SOAR Math Course | Homework Four | Winter, 2002

These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

1. The one part of Lagrange's theorem we didn't prove is the following lemma: *Suppose H is a subgroup of a finite group G. Then, for any element g of G, the order of gH equals the order of H.* Let's prove this as follows: first, notice that $|gH| \leq |H|$. Next, show that $|gH| \geq |H|$ by showing that one can't have $gh_1 = gh_2$ for two distinct elements h_1 and h_2 of H.

2. Complete a *Cayley table* (that is, a multiplication table) for D_3 and D_4, the sets of symmetries of an equilateral triangle and square. Use the notation that $D_n = \{1, r, \ldots, r^{n-1}, m, mr, \ldots, mr^{n-1}\}$, where r is a minimal counterclockwise rotation and m is a mirror reflection, as in class. (This is mostly so you'll have these handy for the next two problems.)

3. Consider the element m in D_3. We say that an element $y \in D_3$ is *conjugate* with m if $y = x^{-1}mx$ for some $x \in D_3$. So mr^2 is conjugate with m since $r^{-1}mr = mr^2$.

   (a) Find all elements in D_3 that are conjugate with m. We call this set of elements the *conjugacy class* of m.
   (b) Find all other conjugacy classes in D_3.
   (c) Do any of your conjugacy classes intersect?
   (d) Does any element not belong to a conjugacy class.

4. Find all the conjugacy classes of D_4. Can you determine the conjugacy classes of D_n?

5. Prove the following facts for any group, G, and elements x, y and z in G.

   (a) If $x \in G$ then x is conjugate with itself.
   (b) If x and y are conjugate, then y and x are conjugate.
   (c) If x and y are conjugate and y and z are conjugate then x and z are conjugate.

   These properties are called the *reflexive*, *symmetric* and *transitive* properties. Any relation, such as conjugacy, which possesses these properties is said to be an *equivalence relation*.

6. Write a short computer program to determine all possible vertex types. Recall that a vertex type is an integer solution $\{n_1, n_2, \ldots, n_k\}$ of the equation

$$\frac{n_1 - 2}{n_1} + \ldots + \frac{n_k - 2}{n_k} = 2$$

   or, equivalently,

$$\left(\frac{1}{2} - \frac{1}{n_1}\right) + \ldots + \left(\frac{1}{2} - \frac{1}{n_k}\right) = 1.$$

   (The n_j are all at least 3, and may repeat values.) You may assume what we proved in class: there are no solutions with any n over 42.