = pq$, where p and q are primes. (This approach – writing the prime decomposition of n – is not recommended in general, but this is a simple case.)